Digital Signal Processing

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Lecture – 35

IIR Design by Bilinear Transformation

This is our 35th lecture and in this lecture, we propose to study Bilinear Transformation. In the previous lecture, we explored the possibility of filter design by the Impulse Invariant Transformation (IIT).

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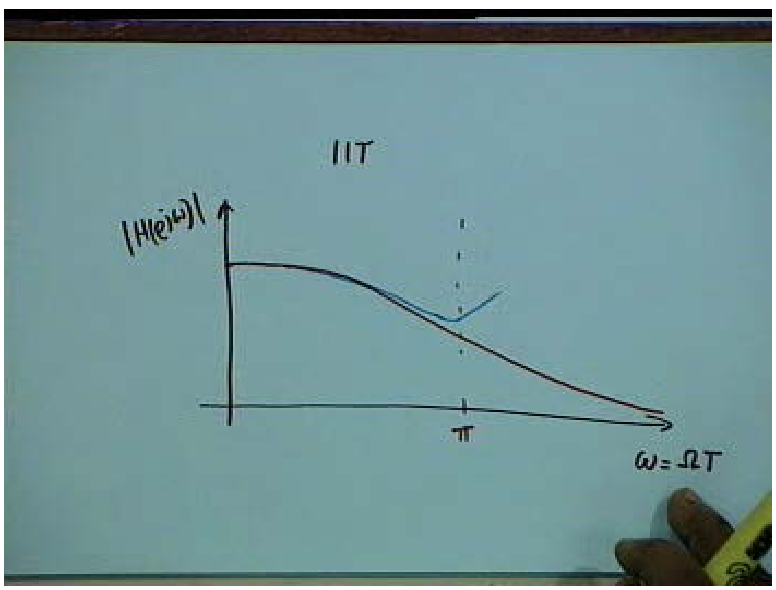

One point that I want to emphasize in this connection is that the obtained frequency response we plot versus small omega which is capital omega × T. Therefore if the analog response, as we saw for the third order filter goes like that shown in the Figure, then when capital omega T is pi, the digital filter response, after showing some aliasing error, goes up again like that shown in the Figure. The scale is the same for the frequency axis, because ω and Ω are linearly related. In the Bilinear Transformation technique (BLT), we shall compress the analog frequency scale 0 to infinity to 0 to 2pi in the digital filter. That is, we will compress an infinite frequency span to a finite span. The philosophy of BLT is the following: If you are given an analog transfer function $H_a(s)$ you can always simulate $H_a(s)$ in what used to be once upon a time a popular type of computer, called the Analog computer.

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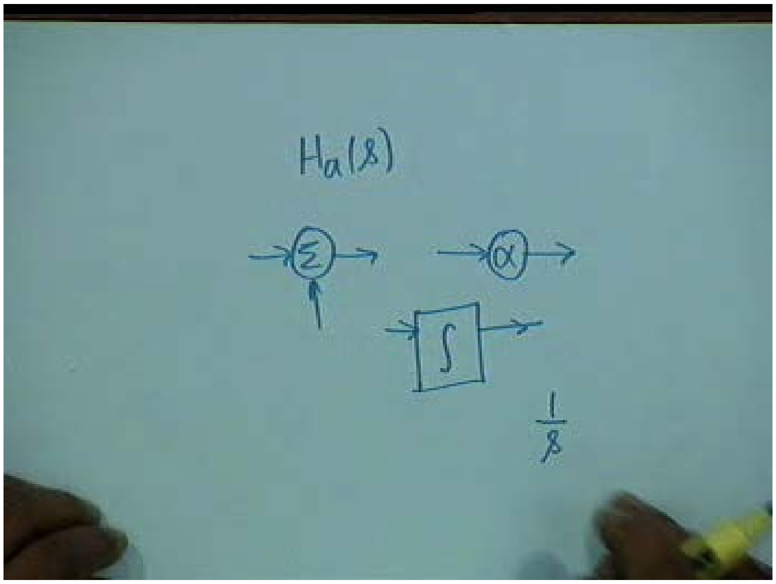

You do not have to know what analog computer is, but the basic fact is that in the simulation of $H_a(s)$, you require summation, multiplication by a constant and a dynamic element, namely an integrator. What people used to do is to use op amps for integration and simulate any given transfer function by op amps only. Multiplication by a constant alpha is either a potentiometer, if alpha is less than 1, or an op amp if alpha is greater than 1. You can take care of both the plus sign and minus signs by inverting op amp and non inverting op amp. Integration is done by putting a capacitor in the feedback loop and the integration is usually associated with a negative sign before the integral. The basic fact is that if I have an adder, a multiplier and a block of transfer function 1/s which describes an integrator, I can simulate any given analog transfer function. If I simulate the given transfer function by adders, multipliers and integrators, then I can convert that diagram into a digital filter because in a digital filter addition and multiplication are the same and there is no change; the only change is that we shall require a digital integrator.

That is the philosophy behind the Bilinear Transformation. It has proved to be the best possible analytical design. Therefore we shall consider BLT in great details. What we are seeking is a digital integrator. If we have an integrator with $x_a(t)$ giving rise to $y_a(t)$ then $y_a(t) = \int x_a(t)dt$.

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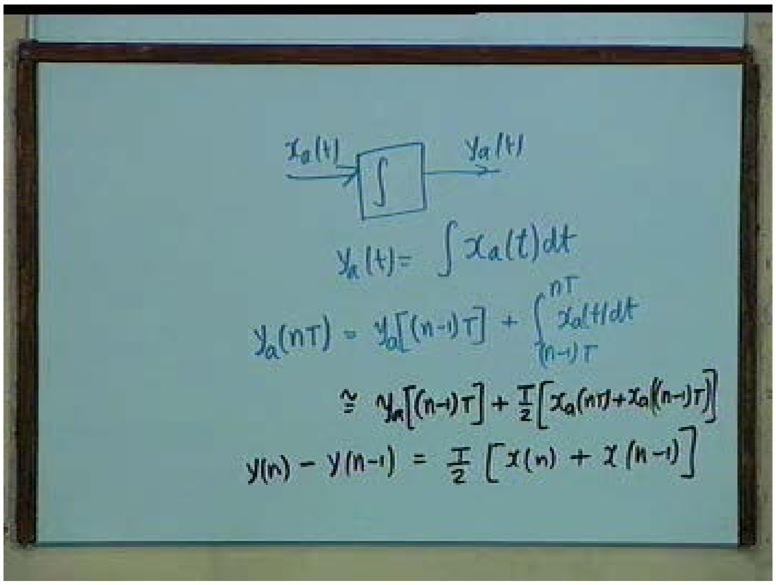

To get a digital integrator let us evaluate $y_a(nT)$. $y_a(nT)$ can be written as the initial condition at $[(n - 1)T]$, n is an integer, plus integral from $(n - 1)$ T to nT $x_a(t)dt$. There is no approximation here; this is exact. What I do now is to approximate this as $y_a[(n - 1)T] + ½[(x_a(nT) + x_a((n - 1)T)]T$. I approximate the integration. In digital simulation, there is no integration. Integration is simply finding the area by approximation. I use the trapezoidal rule i.e. I take the average value over the interval and multiply by the interval to approximate the integration. I have no other alternative, because digital hardware cannot integrate in the manner that an op amp does. This T is a small enough interval so that the spectrum of x is the same in the base band of the sampled signal. We call the quantized version of $y_a(nT)$ as y(n). If I bring $y_a((n - 1)T)$ to the left hand side, then you get $y(n) - y(n - 1) = (T/2) [x(n) + x(n - 1)]$. Now we have got a difference equation relating the output and input of a digital version of an Analog Integrator which we call as a Digital Integrator.

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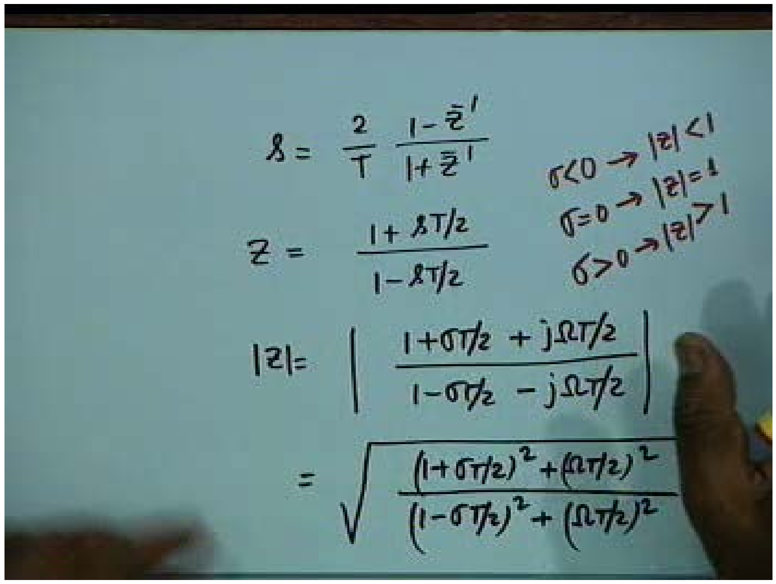

If you take the z transform, then the transfer function $Y(z)/X(z) = (T/2)\ (1 + z^{-1})/(1 - z^{-1})$. This should correspond to $1/s$ which is the integrator in the analog domain. Therefore s corresponds to $(2/T)(1 - z^{-1})/(1 + z^{-1})$. In other words what it says is, if in an analog transfer function, instead of finding the impulse response, sampling it, keeping the impulse response invariant, or finding the step response then sampling it and saying we derive a step invariant transformation, we simply replace s with $(2/T)(1 - z^{-1})/(1 + z^{-1})$. Then that should be a good digital filter because summation and multiplication are invariant whether the domain is analog or digital, and we replaced analog integration by a digital integrator. In other words, $H(z)$ shall be $H_a(s)$ where $s = (2/T)\ [(1 - z^{-1})/(1 + z^{-1})]$. The transformation from s to z is called a Bilinear Transformation because it is a rational function, with a linear term in the numerator and a linear term in the denominator. The technique is called Bilinear Transformation Technique or simply BLT. In order for it to be useful, first condition is that $H(z)$ must be stable. In order to examine whether this transformation is stable or not let us invert the relationship $s = (2/T)(1 - z^{-1})/(1 + z^{-1})$. That is, I find z in terms of s. the result is: $z = [1 + sT/2]/[1 - (sT/2)]$. The magnitude of z would be equal to magnitude of (1 + sigma T/2 + j omega T/2)/(1 – sigma T/2 – j omega T/2). This is equal to square root of [(1 + sigma T/2)^2 + (omega T/2)^2]/[(1 – sigma T/2)^2 + (omega T/2)^2].

If sigma is less than 0, i.e. sigma is negative, then obviously the numerator will be less than the denominator and therefore this corresponds to mod z less than 1. If sigma is equal to 0, then correspondingly, mod z = 1; and sigma greater than 0 corresponds to mod z greater than 1. Therefore it is a stable transformation. That is, left half of s-plane is transformed to inside the unit circle. In order that $H_a(s)$ is useful it must have all its poles in the left half plane. So analog poles are transformed to inside the unit circle in the z-plane. The imaginary axis sigma = 0 is transformed to the unit circle and the right half plane is transformed to outside the unit circle, exactly like the Impulse Invariant Transformation and therefore it is a stable transformation. Here is the figure showing this fact.

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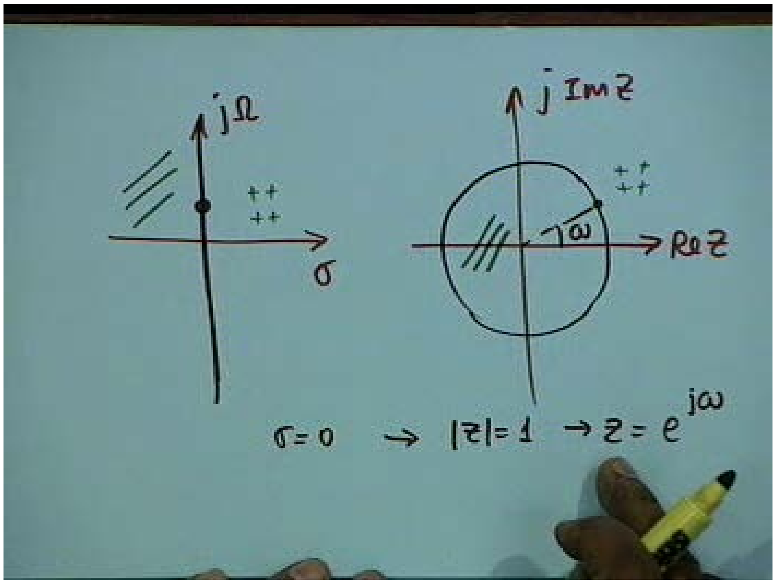

The imaginary axis transforms to the unit circle. The left half plane transforms to the inside of the unit circle and the right half plane transforms to outside the unit circle. This is a picture of the important fact that the BLT is a stable transformation. In particular, sigma = 0 leads to mod z = 1, which means that z = e^{jomega}. In other words, a point on the imaginary axis (sigma = 0) will transform to some point on the unit circle whose angle is omega; only then mod z can be = 1. If I write this in the equation $s = 2/T(1 - z^{-1})/(1 + z^{-1})$, $s = j\Omega$, then I should write $z = e^{j\omega}$, so $j\Omega = (2/T)\,(1 - e^{-j\omega})/(1 + e^{-j\omega})$.

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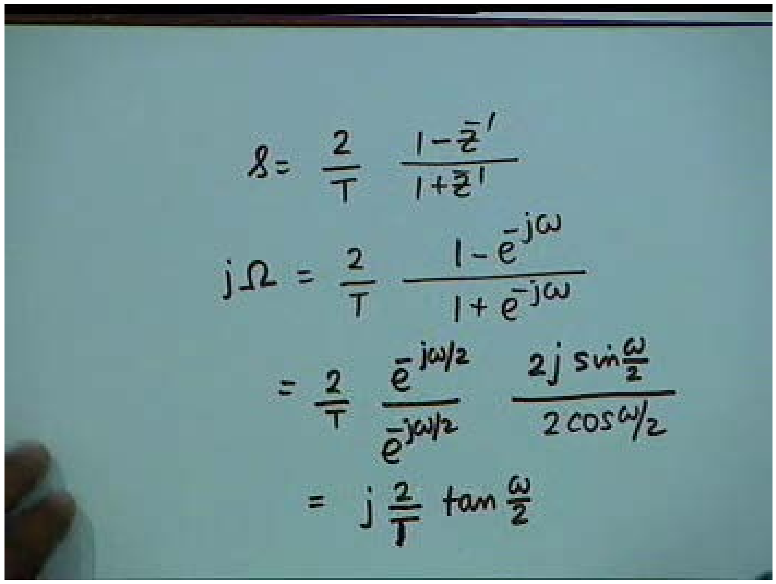

It requires a little bit of manipulation to put it in terms of a real quantity. If you do that, you get $\Omega = (2/T)$ tangent of $\omega/2$.

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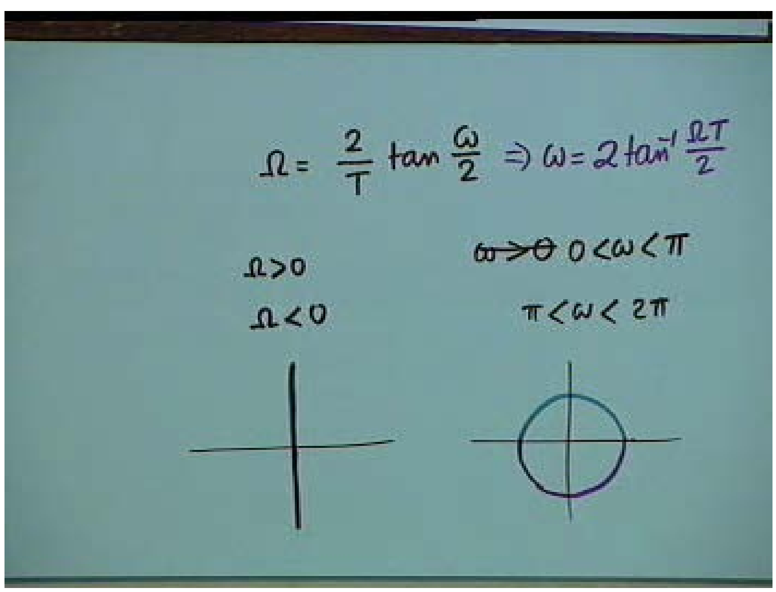

There are several things to observe at this point. One is purely a point of academic interest, that is if $0 \le \Omega \le \infty$, then correspondingly, $0 \le \omega \le \Pi$. Thus the positive jΩ - axis goes to the upper half of the unit circle. Similarly, the negative part of the jΩ - axis will go to the lower half of the unit circle. It has no other implication or import on digital filter design, but is just an observation. The other observation is that this relationship is a nonlinear one, but one to one. In other words from this, you can get ω as equal to twice tan inverse Ω T/2. Now, if you plot ω versus Ω, then the plot becomes like one shown in the figure.

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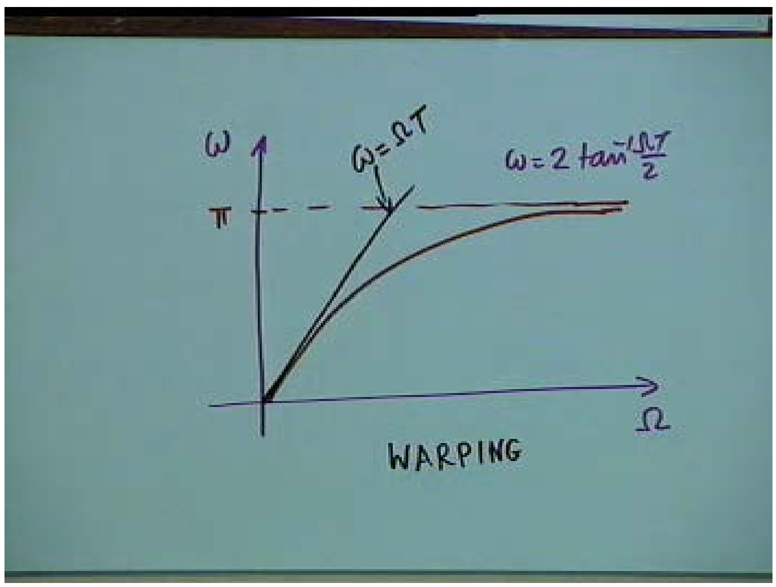

Unlike Impulse Invariant Transformation where the relationship was simply $\omega = \Omega T$ as indicated in the figure, in BLT, there is a deviation from linearity because the relation between Ω and ω is nonlinear. This is how an infinite axis is compressed to a finite axis, that is 0 to infinity is compressed to 0 to pi; this phenomenon is called Warping. So frequency scale is warped which is a disadvantage unless you take care of it in the appropriate manner. We shall do pre-warping or anti-warping so that the effect of warping ultimately is canceled and we get what we want. What we do is the following. Given a digital filter spec, i.e. delta p's, delta s's, omega p's and omega s's, you transform ωp's to Ωp's and ωs's to Ωs's (there can be several of them in a multiple passband and/or multiple stopband filters) by using $\Omega = (2/T) \tan \omega/2$.

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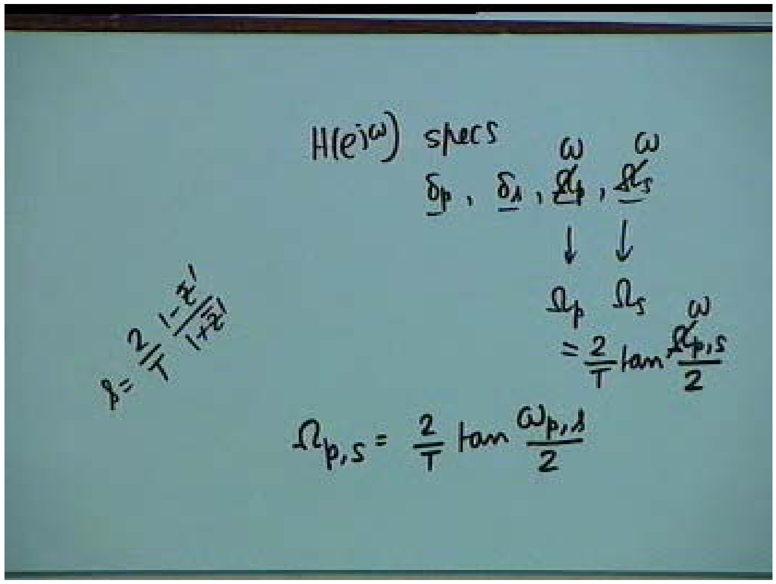

In the figure, I put a bar below each spec to indicate this possibility of multiple specs in δ ’s and ω’s. Delta p’s and delta s’s shall remain invariant, but ωp’s and ωs’s are transformed to Ωp’s and Ωs’s by the relationship Ω = (2/T) tangent of ω/2. This is pre-warping, that is the digital filter frequencies are pre-warped to analog frequencies. Thus you get the specs on the corresponding analog filter. Design the analog filter then put the transformation $s = (2/T)(1 - z^{-1})/(1 + z^{-1})$ which automatically does the warping and gives us the correct results. So we can take care of the warping by pre-warping. Now, in terms of filter design what I said in words is contained in the next diagram, where I have taken the case of an elliptic low-pass filter.

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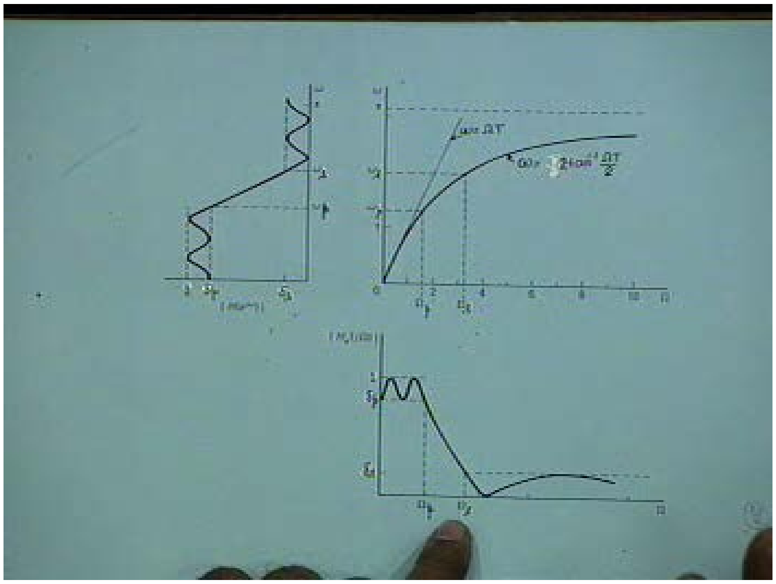

Given the digital filter specs; delta p, delta s, ωp and ωs, as shown in the left diagram, where ω goes from 0 to Π, you project ωp and ωs to the diagram depicting ω versus Ω and show the corresponding frequencies. In practice, you do it analytically, but to understand what is happening I have drawn this diagram. Project Ωp and Ωs on the analog filter response shown in the lower diagram, which is drawn with the same tolerances as in the digital filter magnitude response.

In the analog filter magnitude response, we have shown only one maximum in the stopband; it must be understood that there is another maximum that occurs at $\Omega = \infty$ after one minimum. These have not been shown to keep the diagram within the page. Now, to formalize the procedure for design, let us be concerned only with low pass design because we know how to convert a low pass into any other form.

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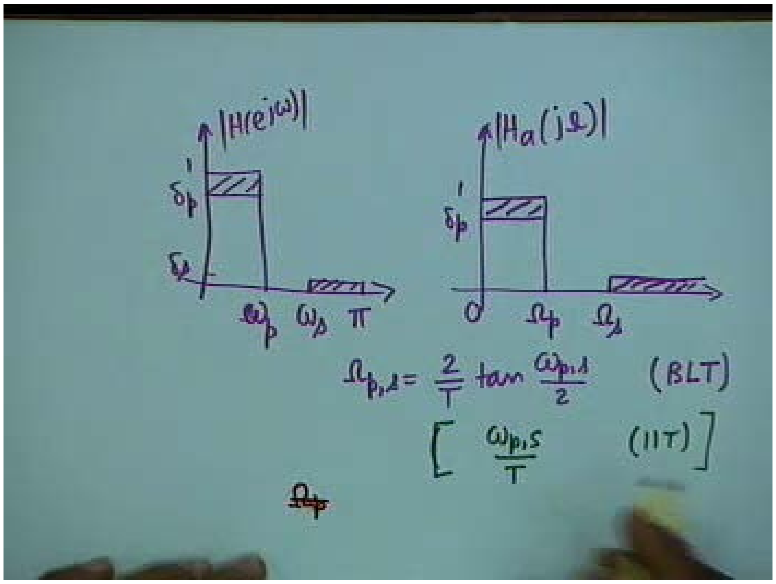

We shall take other examples of design later. But if I have a low pass design problem, typically the specs would be maximum magnitude = 1, delta p, delta s, ωp and ωs. First, you transform this into the analog filter diagram giving $|H_a(j\Omega)|$. You use $\Omega p,s = (2/T) \tan (\omega p,s/2)$. The upper end of the stopband cannot be shown because it goes to infinity. Side by side, I would also like to keep the relationship for Impulse Invariant Transformation which is simply $\Omega p,s = \omega p,s/T$. When we take an example of design, we will do with impulse invariance as well as with bilinear transformation. We have not done any example of design so far with impulse invariance but we will do them together to point out the differences and advantages of BLT over IIT. Now the other thing that I want you to notice is that IIT actually is an approximation of BLT. If ω T/2 is small, then tan inverse ω T/2 is same as ω T/2. Therefore you simply get $\omega = \Omega T$. In other words, the initial part of the nonlinear diagram is coincident with the linear diagram, and is true for IIT, while the rest is valid for BLT. And because of the deviation, there is aliasing.

Now I would also like to point out that there is absolutely no aliasing in Bilinear Transformation because the total transfer function is being transformed. In Impulse Invariant Transformation, only poles were transformed. In step invariant transformation also, only poles were transformed and additionally a zero at infinity was created. We take care of the total base band 0 to pi in BLT and we pre-distort it intentionally to go to the analog domain and then the Bilinear Transformation takes care of distortion in the other direction so that we get what we want. In an analog low pass filter design, you also require this ratio $\Omega s/\Omega p$ to determine the order required of the filter. Here $\Omega s/\Omega p$ = (tangent of $\omega s/2$)/(tangent of $\omega p/2$) in BLT.

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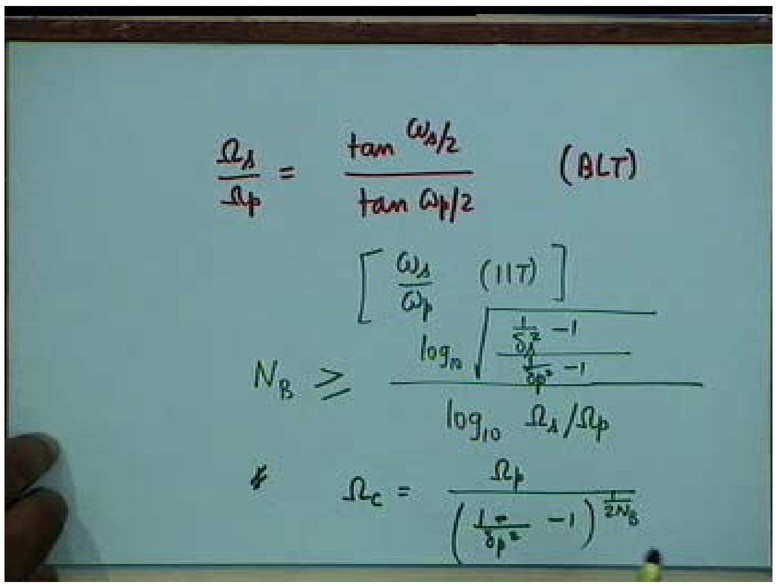

In IIT, it is simply $\omega s/\omega p$ because the relationship is linear. I would like to keep both of them. IIT is an approximation for the BLT relationship because for small theta, tangent theta can be replaced by theta, which gives IIT. For small ω or Ω, IIT and BLT are indistinguishable. For filter design, you also require the fact that N_B, the Butterworth filter order should be greater than equal to $\log_{10}$ of [((1/delta s^2) – 1))/((1/delta p^2) – 1))]/$\log_{10}$ (Ω_s/Ω_p). In Butterworth, we also require the cutoff frequency Ω_c, and that is obtained as Ω_p/[(1/delta p^2) – 1]$^{1/(2N_B)}$. These are the design parameters. You require the order and the cutoff frequency. What you are given is 0 – Ω_p for a tolerance of 1 – delta p. For a tolerance of 3db, Ω_p is Ω_c. Nothing like this is required in Chebyshev.

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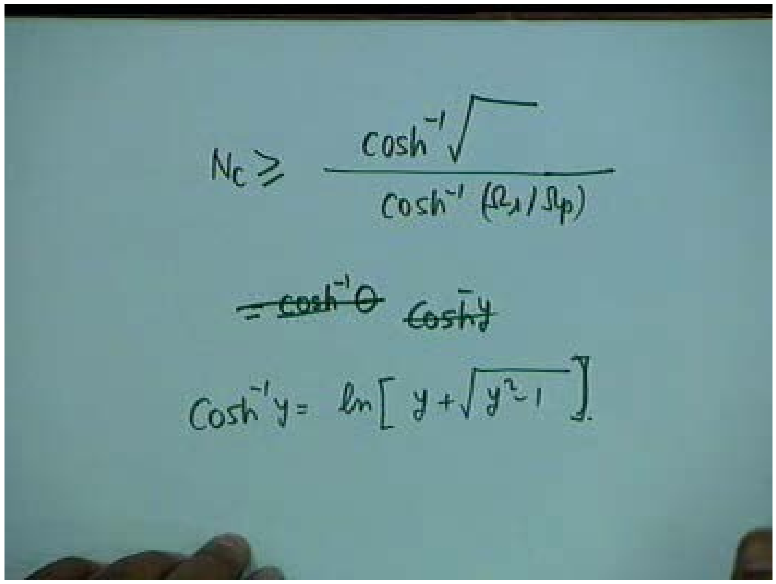

In Chebyshev, N_c is greater than equal to the same expression as in Butterworth with cosh inverse replacing $\log_{10}$. In computing this, you may use the relationship that cosh inverse y is equal to log natural (not to the base ten but to exponential e) (y + square root of (y^2 – 1)). These are all our basic design equations. Now, we take an example and we shall follow this example through for designing Butterworth IIT, Butterworth BLT, Chebyshev IIT and Chebyshev BLT - all the four of them. Now, look at the specs, as shown in the figure.

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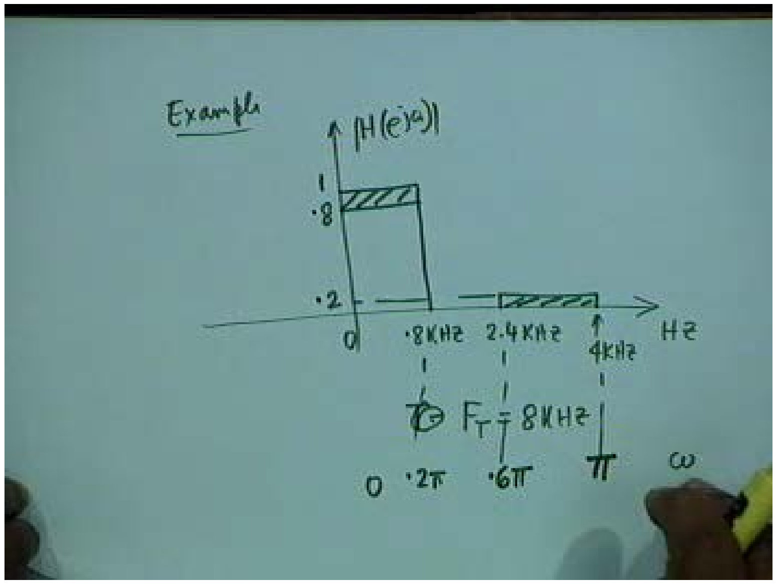

The frequencies are given in Hertz and the magnitude of $H(e^{j\omega})$ is normalized to lie between 1 and 0.8 (better than 3 db) in the passband. I want to pass only up to 800 hertz that is 0.8 kilo Hertz for some reason. And then 2.4 kilo Hertz to 4 kilo Hertz I want to pass with less than 0.2 magnitude. A digital filter is to be designed such that these specs are satisfied.

What is your F_T, the sampling frequency? Obviously it is 8 kilo Hertz. And once you know this then you can convert these specs to normalized digital frequency; 4 kilo Hertz shall correspond to pi so 2.4 kHz will be 0.6 pi, 0.8 kHz will be 0.2 pi and 0 is of course 0. Given the actual frequency specs, we have to convert them to ω. It is a linear conversion; do not confuse with IIT or BLT.

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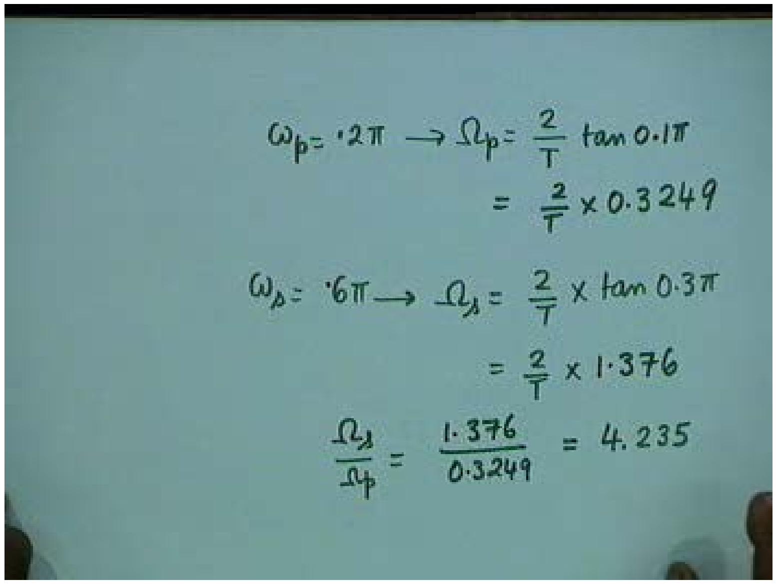

To design the analog filter, we have to convert ω's into analog frequencies; for $\omega_p = 0.2$ pi, Ω_p would be (2/T) tangent of 0.1 pi and this comes as (2/T) × 0.3249. $\omega_s = 0.6$ pi corresponds to Ω_s = (2/T) × tangent of 0.3 pi which is = (2/T) × 1.376. You also require Ω_s/Ω_p which is 1.376/0.3249 = 4.235.

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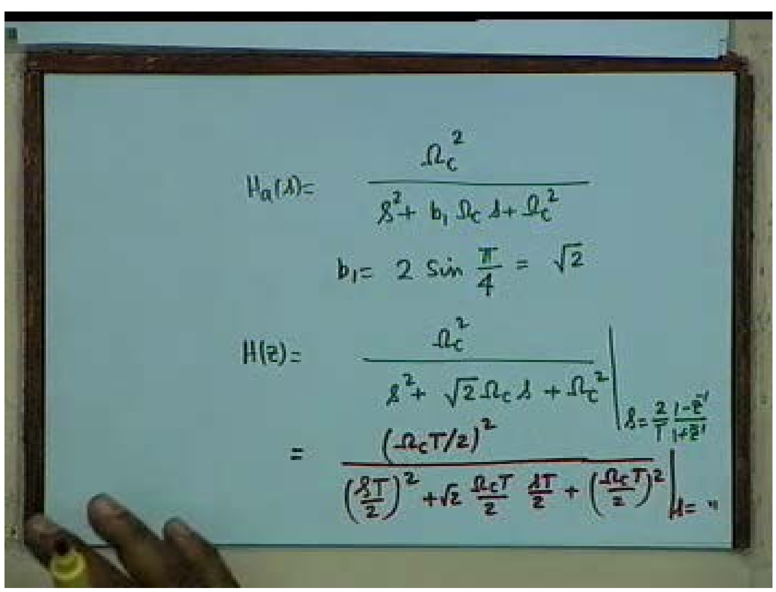

The Butterworth order N_B can be calculated from the formula. It comes out as greater than or equal to 1.3; therefore $N_B = 2$. Once you find N_B, you have to find out Ω_c, the 3 db cutoff frequency, which is equal to $\Omega p/1/(\text{delta } p^2 - 1)^{1/(2 N_B)}$. If you substitute numerical values, you get $\Omega_c = (2/T) \times 0.3752$. If this figure had come lower than 0.3249, then you are assured that you have made a mistake. Now I can write my $H_a(s)$ as $\Omega_c^2/(s^2 + b_1 \Omega_c s + \Omega_c^2)$; b_1 = 2 sine of pi/4 = $2 \times (1/\sqrt{2}) = \sqrt{2}$.

Therefore my H(z) would be = $\Omega_c^2/(s^2 + \sqrt{2}\, \Omega_c s + \Omega_c^2)$ with $s = (2/T)\,(1 - z^{-1})/(1 + z^{-1})$. Now you shall notice that T shall disappear from the result. Now we write one more step with $(\Omega_c T/2)^2$ in the numerator, that is we multiply by T square/4. Then in the denominator I get $(sT/2)^2 + \sqrt{2}(\Omega_c T/2)\,(sT/2) + (\Omega_c T/2)^2)$. Now we have found out Ω_c to be $2/T \times 0.3752$ so $\Omega_c T/2$ is simply 0.3752. Note that sT/2 is simply $(1 - z^{-1})/(1 + z^{-1})$. Therefore from the expression, T disappears. It has to disappear because Bilinear Transformation suffers from no aliasing and the transfer function should not depend on T.

Another way to do this is to assume T = 2 then the factor T/2 or 2/T disappears. You can use this right from the beginning but you must be aware that the value of T does not matter. We arbitrarily assume T = 2. After simplification, the transfer function becomes $0.08421(1 + z^{-1})^2/(1 - 1.0281\, z^{-1} + 0.3651 z^{-2})$.

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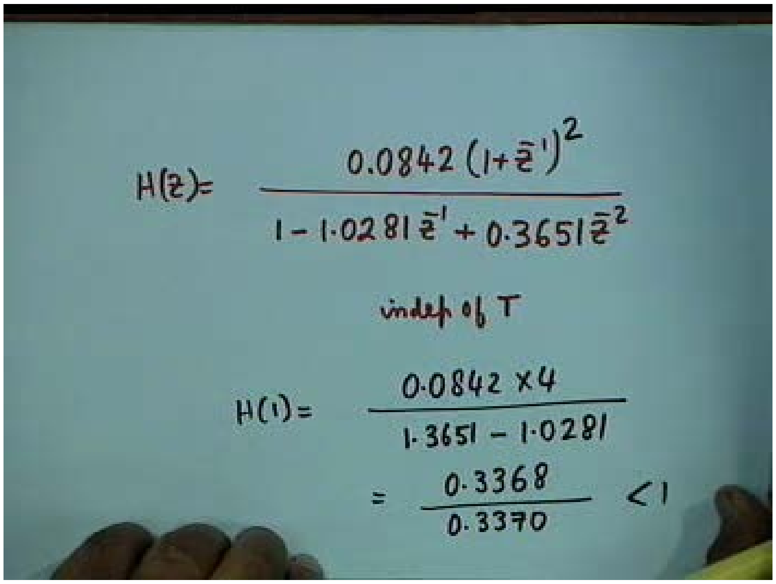

You notice that H(z) is 0 when $z^{-1} = -1$. That corresponds to ω = pi and the 0 is of multiplicity 2, i.e. a second order 0. The analog transfer function was a second order one having two zeros at infinity. Infinite Ω is mapped into ω = pi, and therefore this is quite expected. If in the numerator $(1 + z^{-1})^2$ does not appear, then you have done something wrong. The other thing is, if I calculate H(1), which corresponds to ω = 0, then you get the value 0.3368/0.3370 this is not exactly 1 but it is less than 1. Why has this occurred? It has occurred because of rounding and not because of aliasing. Now let us see a diagram of the magnitude plot.

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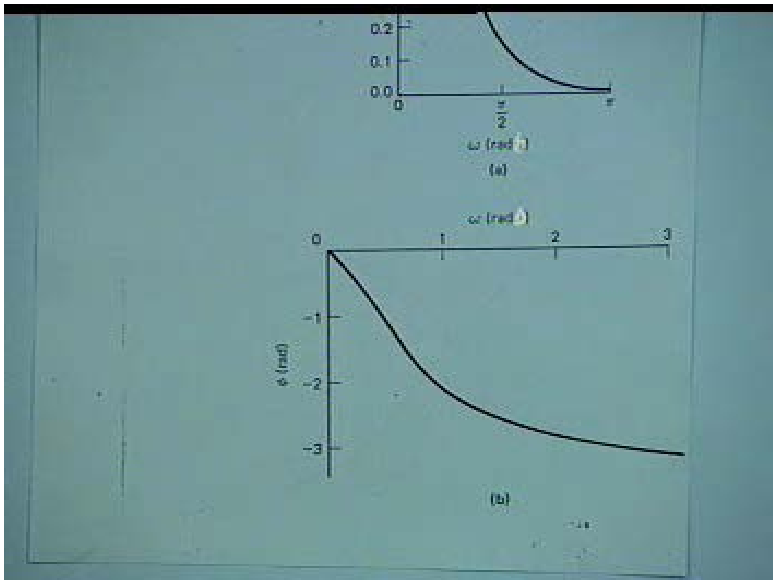

Now at pi, it should be exactly 0; therefore there is a mistake here in drawing the diagram. Here the phase plot is approximately linear upto a certain frequency, about 1 radian and beyond that it is highly nonlinear; it saturates at – pi ultimately. And if you want to use this in a situation where delay distortion is not permitted, then you have to use an all-pass filter in cascade. If possible, use a first order all-pass because that reduces the number of multipliers and the hardware requirement. We will close here, and next time we will continue this discussion with IIT for the same example and Chebyshev Design with BLT as well as ILT.