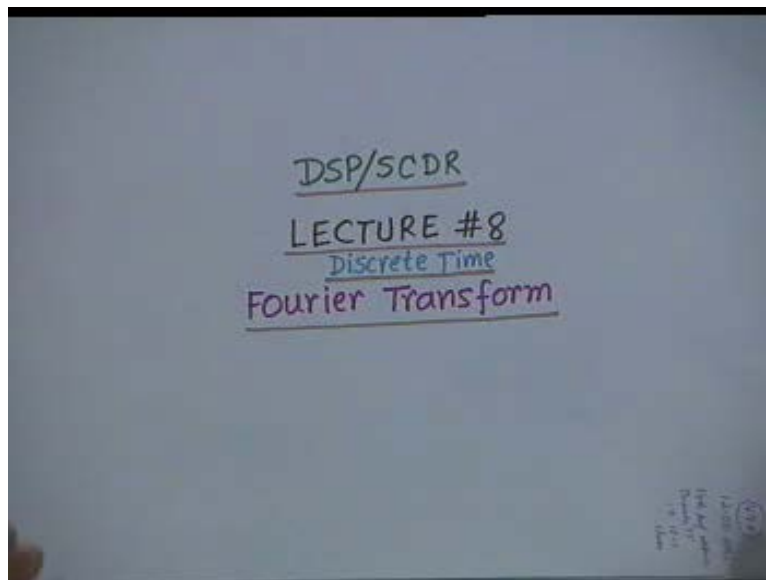**Digital Signal Processing**

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Lecture- 8

Discrete Time Fourier Transform

Digital Systems

This is the 8th lecture on DSP and today's topic is Discrete time Fourier Transform.

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As I have already told you, we shall omit the adjective discrete time; we shall simply call it Fourier Transform. In the last lecture, we had discussed in detail difference equations. I told you that difference equations can be solved by two methods; you can have the complementary function, which is the solution to the homogeneous difference equation, and add to it the particular solution. Then you find the constants in the complementary function from the initial conditions. The other method is: you find out zero input response and then add it to zero state response. In both of them you have to find out the constants. In the zero input response, you have
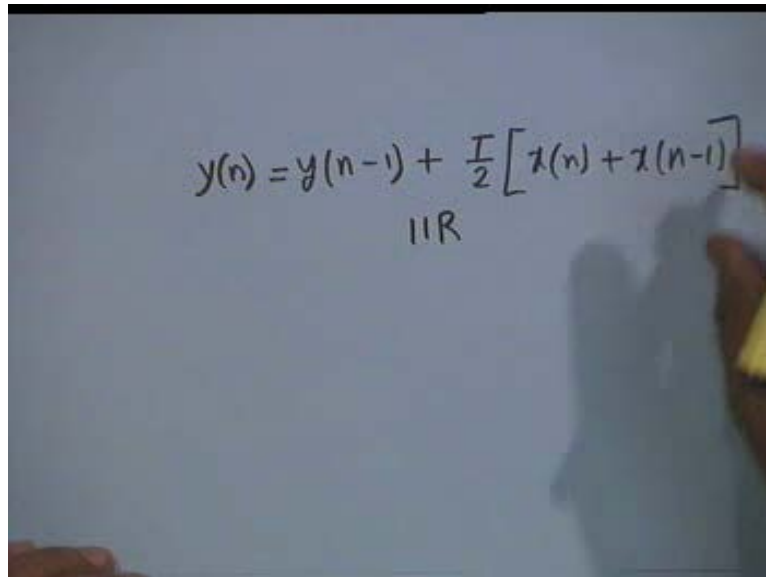
to find the constants from the given initial conditions. In zero state response, you have to find the constants from zero initial conditions. You have to find the constants twice. But since these terms have significance in system theory, they should have been introduced and we have done that.

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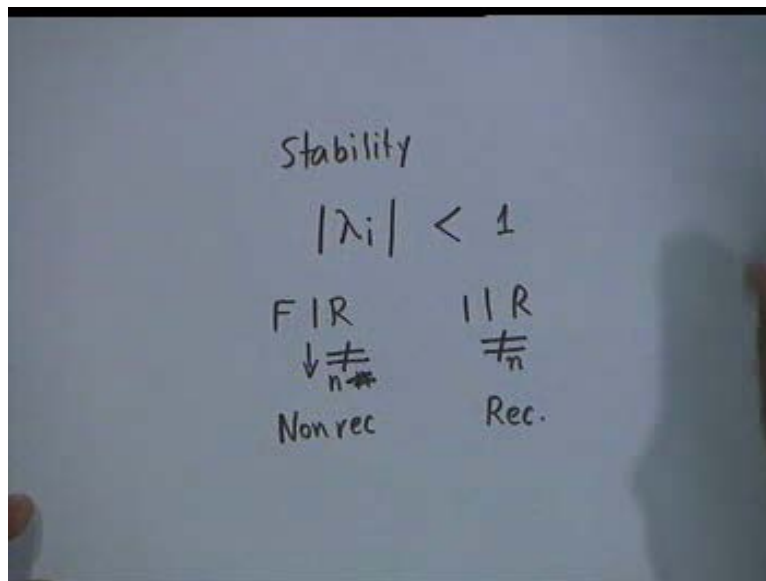
We also said that stability can be obtained in terms of the roots of the characteristic equation, which is the characteristic polynomial equated to 0. Each root magnitude, for stability, should be bounded by unity; it should be strictly less than 1. We discussed FIR and IIR and we illustrated the fact that FIR does not necessarily mean non recursive. Similarly IIR is not necessarily equivalent to recursive.
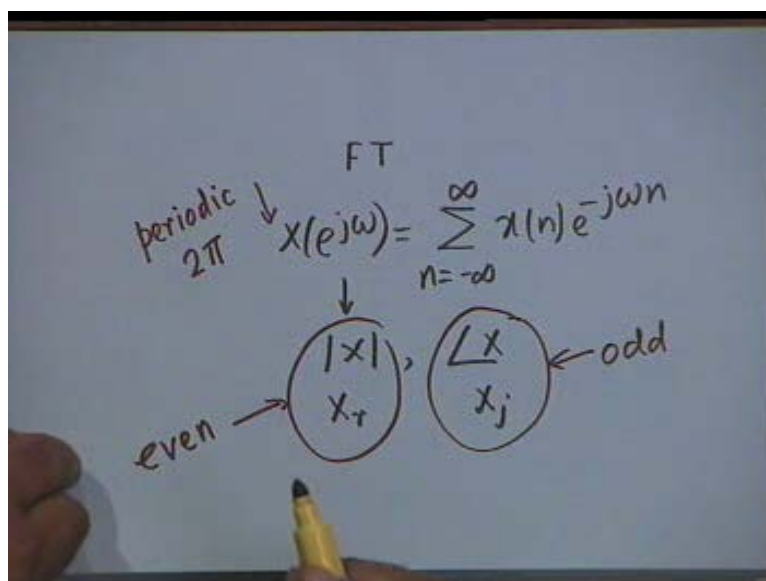
We also discussed an example of digital integrator and the equation that we got was $y(n) = y(n - 1) + (T/2) (x(n) + x(n - 1))$. We showed that it is an IIR system and we also found out its unit impulse response. Then we introduced the Fourier Transform.

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FT is the so called DTFT which we have abbreviated as FT and we defined $X(e^{j\omega}) =$ summation $x(n)\, e^{-jn\omega}$ where $n = -$ infinity to $+$ infinity. In terms of its parts, it can either be described in polar form in terms of magnitude ($| X |$) and angle, or in the terms of real part (X_r) and the imaginary part (X_j). An intriguing fact is that the imaginary part is also a real quantity. It is only by multiplication by j that you make it imaginary. We also stated that both magnitude and the real part are even functions of omega and angle X and X_j are the odd functions of omega. The other point is that the $X(e^{j\omega})$ is a periodic function with a period of 2π. That is the reason we concentrate on $-$ pi to $+$ pi, and because of the evenness and oddness of its parts, we concentrate only from 0 to π. If we know this portion then we know the total spectrum.

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We considered two examples. One was delta (n) whose Fourier Transform is 1 and the other was $(1/2)^n\, u(n)$ whose Fourier Transform is $1/(1 - \frac{1}{2}\, e^{-j\omega})$. In general, if 1/2 is replaced by alpha, some constant, which can be real, or complex, that is, the signal is alpha^n u(n), then the transform can be very easily shown as $1/(1 - \text{alpha}\, e^{-j\omega})$, provided the magnitude of alpha is less than 1. This raises the question of existence of the Fourier Transform. The Fourier Transform summation must converge in order that the Fourier Transform exists.

We concluded the 7th lecture by finding out IFT (Inverse Fourier Transform) and we said that $x(n) = [1/(2\pi)]$ integral $-\pi$ to π [Xe $^{(jomega)}$ e$^{j\,n\,omega}$ d omega]. We also emphasized the fact that Fourier Transform and inverse Fourier Transform are not both matters of definition; one follows from the other. That is, given x(n), we can find out $X(e^{jomega})$ uniquely, or given $X(e^{jomega})$, we can find x(n) uniquely; one follows the other. Now consider the question of convergence of the summation $[X(e^{jomega})]$. Formally if we take this summation[x(n) e$^{-j\,n\,omega}$] where n = $-$ k and + k, obviously, the sum being finite, shall always converge. If we allow k to go to infinity, this must converge for FT to exist.

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$$\text{Lim}_{K\to\infty} \left| X(e^{j\omega}) - X_k(e^{j\omega}) \right| \to 0$$

$$\text{if} \quad \sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

$$\underline{|X(e^{j\omega})|} \leq \sum_{n=-\infty}^{\infty} |x(n)| < \infty$$
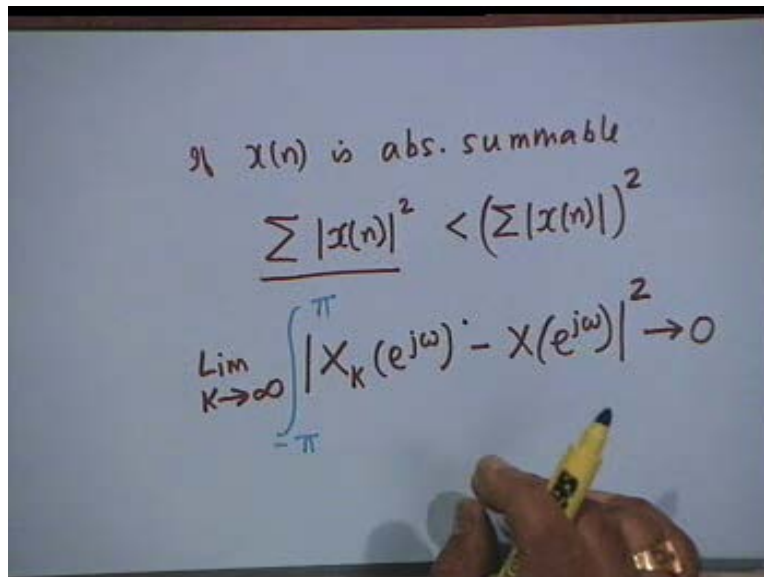
$$\text{Sufficient}$$

One of the criterion is that the magnitude of the error between the actual value $X(e^{j\omega})$ and the finite sum $X_k(e^{j\omega})$, as k goes to infinity, must tend to 0. This criterion for convergence is sufficient but not necessary. One of the sequences which always satisfies this criterion is one which is absolutely summable. The magnitude of $[X(e^{j\omega})]$ will be less than or equal to the summation [magnitude of $(x(n)]$. Thus if x(n) is absolutely summable, then its Fourier Transform exists. But this condition is only sufficient. How do you prove that it is sufficient? Give at least one example of a sequence which is not absolutely summable but its Fourier Transform exists. There are sequences which are not absolutely summable but square summable. One such example is provided by the sequence $x(n) = (1/n) u(n - 1)$.

$$x(n) = \frac{1}{n} u(n-1)$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{does not converge}$$

Now it can be shown that summation $[1/n^2] = \pi^2/6$ where n =1 to infinity. But summation $(1/n)$, in which n = 1 to infinity does not converge. Thus $(1/n)$ u(n – 1) is not absolutely summable, but summation $(1/n^2)$, n = 1 to infinity converges. So this is a counter example showing that x(n) absolute summability is not necessary but a sufficient condition. Now $(1/n)$ u(n – 1) is an example of a sequence whose energy is finite because summation $1/n^2$ defines its energy.

The next statement is, if x(n) is absolutely summable then its energy, summation $|x(n)|^2$ is less than infinity because $a^2 + b^2$ is always less than $a^2 + b^2 + 2ab$. So an absolutely summable sequence has a finite energy. An absolutely summable sequence is also a square summable, that is, its energy is finite, but converse is not true. In other words, a finite energy sequence is not necessarily absolutely summable.

The example we took is that x(n) = (1/n) u(n – 1) has finite energy but is not absolutely summable. However such sequences which are square summable do have Fourier Transforms. In this case we take $/(X_k(e^{jomega}) - X(e^{jomega}))/^2$ and integrate from – pi to + pi. This is our range of vision and within this range this error in energy must go to 0 as k goes to infinity. This is another way of satisfying the existence of Fourier Transform. First, we said that absolutely summable sequences have a Fourier Transform, but it is only a sufficient condition. Then square summable sequences also have a Fourier Transform and the error criterion is that as k goes to infinity, total error energy in the frequency domain tends to 0. There are sequences which are neither absolutely summable nor square summable but the Fourier Transform does exist.
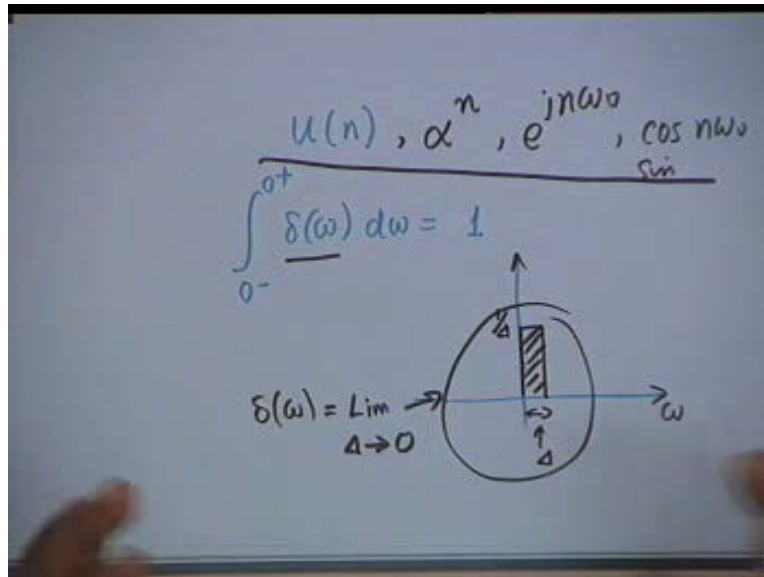
For example, is u(n) absolutely summable? The sum goes to infinity, and since $1^2 = 1$, u(n) is not square summable either. However, the Fourier Transform does exist. In other words square summabiltiy is also a sufficient condition, like absolute summability.

There are sequences which are neither absolutely summable nor square summable and for them also Fourier Transform exists. For the existence of Fourier Transform, we have to appeal to the analog delta function. The Fourier Transform of such sequences which are neither absolutely summable nor square summable shall have the analog delta function in its FT. The analog delta function is defined by integral (0 – to 0 +) delta (omega) d (omega) = 1 where delta (omega) is nothing but the limit of a square pulse whose duration is d and whose amplitude is 1/d, so that the area under the pulse is unity, when d tends to 0. This is the definition of an analog impulse function. Note that analog as used here is not in the time domain, it is in the frequency domain.

The absolute summability or square summabilty criterion is not valid for x(n) = (alpha)^n not multiplied by u(n), where alpha can be any quantity. $e^{jn\varpi_0}$ is also neither square summable nor absolutely summable. If $e^{jn\varpi_0}$ has a Fourier Transform then its real part and imaginary part which are sinusoidal that is cosine(n omega$_0$) or sine(n omega$_0$) also should have a Fourier Transform

because one is the real part and the other is the imaginary part. We shall show that for such functions, which are neither absolutely summable nor square summable, the Fourier Transform will have an analog delta function. Once you admit the existence of this function, delta (omega), then they are Fourier transformable.

Therefore, if we take the help of delta function a large range of functions are Fourier Transformable and u(n) and $e^{jn\varpi_0}$ are examples.

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For example, let us take $e^{jn\varpi_0}$. You can show that the Fourier Transform of this is given by summation [2π analog delta(omega – omega $_0$ + 2π k)] where k goes from – infinity to + infinity. This is an example; the FT of $e^{jn\ omega0}$ contains a chain of delta functions, at $omega_0$, $omega_0 + 2\pi$, $omega_0 – 2\pi$ and so on. Proving this in a straight forward manner by applying the definition of Fourier transform is not easy. You have to bring in the theory of distribution. As you know, Fourier Transform is a one to one transformation, that is, if you prove that the inverse Fourier Transform of $X(e^{j\omega})$ is x(n), then this is a good enough proof because this is necessary as well as sufficient; it has to be one to one transformation. And in such cases, it is easier to find the inverse Fourier Transform. The inverse Fourier Transform will be: x(n) = [1/(2π)] integral

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[summation (2π delta ($\omega - \omega_0 + 2\pi$ k))e$^{jn\ omega}$ d (omega)]; here integral ranges from – pi to + pi and k goes from – infinity to + infinity. In this summation, we have to consider only one term i.e. k = 0, because integration is from – pi to + pi, this being the base band. Therefore, 2π and 2π cancel, the integral of [delta (omega –omega$_0$) e$^{jn\ omega}$ d (omega)] ranges from – pi to + pi. This function exists only at omega = omega$_0$; it then follows that the inverse Fourier Transform of this summation we started with, is $e^{jn\varpi_0}$.

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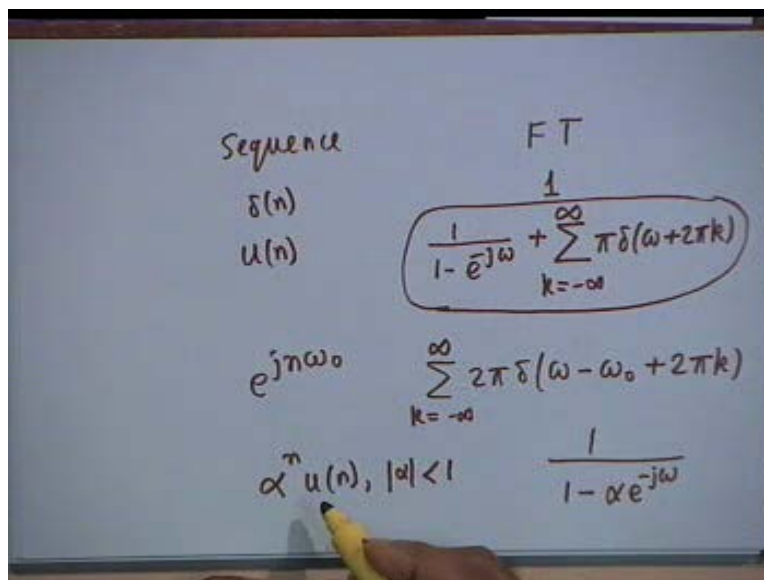
We now write a few sequences and their Fourier Transforms. For delta(n), the FT is1; u(n) has a rather complicated transform which is [1/(1 – e$^{-jomega}$)] + summation π delta (omega + 2π k) where k goes from – infinity to + infinity. If you apply the formula for FT of (alpha)^n u(n) with alpha = 1, you should simply get 1/(1 – alpha e$^{-j\ omega}$). There are two short comings in this application. First, the formula is valid for $|\alpha| < 1$; secondly, it does not take care of the average value of u (n). Average value is 1/2 and the average value is a constant, whose Fourier transform is this summation term. For proof, you know that the inverse Fourier transform of this quantity is equal to u(n). In a similar manner, you can show that $e^{jn\varpi_0}$ transforms to, summation 2π delta(omega – omega$_0$ + 2π k), where k goes from – infinity to + infinity, alpha^n u(n), magnitude

alpha less than 1, transforms to $1/(1 - \text{alpha } e^{-j\text{omega}})$. This table gives you the basic sequences and transforms. Any other sequence can be expressed in terms of these sequences. I suggest that you verify this table for yourself with questions like what will be the Fourier Transform of $u(-n)$ or of $\text{alpha}^n u(-n)$. Once we admit the existence of a Fourier transform, all properties shall be valid.

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Next, I take the example of the up sampler: $y(n) = x(n/L)$ where $n = 0, (+ -) L, (+ -)2L$ and so on and 0 otherwise. The Fourier Transform of $y(n)$ is $Y(e^{j\text{omega}}) = $ summation $[x(n/L) e^{-jn \text{ omega}}]$. That is the definition, where $n = 0$ or $\pm L, \pm 2L, \ldots$ up to infinity, but only for discrete values of n. We are not taking the in between samples. In between samples at, e.g. $n = 1, 2 \ldots L - 1$, will contribute 0 to the summation. What does this summation lead to? Let us put $n/L = m$, (some other quantity, which is an integer). Then my summation becomes summation $x(m) e^{-jmL \text{ omega}}$ where $m = -$ infinity to $+$ infinity, and this equals $X(e^{jL \text{ omega}})$.
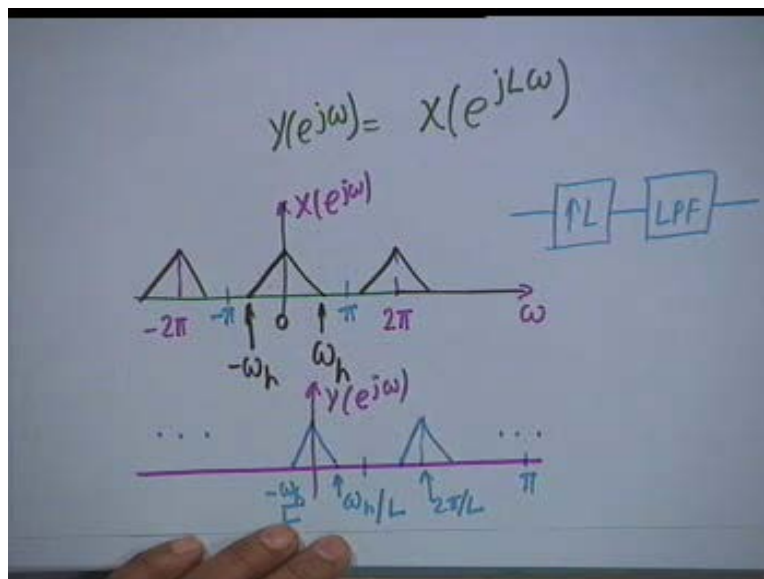
Therefore for an up sampler the spectrum $Y(e^{j\omega mega})$ is related to the original spectrum of the input sequence by $X(e^{jL\ omega})$. Since, a picture is worth 1000 words, let us draw the picture. Let us suppose that X has the spectrum shown in the figure, it has to be repetitive. We have marked $0, -\omega_h, \pi, -\pi, 2\pi, -2\pi$ (repetition after every 2π) etc. Note that I have taken $\omega_h \langle \pi$. If I take the spectrum of $Y(e^{j\omega mega})$, how does it change? You see that it would have the same shape, except that ω_h will be replaced by ω_h/L; next one will be centered at $2\pi/L$ and extend from $(2\pi/L) - \omega_h/L$ to $(2\pi/L) + \omega_h/L$ and so on. So the spectrum is compressed; what existed between $-\omega_h$ and $+\omega_h$, is now compressed to $-\omega_h/L$ and $+\omega_h/L$. How many such repetitions shall be there between $-\pi$ and π? There are $2L - 1$ repetitions, whereas from $-$ pi to $+$ pi there was only one sample in $X(e^{j\omega mega})$. In the up sampled version, you get repetitions between 0 and pi, $L - 1$ of them, and this creates problems in up sampling. In up sampling, you have to follow the up sampler by a low pass filter to get rid of all the repetitions and the low pass filter cut off frequency must be ω_h/L or slightly more than ω_h/L but before the next spectrum starts. The up sampler is also a spectrum compressor and the repetition may create aliasing if not properly filtered. As you may recall higher frequencies posing as low frequencies is aliasing distortion. Thus up sampler has to be necessarily followed by a low pass filter whose cut off frequency is such that only the base band signal of the original

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sequence is retained. All the rest is deleted; otherwise your further processing will give you unreliable results.

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Let us take another example: you are required to find the inverse Fourier Transform of this sequence $X(e^{j\omega}) = j$ sine omega$[(3 + 4$ cosine omega $+ 2$cosine^2 omega$)]$. In a general case we shall have to evaluate the integral $[1/(2 \text{ pi})]$ ($-$ pi to pi) $X(e^{j\omega})$ $e^{jn\omega}d\omega$. Here, we require to express this as summation$[x(n)e^{-jn \text{ omega}}]$ with appropriate limits of n. In general n $= -$ infinity to $+$ infinity. That is, we have to find the coefficients of $e^{-n \text{ omega}}$. This is an example where $X(e^{j\omega})$ $= [(e^{j\omega} - e^{-j\omega})/2]$ multiplied by $[(3 + 4 [(e^{j\omega} + e^{-j\omega})/2] + 2((e^{j\omega} + e^{-j\omega})/2)^2]$. We have to simplify this and find out the coefficients of $e^{-jn \text{ omega}}$. What will be the highest power of exponential? It is 3, is it $+$ or $-$? It is both. If you can write this form, then the sequence x(n) shall be obvious.

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$$x(n) = \left\{ \frac{1}{4}, 1, \frac{4}{7}, \underset{\uparrow}{0}, \frac{9}{2}, -1, -\frac{1}{4} \right\}$$

$$n=0$$

Properties

Name	sequence	FT
	$g(n)$	$G(e^{j\omega})$
	$h(n)$	$H(e^{j\omega})$
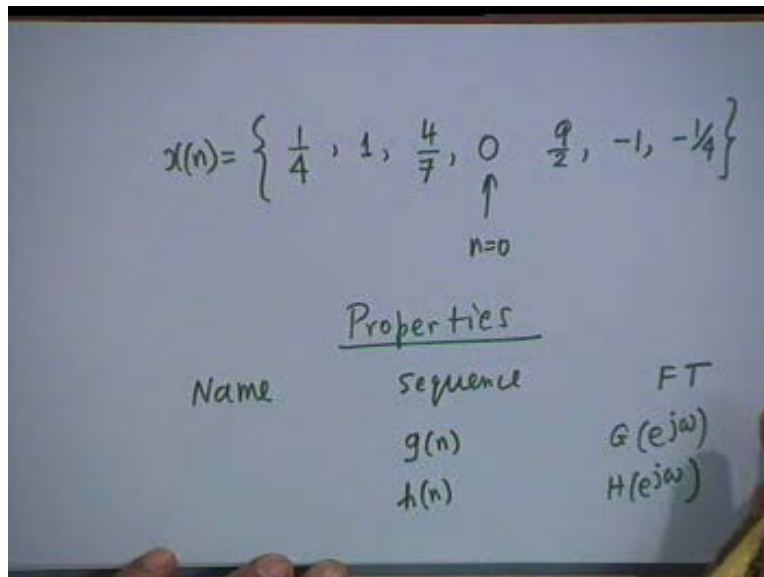
I suggest you to do this simplification and verify that the sequence becomes x(n) = (1/4,1,4/7.0 (and this is n = 0) 9/2,–1,–1/4). So given a problem either to find the DFT or the IDFT or prove a given Fourier Transform pair, you have to decide which way and how to go about it.

Now, we talk about the properties of Fourier Transforms. And to do this, we make a table: name of the property, sequence and its Fourier Transform. We shall use two sequences g(n) which has a Fourier Transform $G(e^{j\omega})$ and h(n) which has a Fourier Transform $H(e^{j\omega})$.
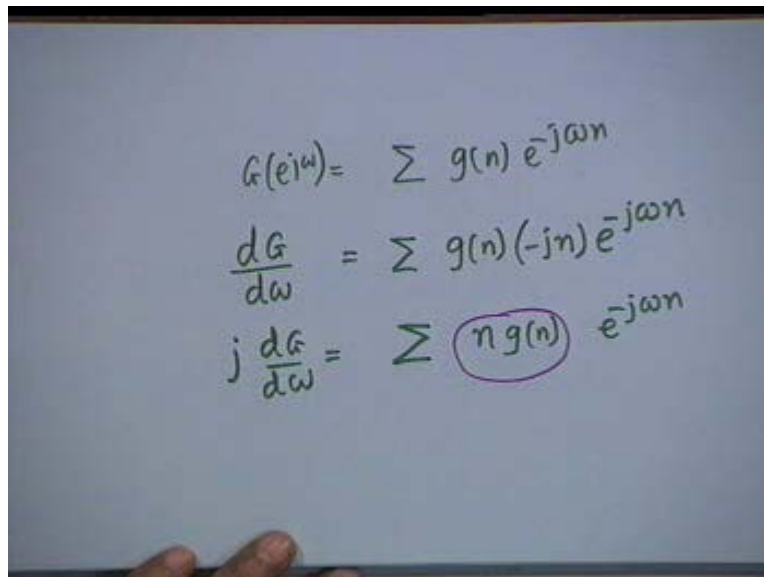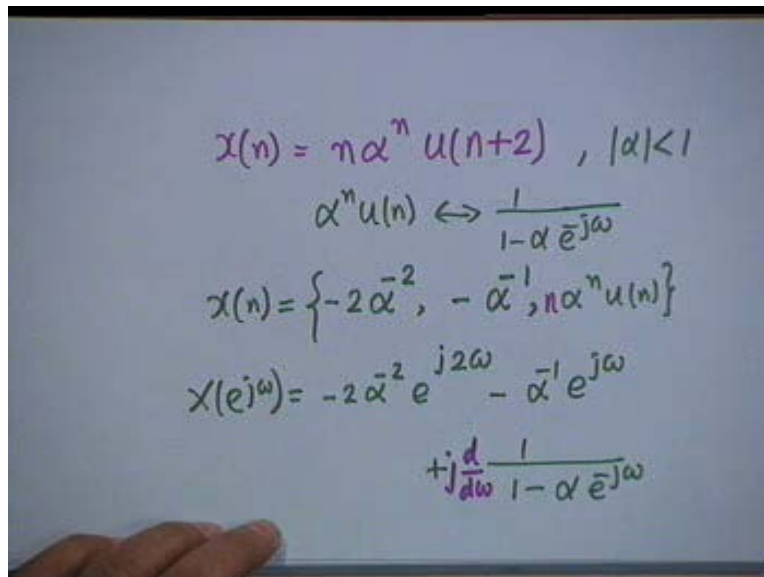
First property is Linearity, Fourier Transform operation obviously is a linear operation. In other words, if you take alpha g + beta h, this shall have a Fourier Transform alpha G + beta H (it is a linear transformation because the definition is simply a summation). Then comes time shift: if you take g $(n - n_0)$, then the Fourier Transform would be $e^{-jn_0 \, \text{omega}} \times G(e^{j\text{omega}})$; this also follows from the definition. Frequency shift property is that if you multiply a sequence g(n) by an exponential sequence $e^{jn\,\varpi_0}$, then the Fourier Transform is $G(e^{j(\text{omega} - \text{omega}_0)})$; this also follows simply by writing the definition. Then comes differentiation in the frequency domain: multiplication of g(n) by n leads the Fourier Transform j d $G(e^{j\text{omega}})$/d omega; this can also be proved from the definition itself.

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Write $G(e^{j\omega}) = $ summation $g(n)$ $e^{-jn\ \omega}$ for n = – infinity to + infinity, and differentiate with respect to omega. Then in the right hand side we get summation $[g(n)(- j\ n)e^{-jn\ \omega}]$. If I multiply both sides by j, j d G/d omega becomes equal to summation n $g(n)$ $e^{-jn\ \omega}$ which clearly shows that $jdG/d\omega$ is the Fourier Transform of the new sequence n $g(n)$. Similarly, all other properties simply follow from the definition.

Let us take an example: let $x(n) = n \text{ alpha}^n u(n + 2)$. We know that the Fourier Transform of $\text{alpha}^n u(n)$, magnitude alpha less than 1, is $1/(1 - \text{alpha } e^{-j\omega})$. How does the given sequence differ from this? There are just two additional samples in $x(n)$ and we have to take care of them. We write $x(n) = \{-2 \text{ alpha}^{-2}, - \text{alpha}^{-1}, n \text{ alpha}^n u(n)\}$; therefore $X(e^{j\omega}) = - 2 \text{ alpha}^{-2} e^{j2\omega} - \text{alpha}^{-1} e^{j\omega} + j (d/d \text{ omega}) [1/(1 - \text{alpha } e^{-j\omega})]$. Try to understand the problem; if you understand the problem, then half of it is solved and the rest is pure algebra and calculus.

The final result is: $X(e^{j\omega}) = -2\text{alpha}^{-2} e^{j2\omega} - \text{alpha}^{-1} e^{j\omega} + \text{alpha } e^{-j\omega}/(1-\text{alpha } e^{-j\omega})$.

The next property is convolution: it says that if you have a sequence which is the convolution of two sequences, that is if $x(n) = g(n)*h(n)$, then, in the frequency domain, the Fourier Transform is simply the product of $G(e^{j\omega})$ $H(e^{j\omega})$ and this is a great simplification. Finding convolution in the time domain may not be very simple. But in the frequency domain, if you know the two Fourier Transforms, then you simply multiply them and then find the Fourier inverse. Inversion may not require evaluating an integral. But if you have to do it, then you do it!

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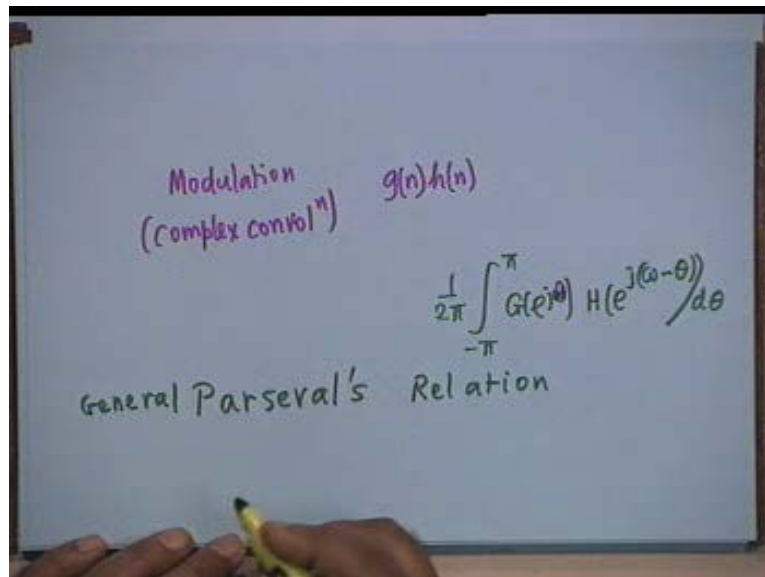
Next comes the property of Modulation; it is also known as Complex Convolution. This property says that if two sequences are multiplied in the time domain (the previous case was multiplication in the frequency domain and it was convolution in the time domain), in the frequency domain, it is a convolution. The expression is $[1/(2\pi)]$ integral $(-\pi$ to $\pi)$ $G(e^{j\,theta})$ H $(e^{j(\omega - theta)})$ d(theta). We introduce a dummy variable theta. $H(e^{j(\omega - \theta)})$ a delayed and folded version; so this is called Complex Convolution, because it is a convolution of two complex quantities.

$$\sum_{n=-\infty}^{\infty} g(n)\, h^*(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} G(e^{j\omega})\, H^*(e^{j\omega})\, d\omega$$

$$g(n) = h(n)$$

$$\sum_{n=-\infty}^{\infty} |g(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |G(e^{j\omega})|^2\, d\omega$$

Fourier Transform also gives an easy way of calculating the energy of a sequence and this is given by the so called Parseval's relation. We will first state the general Parseval's Relation: summation g(n) h*(n) = [1/(2 pi) integral – π to π [G(e^{jomega}) H*(e^{jomega}) d omega]. This can be proved from the Modulation Theorem just discussed. Try proving it. This is the generalized Parseval's relation where we have assumed that both g and h can be complex and that is why we put *. Now a special case of this is g(n) = h(n) then you see the left hand side becomes summation g(n) magnitude squared where n = – infinity to + infinity, and this would be [1/(2 pi)] integral – π to π G(e^{jomega}) magnitude squared d omega. The left hand side is energy in the time domain and the right hand side is the energy in the frequency domain. The two energies must be the same because Fourier Transform is a one to one transformation.

I shall conclude the class with some symmetry relations. I strongly urge you to prove each of them. Proof is simple, particularly in the first two cases but it may not be so in the later ones. It says that if FT of x(n) is $X(e^{j\omega})$ then the FT of x(– n) is $X(e^{-j\omega})$. Is it obvious? In x(– n) $e^{jn\omega}$, if you put – n = r then obviously that becomes $X(e^{-j\omega})$. The next relation is that, if you take the complex conjugate of x(n), then the corresponding Fourier Transform is $X*(e^{j\omega})$; this is also not difficult to prove. The real part of x(n), if x(n) is complex, would be given by [x(n) + x*(n)]/2. So the FT is $(1/2) (X(e^{j\omega}) + X*(e^{j\omega}))$. Similarly the imaginary part, j imaginary x(n), shall have the FT $\frac{1}{2}(X(e^{j\omega}) – X*(e^{j\omega}))$. Unfortunately, the nomenclature is such that the imaginary part is also a real quantity.