Digital Signal Processing

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Lecture – 29

IIR Realizations

This is the 29th lecture on DSP and our topic today is IIR Realizations. In the previous lecture, we talked about the process of transposition and then FIR realization. We discussed the direct form structure, its transposed structure, the cascade structure and the parallel structure. The parallel is obtained by polyphase decomposition and parallel in FIR does not lead to a higher speed. Even if it is parallel processing, the speed cannot be increased but the realization can be made canonic by sharing delays. We took an example to illustrate this. Polyphase decomposition is not normally resorted to because it does not speed up processing, but it is very useful in multi rate signal processing where decimation and interpolation do reduce the computational complexity. Then we said that in the linear phase realizations, because of symmetry or anti-symmetry, the number of multipliers can be reduced approximately a factor of half, exactly half if the length is even or order length + 1 divided by 2 if the order is odd. Today we will discuss about IIR Realizations.

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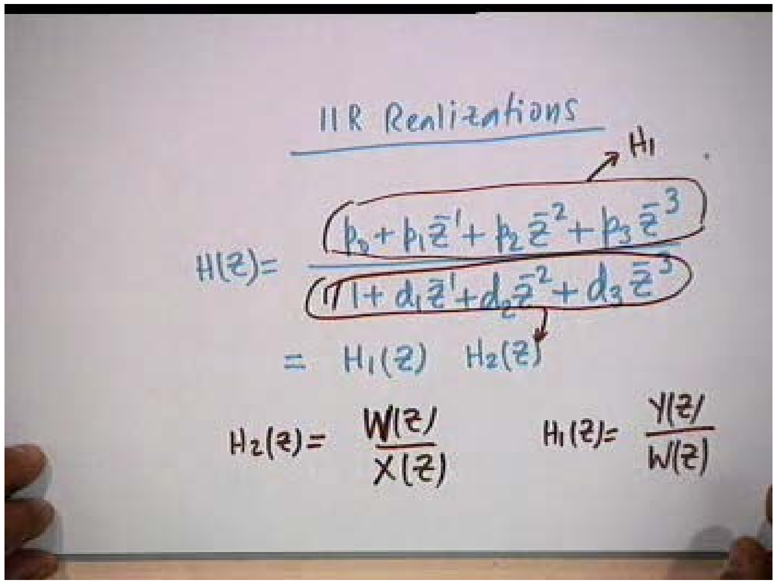

We have already looked at IIR Realization of the first order, where you saw how to make it canonic. A direct form canonic structure can be extended to a general order. For example, let us take a third order IIR Filter $(1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3})^{-1} (p_0 + p_1 z^{-1} + p_2 z^{-2} + p_3 z^{-3})$. Then we decompose this into two transfer functions $H_1(z)$and $H_2(z)$ where H_1 is an FIR, and $1/(1 + d_1 z^{-1} + d_2 z^{-2} + d_3 z^{-3})$ is H_2. We look upon it as a product of FIR and IIR. Then we realize $H_{2.}$ Obviously if I write $H_2(z)$ as $W(z)/X(z)$, then $H_1(z) = Y(z)/W(z)$.

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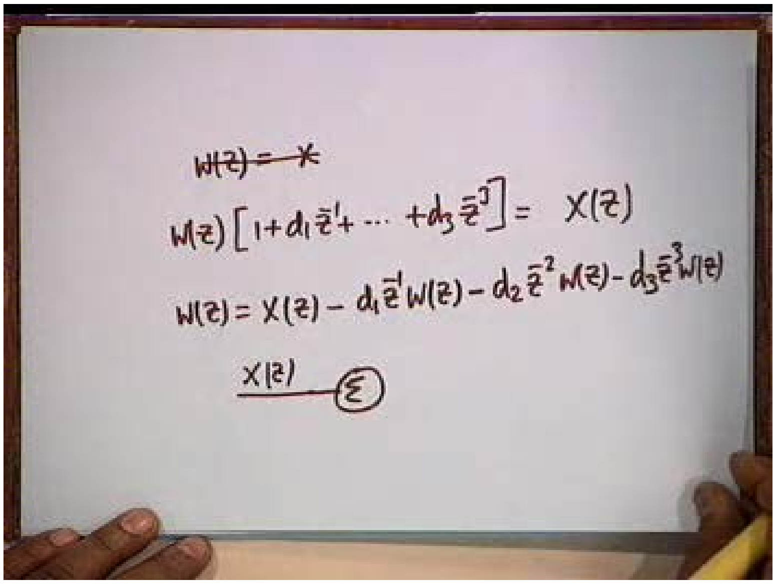

So if I realize $H_2(z)$ first we shall get the equation $W(z)\times(1 + d_1 z^{-1} + \ldots + d_3 z^{-3})$ X(z). So W(z) is just an intermediate variable and it is given by $W(z) = X(z) - d_1 z^{-1} W(z) - d_2 z^{-2} W(z) - d_3 z^{-3} W(z)$.

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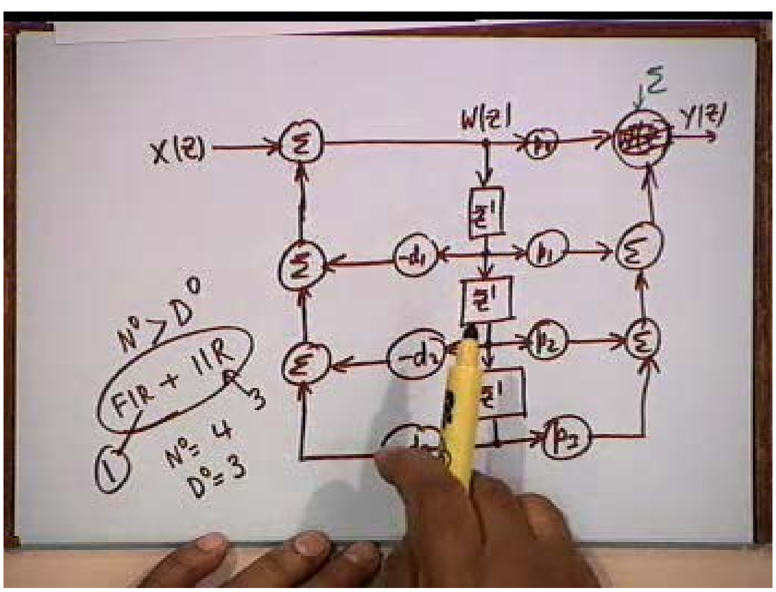

For W(z), I require three delays corresponding to $d_1 z^{-1}$, $d_2 z^{-2}$, $d_3 z^{-3}$ and I have to apply feedback after every delay by multiplying by a constant. The realization of $H_2(z)$ is shown in the left side of the Figure, indicating the location of the signal W(z). If I want Y(z), then I have to also realize $H_1(z)$ which is FIR and that is very simple. Take signals from W(z), z^{-1}W(z), z^{-2}W(z) and z^{-3}W(z), multiply by p_0, p_1, p_2 and p_3 respectively, and add them two signals at a time. This is the general way of obtaining a direct canonic form for any IIR filter.

It is obvious that if the numerator contained a higher degree, that is, if the numerator had $p_4 z^{-4}$ then we would have required another delay, a multiplier p_4 and another adder. You could also handle that problem, that is if the numerator degree is one or two higher than the denominator degree, by writing it as an FIR plus a proper rational function. So the FIR would be in parallel with IIR realization and that would speed up the process. So whenever you get a numerator degree higher than the denominator degree, this procedure is preferable rather than adding more delays and taking feed forward through multipliers. These are some practical points which one should remember.

For the particular problem of a 4th degree numerator and 3rd degree denominator, if you realize by 1st order FIR in parallel with 3rd order IIR, the first output sample would be available after 3 delays, not 4. Hence the process is speeded up. If I take the transpose of the H_2H_1 realization, what kind of structure shall we get? We will get H_1 first that is the feed forward or the FIR part, and then we shall get the feedback part or H_2. It is very easy to draw the diagram, as shown in the Figure. In this, I have shown three signals in the middle two adders; one should replace each of them by two adders.

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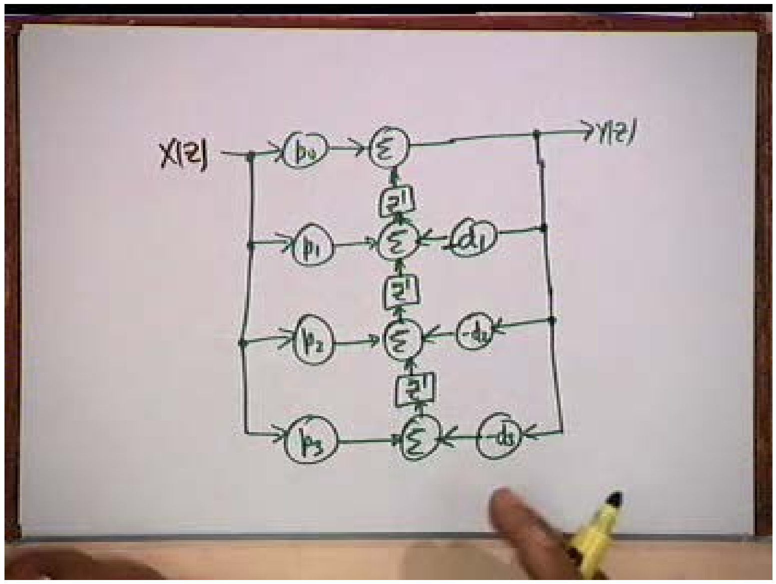

As I said there is a value for obtaining alternative structures, because every structure has its own characteristic signature on word length effects and overflow phenomena. You should explore the possibility of minimizing these two phenomena. If there is overflow, then you can either take a costly solution, that is change from fixed point to floating point, or you could scale. The latter is a better solution because scaling does not require multiplication. Multiplication operation is the most costly one in terms of the speed of processing. Multiplication is repeated addition and therefore it takes time.

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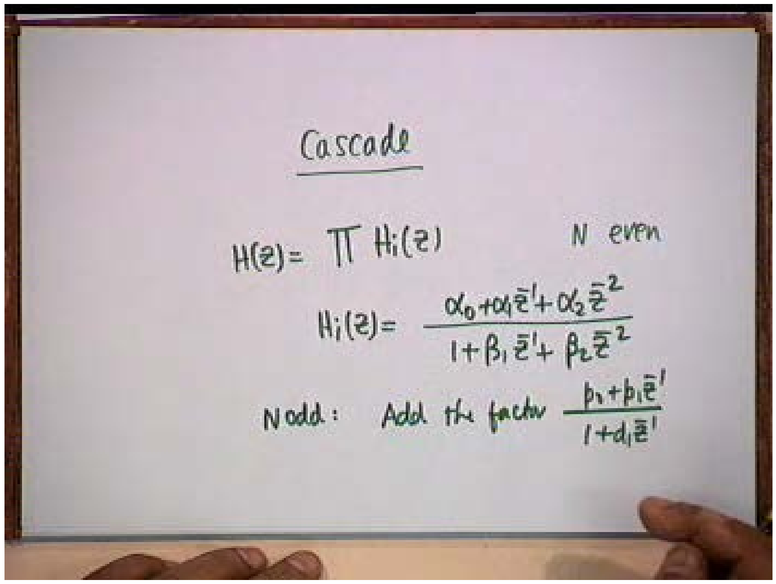

Next we consider cascade realization. IIR, like FIR, can be realized in cascade form. We are now assuming that the given IIR is a proper rational function. So if the order is even then you require product of factors like this $H_i(z)$ where $H_i(z)$ is of the form $(1 + \beta_1 z^{-1} + \beta_2 z^{-2})^{-1} (\alpha_0 + \alpha_1 z^{-1} + \alpha_2 z^{-2})$. This is not quite a proper rational function, you can take a constant out of it and have the numerator as a linear polynomial and the denominator as a quadratic one. But this does not hurt the canonicity because you are already using two delays for the denominator. So there is no harm in having a quadratic factor in the numerator. This is the case if N is even; on the other hand if N is odd then in addition to this you add a bilinear function of the form $(1 + d_1 z^{-1})^{-1} (p_0 + p_1 z^{-1})$. I have already told you that all components cannot be bilinear because poles and zeros may be complex. You should insist on real coefficients only.

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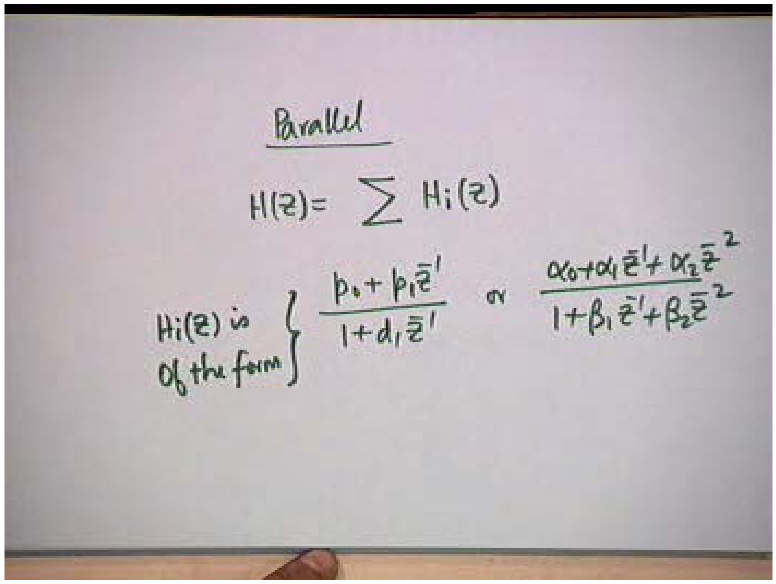

We can also have parallel realization, which now assumes importance. Unlike FIR, it leads to speeding up the processing. In this, you decompose H(z) into ∑ $H_i(z)$ where each component transfer function is no more complicated than a biquadratic. That is $H_i(z)$ is either a bilinear or a biquadratic function. So at the most, you require two delays in each component $H_i(z)$. Even if the order of this filter is 10, the minimum processing time, excluding the time required for multiplication, is not 10 T, but it will be simply 2T. T is the sampling interval and that is the virtue of parallel realization in IIR. In FIR, it does not afford any advantage with regard to speed.

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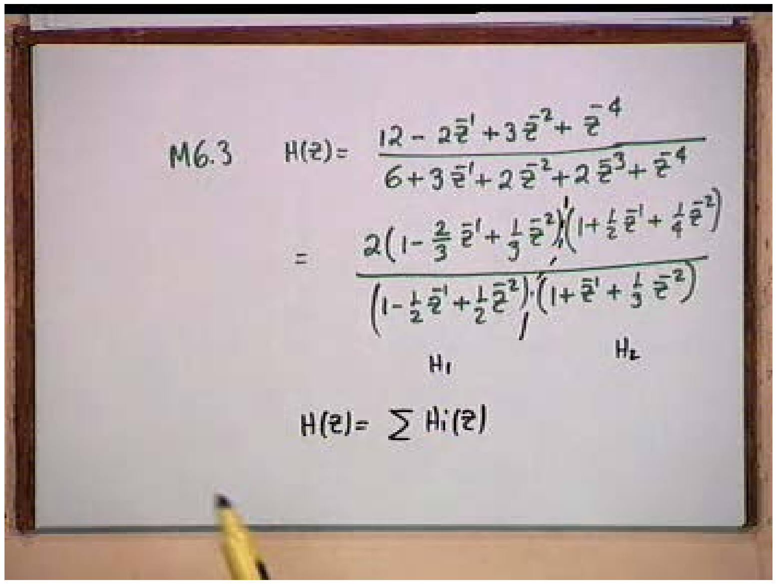

Here is an example from Mitra, Problem M 6.3; the transfer function given is $H(z) = (12 - 2z^{-1} + 3z^{-2} + z^{-4})/(6 + 3z^{-1} + 2z^{-2} + 2z^{-3} + z^{-4})$. If, for a synthesis problem, there exists one solution, then there exists indefinite number of solutions. One of the straightest things that you can do is take out a constant from here and make the numerator of degree 3; the denominator degree remains at 4. Direct form is one possible realization. But let us see what we can do about cascade realization. First thing you do is to factorize the denominator. This is required in parallel realization also.

In parallel realization, you require factorizing the denominator only, whereas in cascade you require factorizing the denominator as well as the numerator because you have to assign numerator factors to second order transfer functions and to the first order transfer functions, if the overall order is odd. In the sense that the co-efficients you use for multiplication are not the ones that are given to you, but they are derived, cascade and parallel realizations are called indirect. In both parallel and cascade cases, you require to factorize one or two polynomials, and these factors are always written with a constant term equal to 1 that affords easy realization.

You can take out the overall scaling factor and always use this as a multiplier. In this case you can see that the scaling factor is 2. 2 is simply a shift and therefore it is not a multiplication. I have done this factorization. The result is: $H(z) = 2[(1 - (2/3) z^{-1} + (1/3) z^{-2}) (1 + (½) z^{-1} + (¼) z^{-2})]/[1 - (1/2) z^{-1} + (1/2) z^{-2})(1 + z^{-1} + (1/3) z^{-2})]$.

In cascading you can take either factor in the numerator and associate either factor in the denominator with it, forming the transfer function H_1. The remaining two factors constitute H_2. There are quite a few possibilities, as you can see. You could also assign the factor 2 to either the first one or the second one, at the beginning or at the end. So there are many different realizations which are possible. The individual transfer functions are realized in one of the direct forms.

On the other hand, for the parallel realization, you require to decompose H(z) as $\sum H_i(z)$ where $H_i(z)$ is no more complicated than a bilinear or a bi-quadratic function. How to break it up? One of the simple ways is to use partial fraction expansions and that will ensure that you have real polynomials in the numerator as well as the denominator.

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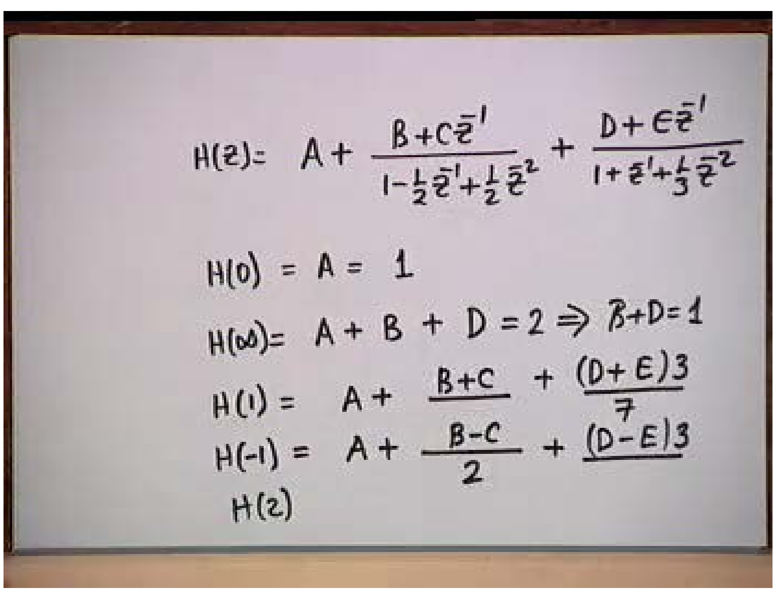

In this particular case, for example, our partial fraction expansion, shall, in the first step, contain a constant term added to a cubic/quartic term. So our partial fraction expansion shall be of the form A + [(B + C z^{-1})/(1 – $^1/_2$ z^{-1} + $^1/_2$ z^{-2})] + [D + E z^{-1})/(1 + z^{-1} + $^1/_3$ z^{-2})]. It is a straightforward partial fraction decomposition.

Then, in order to find the constants, do not go to the complex calculations. Instead, what you do is you solve a set of simultaneous equation. You notice that H(0) = A = 1. When you find H(∞), it is obviously equal to A + B + D, and from the given function H(∞) = 2 and therefore you have reduced five constants into four. So you have to solve only four simultaneous equation which means that B + D = 1 and by replacing D by 1 – B, you reduce the problem to solving a set of three linear equations. Then you require three more equations and the easy things to do are put z = ± 1; if you put z = 1 then what you get is H(1) = A + B + C + (D + E)(3/7) and you replace D by B – 1. The other one is H(– 1) = A + [(B – C)/2] + (D – E)3. You can take any third number, may be +2 or –2, and write another equation. By solving them, the final result is as follows:

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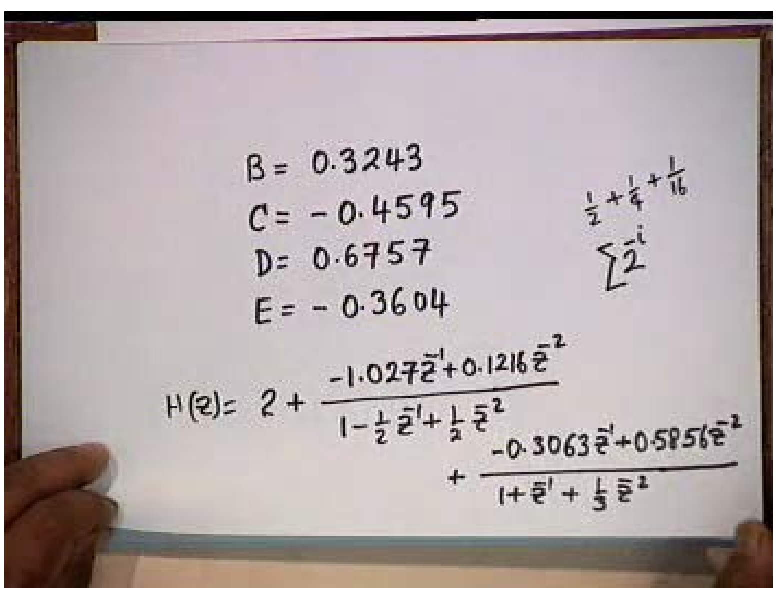

B is 0.3243, C is – 0.4595 (I went up to four places of decimals), D is 0.6757, and E is – 0.3604; so you get one parallel realization. How can you make a variation in this? You can distribute this constant 1 into these two factors in any manner you like. So, theoretically there is an infinite number of variations. If you can distribute it in such a manner that the coefficients B, C, D, and E become fractions which can be obtained by shifting then you have done without multipliers. The aim is to reduce the number of multipliers as much as possible. Multiplication is a time consuming process. For example, one of the solutions can be that you take the constant term as 2 and make appropriate subtractions from these two terms. This gives a solution as shown in the slide. Your starting point is a partial fraction expansion taking care of the degrees of the numerator and denominator. Suppose the numerator here was of fifth degree, then instead of A, you shall take out A + B z^{-1} and then you then make the partial fraction expansion of the proper rational function.

I would like to point out to you that coefficients like 1/3 (= 0.3333 recurring) are nuisances, because of necessity, you have to truncate it, but coefficients like ½, ¼, 1/8, 1/16 or a combination like (1/2) + (1/4) + (1/16) can be obtained by shifts. If you can express a coefficient as $\sum \alpha_i \, 2^{-i}$, where $\alpha_i = 0$ or $\pm$ 1, you are very lucky because you have speeded up the processor.

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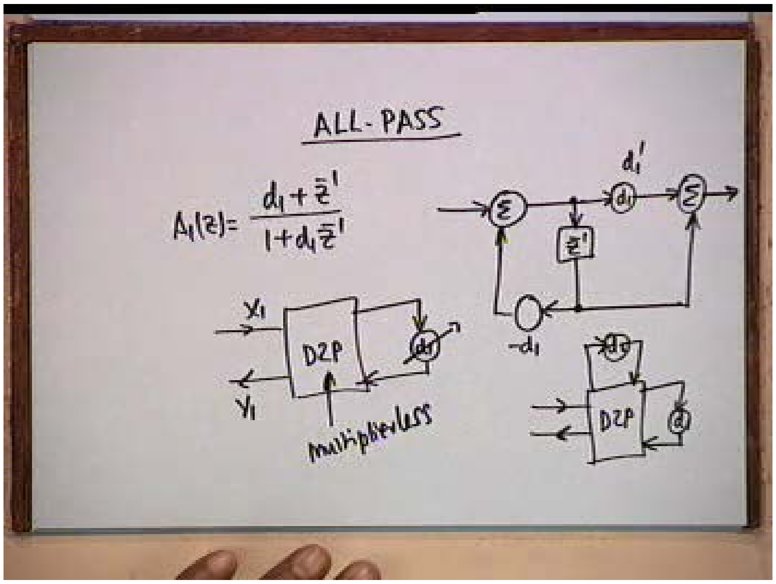

We now consider all pass realizations. All pass FIR is trivial, it is simply z^{-N}. All pass IIR has the problem that the direct or cascade or parallel realization cannot be made canonic in multipliers. For example, if I take an all pass filter of first order, $A_1(z) = (1 + d_1 z^{-1})/(d_1 + z^{-1})$, then the canonic realization should require one delay and, one multiplier, i.e. d_1. But in the direct canonic realization for example, you require two multipliers, $-d_1$ and d_1.

Now, – 1 is not a multiplier; we are simply changing the sign bit. But suppose due to quantization or any other reason, the multipliers are not exactly equal and opposite, then the all pass property is disturbed. The pole and zero will no longer be reciprocal pairs and therefore the all pass property is destroyed.

On the other hand, if we can make a multiplierless digital two pair (D2P) in which the termination is d_1, then even if d_1 changes it does not matter and the filter still remains all pass. There will be slight distortion in the delay but the all pass property shall be kept intact. So, how to obtain the required digital two pair in which there are no multipliers? If you can do that, then we shall obtain a single multiplier first order all pass realization. This can be obtained by the so-called Multiplier Extraction Approach.

For example, if we had a second order then we would need a multiplierless digital three pair which has the two multipliers d_1 and d_2 as terminations. Fortunately, we can do with digital two pair concept only, because instead of multipliers D2P, we can absorb one multiplier inside the D2P and use the other as termination by 2.

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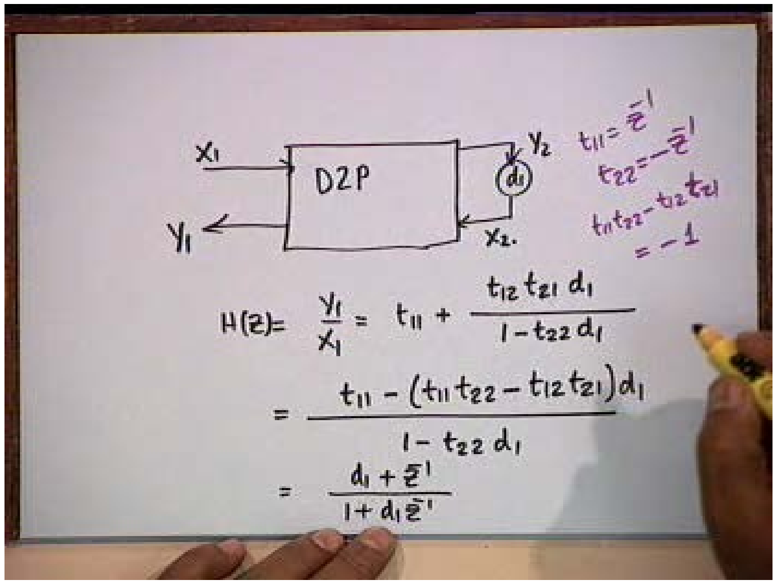

Let us consider the first order case and draw the digital two pair with the termination d_1, as shown in the figure. For a terminated digital two pair, the transfer function in terms of the transmission parameters is $t_{11} + (t_{12}\ t_{21}\ d_1)/(1 - t_{22}\ d_1)$. We shall try to identify the transmission parameters and then synthesize the digital two pair. So what we do is to simplify this to $(1 - t_{22}\ d_1)^{-1}\ [t_{11} - (t_{11}t_{22} - t_{12}\ t_{12})d_1]$. We compare this with what is given here i.e. $(1 + d_1z^{-1})^{-1}(d_1 + z^{-1})$. From this we try to identify the transmission parameters. Obviously one choice is $t_{11} = z^{-1}$, and $t_{22} = -z^{-1}$; then $t_{11}\ t_{22} - t_{12}\ t_{21} = -1$. We have already fixed t_{11} and t_{22} and therefore we can find t_{12} and t_{21} from this last relation.

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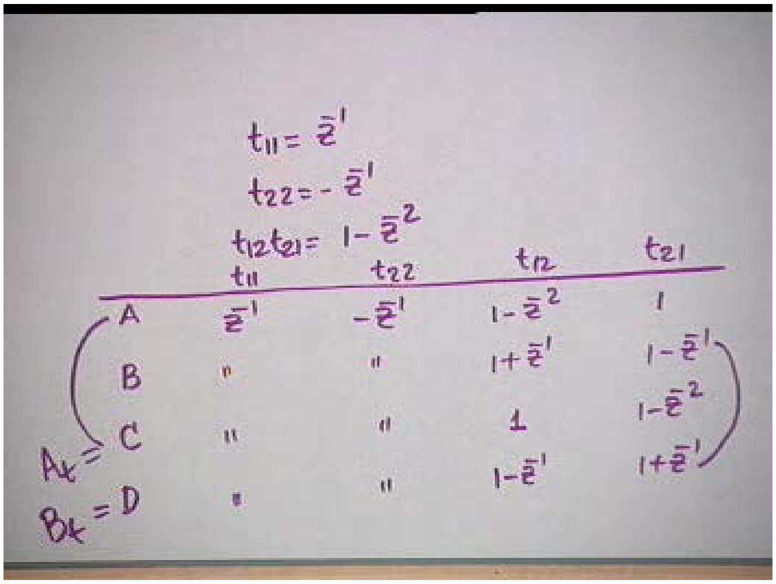

Our choice is $t_{11} = z^{-1}$, $t_{22} = -z^{-1}$, and $t_{12}\, t_{21} = 1 - z^{-2}$. Obviously, we have a lot of flexibility with regard to the choices of t_{12} and t_{21}. Let me list four choices A, B, C, and D. we have $t_{11} = z^{-1}$, and $t_{22} = -z^{-1}$ for all of them. Only t_{12} and t_{21} differ.

Obviously these choices will give rise to four different structures and you can see and anticipate that because t_{11} and t_{22} are the same for A and C (it is same for all of them) and t_{12} and t_{21} are interchanged, the structures A and C will be transposes of each other i.e. C = A transpose. You can draw the structure and then verify this. Similarly for B and D, D = B transpose. We shall close this lecture here.