Digital Signal Processing

Prof. S. C. Dutta Roy

Department of Electrical Engineering

Indian Institute of Technology, Delhi

Lecture - 25

Analog Filter Design (Contd.); Transformations

This is the 25^{th} lecture on DSP and our topic today is continuation of analog filter design. We shall also introduce transformations from low pass to all other kinds of filters. In the last lecture, we talked about Butterworth filters and determination of Butterworth order. We also talked about Chebyshev filters, their pole locations and the type of factors they have in the denominator; in the numerator, there is a constant which you have to adjust. You have to use a factor $1/\sqrt{(1 + \varepsilon^2)}$ if the order is even and no such factor is needed if the order is odd. The orders were found out from a relationship for both Butterworth and Chebyshev which are very similar as one uses log and other uses $\cosh^{-1}$.

For the Butterworth filter it is $\log_{10}$ of $\sqrt{[((1/\delta s^2) - 1)/((1/\delta p^2) - 1)]}/\log_{10}(\Omega_s/\Omega_p)$. This ratio Ω_s/Ω_p is sometimes denoted as the transition ratio. Obviously, it is a transition from pass band edge to stop band edge. There are several other terms which have been used but you can ignore them. For the Chebyshev order N_c, all that changes is that $\cosh^{-1}$ replaces $\log_{10}$. I also told you that if $\cosh^{-1}$ tables are not available, then you may have to use a handbook or you use the alternative fact that at the edge of the stop band, $1 + \varepsilon^2\, C_N^2(\Omega_s/\Omega_p) = \delta s^{-2}$. But as you know, N found from this relationship may or may not be an integer. That is why there is an inequality sign. The transmission at Ω_s is $\leq \delta s$ if you use the greater than sign.

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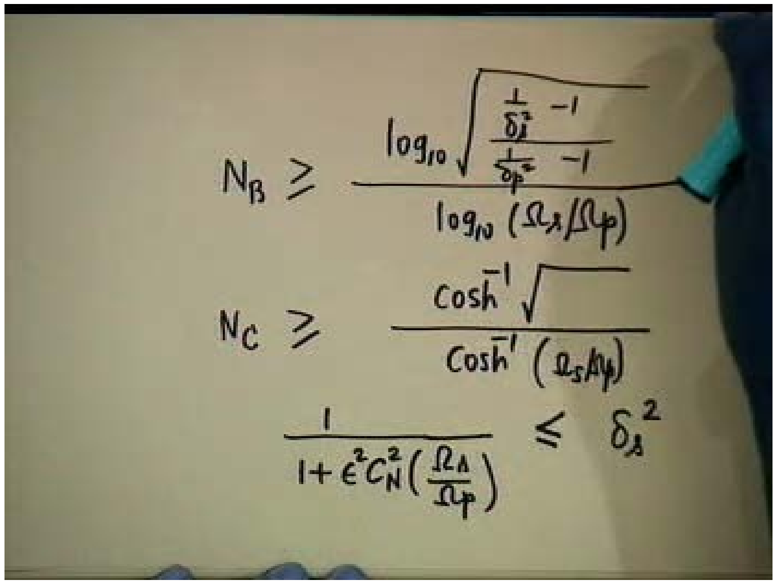

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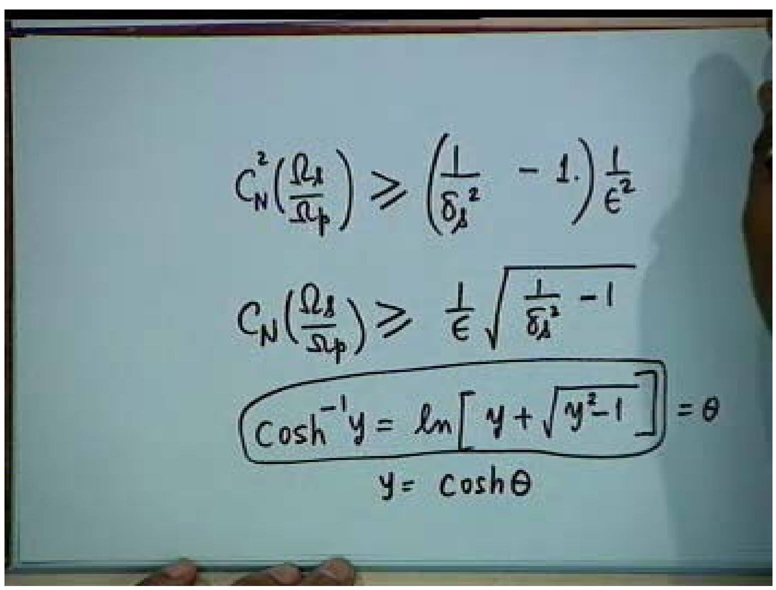

In the relationship $C_N^2(\Omega_s/\Omega_p) \geq (1\ \varepsilon)/(1/\delta s^2) - 1$, ε, δs and the transition ratio Ω_s/Ω_p are known, therefore one can use trial and error for determining N. That is, estimate a value of N and then find out the relationship between the left hand side and the right hand side and increment N till the inequality is satisfied. Now in this calculation, one has to be very careful. Suppose the inequality sign is satisfied critically and both the sides are just equal, for example, then you look at it with suspicion. You increase the accuracy of computation, look at it with a magnifying glass and see if the inequality is exactly satisfied or not. If not then you increment N by 1. If it is exactly satisfied then in all probability the hardware construction will conspire so that it is not satisfied. So even if it is exactly satisfied you increase it by 1 to be on the safe side. This will not cause any harm, you are only increasing the cost incrementally. If the filter cost was 100 rupees, may be another 10 rupees will be added to it, so it is not a problem.

Now this procedure of trial and error succeeds only if N is not very large; otherwise it can become very laborious because you shall have to calculate $C_N(\Omega_s/\Omega_p)$ at a large number of values of N. A third alternative is to use the relationship $\cosh^{-1}y$ = log of $[y + \sqrt{(y^2 - 1)}]$. The proof of this relationship is extremely simple, if $\cosh^{-1}y = \theta$, then obviously y is equal to cosh θ. Also, $\cosh^2 \theta - 1 = \sinh^2\theta$ and therefore the right hand side will be sin hθ plus cosh θ, that is, e^{θ}, and log of $e^{\theta} = \theta$. It is sometimes expedient to use this relation. You shall have to use this in both the numerator and the denominator. I have given you three methods and whichever is suitable or convenient, you use that. For large orders, one invariably uses the third method because for large orders, you may not get the $\cosh^{-1}$ values tabulated at sufficiently narrow intervals.

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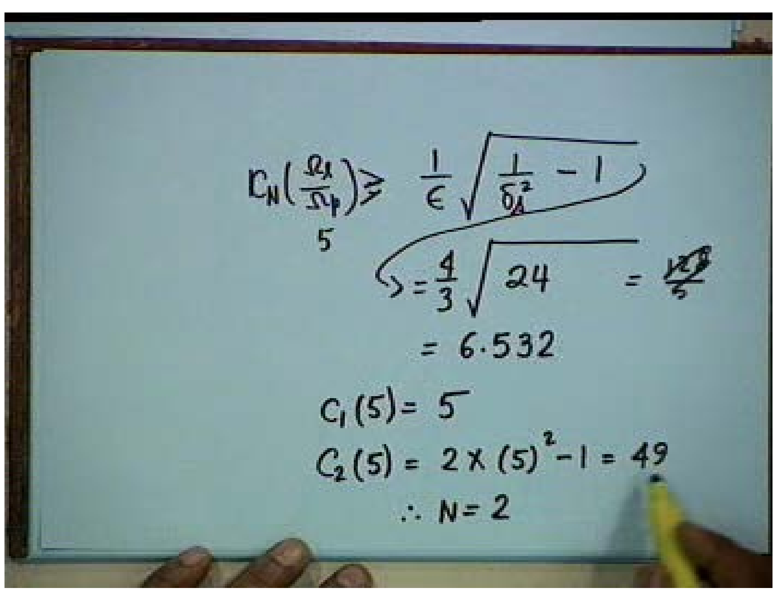

Now let us start with some examples: the first one is an example of Chebyshev filter approximation. Suppose our specs are: 1 less than equal to magnitude less than equal to 0.8, between 0 to 1 kHz and the transition ratio is 5. In other words, the stop band starts at 5 kHz and the stop band transmission must be ≤ 0.2. You are required to find out the required Chebyshev transfer function. The first thing you do is to calculate ε. Now, $1/\sqrt{(1 + \varepsilon^2)}$ must be = 0.8; so ε from here calculates out as ¾ or 0.75. Wherever possible, you maintain a fraction because fraction does not truncate the number. You have found out ε and the next thing is to find N. Let us use this relation $C_N (\Omega_s/\Omega_p) \geq (1/\varepsilon) \sqrt{1/[(\delta s^{-2} - 1)]}$. Now $1/\varepsilon = 4/3$ and $\delta_s = 0.2$ which makes the right hand side = 6.532. So you try a few values of N. Here $\Omega_s/\Omega_p = 5$. Now $C_1(5) = 5$ and $C_2(5) = 49$. Therefore N = 2, satisfies the inequality; obviously, the stop band will be much over satisfied but there is no other way.

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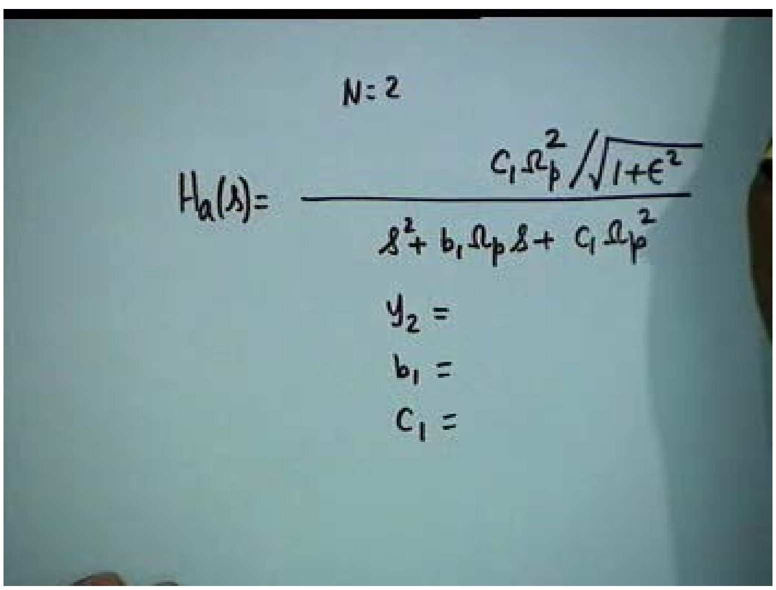

If N = 2 then you can immediately write $H_a(s) = [C_1\Omega_p^2/\sqrt{(1 + \varepsilon^2)}]/(s^2 + b_1\Omega_p s + C_1\Omega_p^2)$. Here $\Omega_p = 2\pi(10)^3$ radians/sec. N is even and therefore you must have the factor $[1/\sqrt{(1 + \varepsilon^2)}]$ in the numerator. For finding out b_1 and C_1 you have to find out y_2 first.

In the second example, let us take a more practical spec that is in decibels; let the pass band tolerance be 1 decibel extending from 0 to 1 kilo Hertz.

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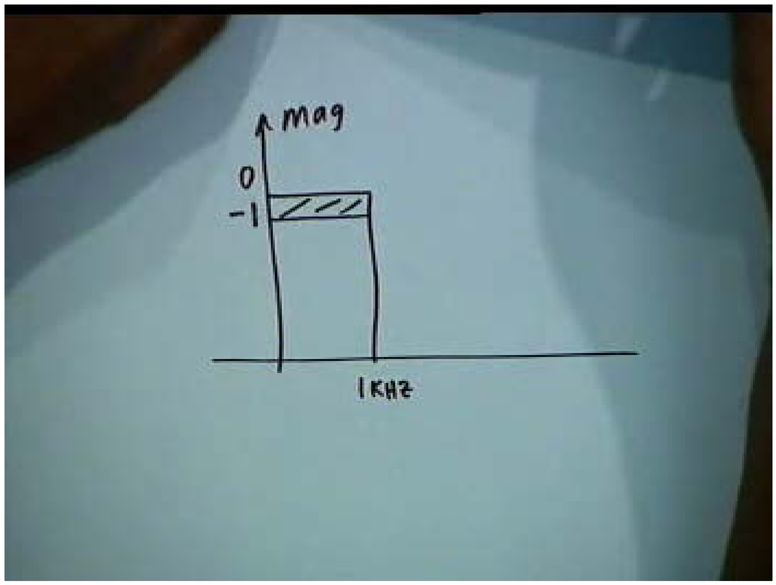

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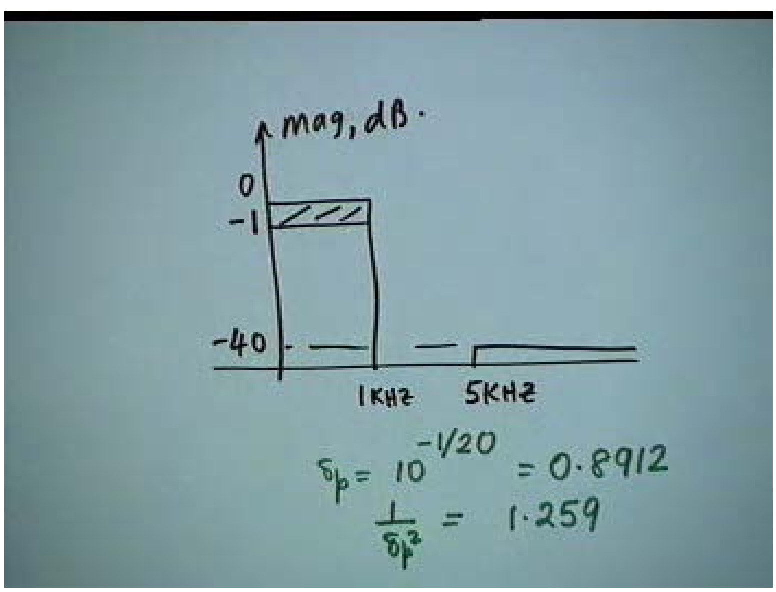

As in the previous example, let the stop band start from 5 kHz and let the minimum attenuation be – 40 decibels. So the first thing you do is to find out δp and δs. Here $\delta p = 10^{-1/20}$ and that comes out as 0.8912. What we need is $1/\delta p^2$, which calculates out as 1.259.

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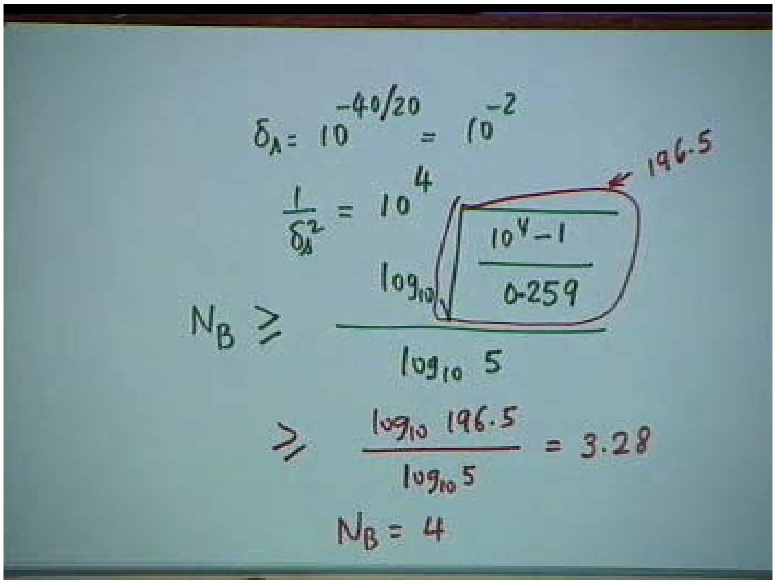

Also $\delta s = 10^{-40/20} = 10^{-2}$ so $1/\delta s^2 = 10^4$. Now it is interesting to find out what order of Butterworth will satisfy these specifications. Then you should be able to compare Butterworth and Chebyshev. If I want to satisfy this specs by Butterworth, then $N_B \geq \log_{10}(\sqrt{[(10^4 - 1)/.259]}/\log_{10}(5)$. This quantity calculates out to $196.5/\log_{10}(5)$. So $N_B \geq \log_{10} 196.5/\log_{10}(5) = 3.28$, therefore $N_B = 4$.

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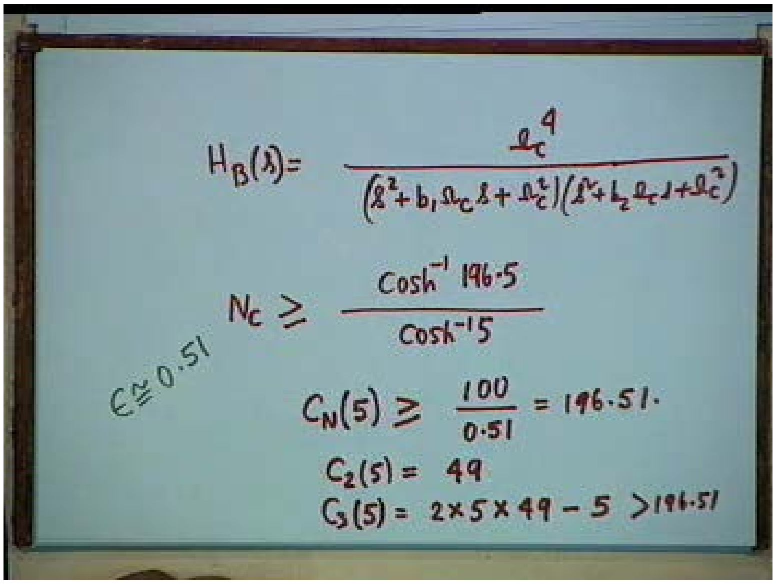

In Butterworth, we have to calculate Ω_c at which the attenuation is – 3 decibels. Then the denominator factor of $H_B(s)$ would be $(s^2 + b_1\, \Omega_C s + \Omega_C^2)$ and $(s^2 + b_2\, \Omega_C s + \Omega_C^2)$; the numerator will be simply Ω_C^4. Next you calculate b_1 and b_2. For Chebyshev, use the formula $N_c \geq \cosh^{-1} 196.5/\cosh^{-1} 5$. Instead of using this, we can use the inequality for $C_N(5)$. Calculate ε from $\delta_p = 1/\sqrt{(\varepsilon^2 + 1)}$. Here ε is approximately 0.51. So $C_N(5) \neq 100/0.51 = 196.51$. Now $C_1(5)$ is 5, $C_2(5) = 49$ while $C_3(5) > 196.51$. Therefore compared to the Butterworth, Chebyshev uses a lower order, and it reduces the cost. They can also be identical but in this particular case the specification is such that a lower order Chebyshev does the job.

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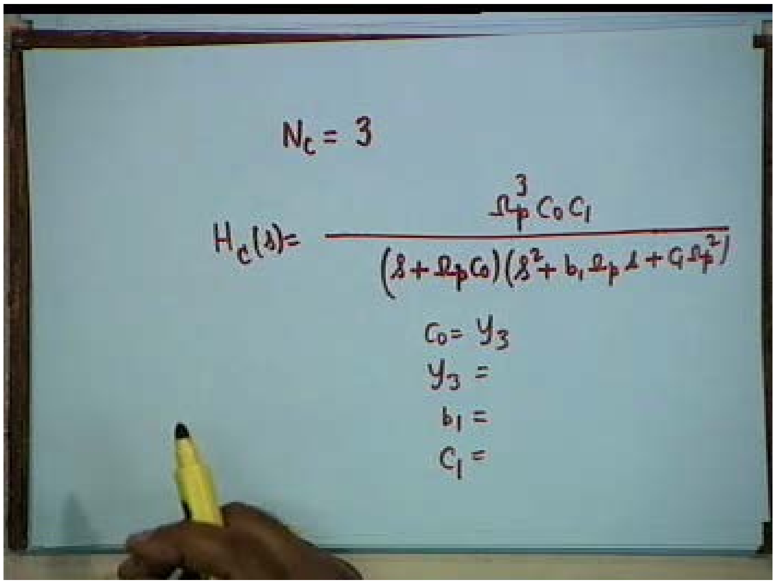

And once you know N_c, then you can write down $H_c(s)$ denominator factors as $(s + C_0\Omega_p)$ and $(s^2 + b_1 \Omega_p s + C_1\Omega_p^2)$; the numerator would be $\Omega_p^3 C_0 C_1$. No factor involving ε is needed because the order is odd. And you can calculate $C_0 = y_3$ and then b_1 and C_1. Chebyshev is always preferred because the cut off slope shall be higher than that of Butterworth. But if you want to keep life simple, then you do not have to even think of Chebyshev; you design a Butterworth filter. If it is one or two filters for a dedicated job, it does not matter. But in an industry, they never fabricate one or two filters because it requires a lot of engineers' time to design a particular filter and therefore they produce in 1000s if not 100,000s.

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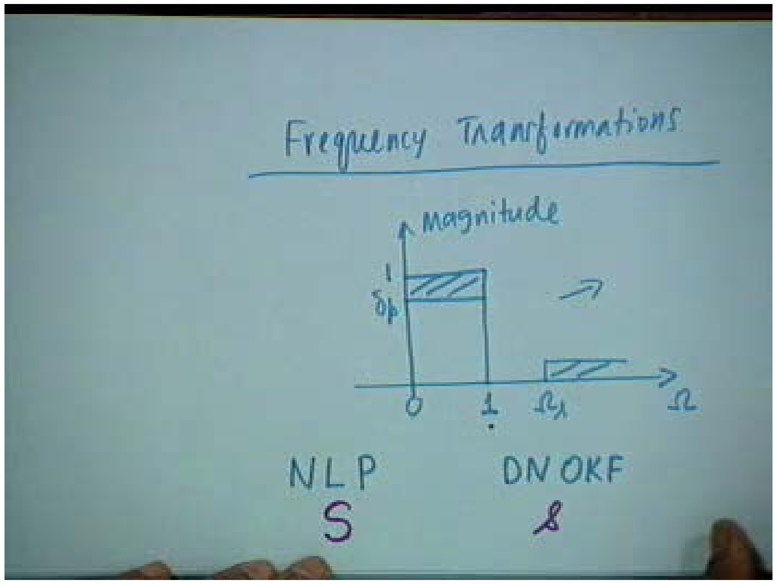

We next consider frequency transformation in analog filters. What we wish to do is to transform a normalized low pass filter with cutoff at 1 radian/sec into other kinds of filters in which the cutoff frequency is Ω_p radian/sec. This cutoff frequency could be 3dB frequency for Butterworth or it could be 1dB frequency for a Butterworth, or it could be .5dB for Chebyshev.

For example, if we wish to transform this to another low pass filter whose cutoff frequency is Ω_p instead of 1 rad/sec, then we simply substitute s by s/Ω_p. Obviously stop band of the transformed filter will start from $\Omega_s \times \Omega_p$. In general, I want to transform a normalized low pass filter (NLP) to a de-normalized other kind of filter. I want to design an actual filter with actual specifications starting from a normalized low pass filter. In order to keep the symbols tractable, we shall use S as the complex frequency variable for the normalized low pass filter, and s for the de-normalized other kind of filter.

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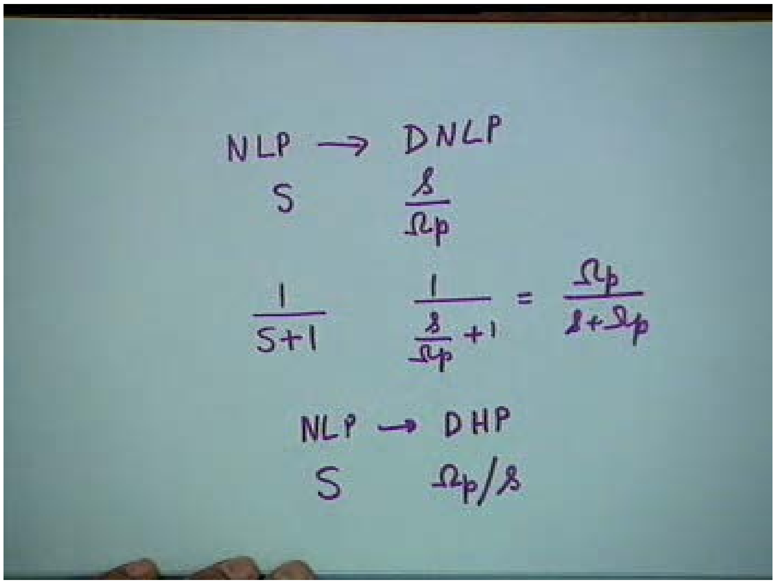

Therefore if we go from normalized low pass filter to de-normalized low pass filter then we shall replace S by s/Ω_p. For example, if I have a first order filter $1/(S + 1)$ having a cutoff frequency at 1 radian per sec (3dB cutoff) and if you want cutoff of Ω_p then the de-normalized low pass filter would be $1/(s/\Omega_p + 1) = \Omega_p/(s + \Omega_p)$. It is a first order filter with cutoff frequency at Ω_p. When you go to other kinds of filters, things are not so simple. For example, let us consider transformation from normalized low pass to de-normalized high pass (HP). Then S has to be replaced by Ω_p/s. What is the justification of this transformation?

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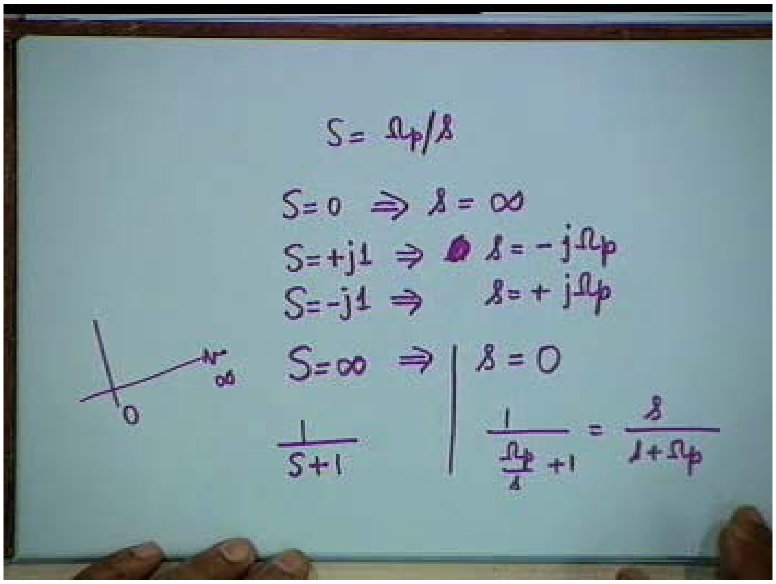

With S = Ω_p/s, S = 0 corresponds to s = ∞. In other words, a low pass filter has gone into a high pass filter. Not only that, S = + j which is the cutoff frequency of the normalized low pass, goes to s = – j Ω_p. Also S = – j corresponds to s = + j Ω_p. The magnitude characteristic is even, so if you plot for negative frequencies, you get the same characteristic as for positive frequencies, which means the cutoff frequency is Ω_p and it is now a high pass filter.

In essence, we have simply flipped over the low pass characteristic with Ω_p as our pivot. For example, if I have a low pass first order low pass filter 1/(S + 1), then replacing S by Ω_p/s, we get s/(s + Ω_p) as the high pass filter. The positive frequencies in LP go to the negative frequencies in HP; it is not a matter of concern because the magnitude characteristic is even.

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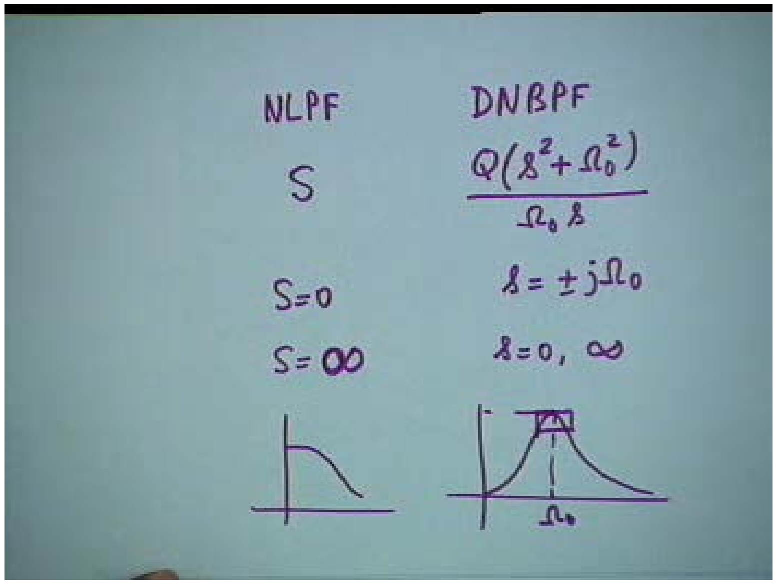

Now comes the question of band pass filter. That is, we want to convert a normalized low pass filter to a de-normalized band pass filter. Then you replace S by $Q(s^2 + \Omega_0^{2)}/(\Omega_0 s)$. First let us see whether it is a band pass filter or not. S = 0 (where the magnitude of the low pass filter is unity for a Butterworth filter (not necessarily for Chebyshev) corresponds to s = $\pm j\ \Omega_0$ and therefore whatever transmission I have at dc in the low pass filter transfers to the non zero finite frequency Ω_0. S = ∞ is possible in two ways: s can go to 0 or ∞.

In other words, at both dc and infinite frequency, the characteristic of the low pass filter at infinity is reproduced. Low pass filter at infinite frequency has to give zero or negligible transmission and therefore in the band pass filter, both dc and infinite frequency are in the stop band. Now you should have been convinced that this is a low pass to band pass transformation.

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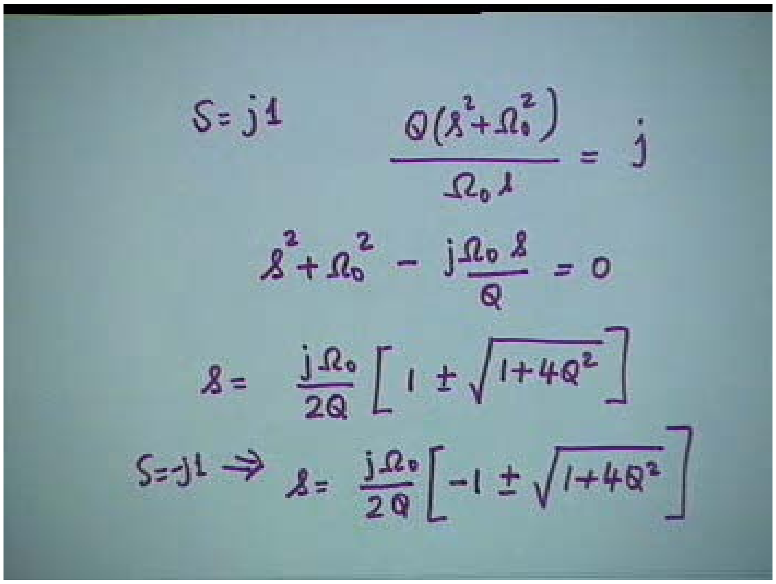

What happens to the cutoff frequency? The low pass cutoff frequency corresponds to S = ± j; let us take the (+) sign first, we can derive other results very simply. Then S = + j will correspond to $Q(s^2 + \Omega_0^2)/(\Omega_0 s) = j$; in other words, $s^2 + \Omega_0^2 - j\Omega_0 s/Q = 0$. This is a quadratic equation and therefore there are two solutions. If we solve them, we get $s = [j\Omega_0/(2Q)]\ [1 \pm \sqrt{(1 + 4Q^2)}]$. There are two solutions; obviously one of them is positive and another is negative. Similarly if you take S = – j then the solutions would be $[j\Omega_0/(2Q)].\ [-1 \pm \sqrt{(1 + 4Q^2)}]$. Thus two solutions S = ± j go over to the band pass filter to four frequencies; two of them are positive and two of them are negative. We are interested in the positive frequencies.

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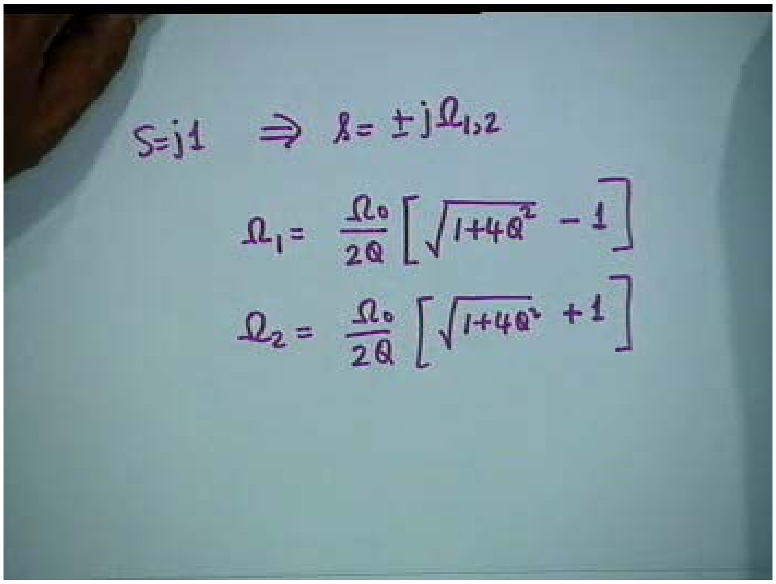

The two cutoff frequencies are therefore s = + $j\Omega_1$, where $\Omega_1 = [\Omega_0/(2Q)]\ [\sqrt{}(1 + 4Q^2) - 1]$ and $\Omega_2 = [\Omega_0/(2Q)]\ [\sqrt{}(1 + 4Q^2) + 1]$. The other two solutions are $-\ \Omega_1$ and $-\ \Omega_2$. What is the nature of the transformation? It is not a linear transformation, it is a quadratic transformation.

The interesting thing is that the product of Ω_1 and Ω_2 is Ω_0^2. In other words, the two pass band edges of the band pass filter have geometric symmetry with regard to centre frequency Ω_0. Also $\Omega_2 - \Omega_1$ which is the bandwidth B is Ω_0/Q. And if you remember, this is one of the definitions of Q.

Note that the two cutoff frequencies are not in arithmetic symmetry; they are in geometric symmetry. In fact, geometric symmetry also reflects in the characteristic all through. If you take any level of magnitude, let us say δ, then there are two frequencies Ωs_1 and Ωs_2 at which this level is attained; these are also geometrically symmetrical with respect to the centre frequency. That is, any two frequencies on two sides at which the transmission is the same satisfy the relation that their product is = Ω_0^2. This applies to pass as well as stop bands. Thus if Ω_{p1} and Ω_{p2} are the pass band edges and Ω_{s1} and Ω_{s2} are the stop band edges, then $\Omega_0^2 = \Omega_{s1}\Omega_{s2b} = \Omega_{p1}\Omega_{p2}$. Now you cannot compel your consumer to make such a specification; the consumer makes specifications according to what he desires. It is the duty of the filter designer to satisfy the specs. But since the tool in his hand is geometric symmetry**,** he must modify the specs and this is where an important point comes. He must modify the specs such that the pass band is exactly satisfied, but he can play with the stop band and modify in such a manner that the geometric symmetry is satisfied; this will make the stop band over satisfied. This is the point I would come to a little later, but let us get some experience of this transformation of the normalized low pass to a de-normalized band pass filter.

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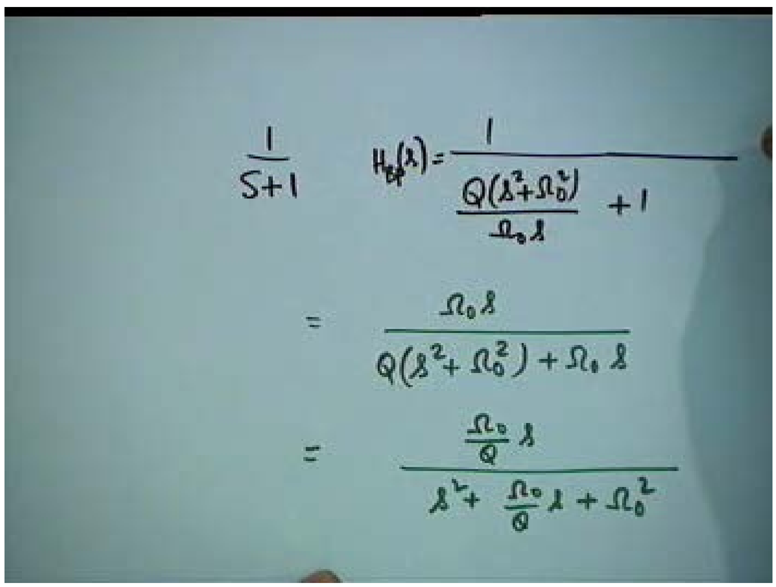

Suppose I take the first order filter 1/(S + 1) and transform it to a band pass filter $H_{BP}(s)$ by using the transformation $S = Q(s^2 + \Omega_o^2)/(\Omega_o s)$. Simplifying this, you see that the order is doubled from first order to second order. This is so because the transformation is a bi-quadratic function, it is a second order transformation. You cannot make a first order band pass filter. If I simplify this, then I shall get $H_{BP}(s) = (\Omega_o/Q)s/[s^2 + (\Omega_o/Q)s + \Omega_o^2]$; this is second order band pass filter. At s = 0 magnitude = 0; at s = ∞ the magnitude = 0, and when $s = \pm j\Omega_o$ then the magnitude = 1. And you can find out the two cutoff frequencies, 3dB or otherwise. Note that (Ω_o/Q) is the bandwidth.

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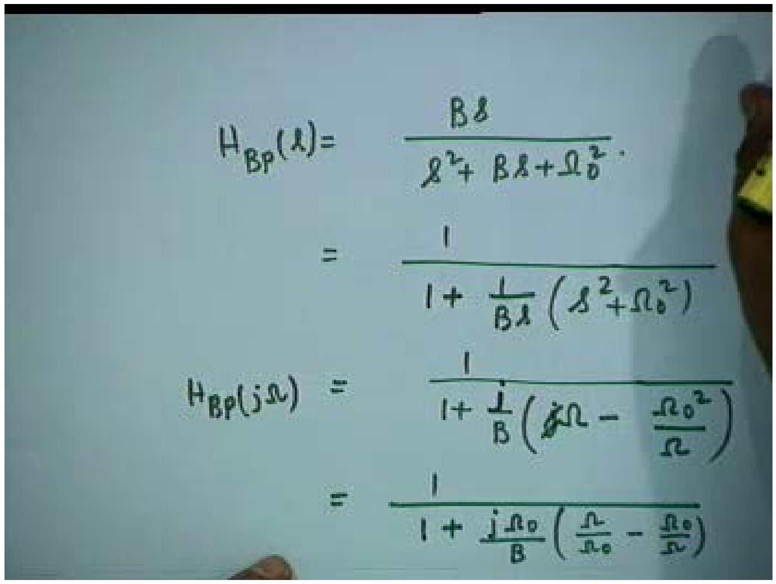

You can write this as $H_{BP}(s) = Bs/(s^2 + Bs+ \Omega_o^2) = 1/[1 + (1/(Bs)) (s^2 + \Omega_o^2)]$. If I take the value on the imaginary axis, then $H_{BP}(j\Omega)$ can be written as $1/[1 + j(\Omega_o/B) ((\Omega/\Omega_o) - (\Omega_o/\Omega))]$.

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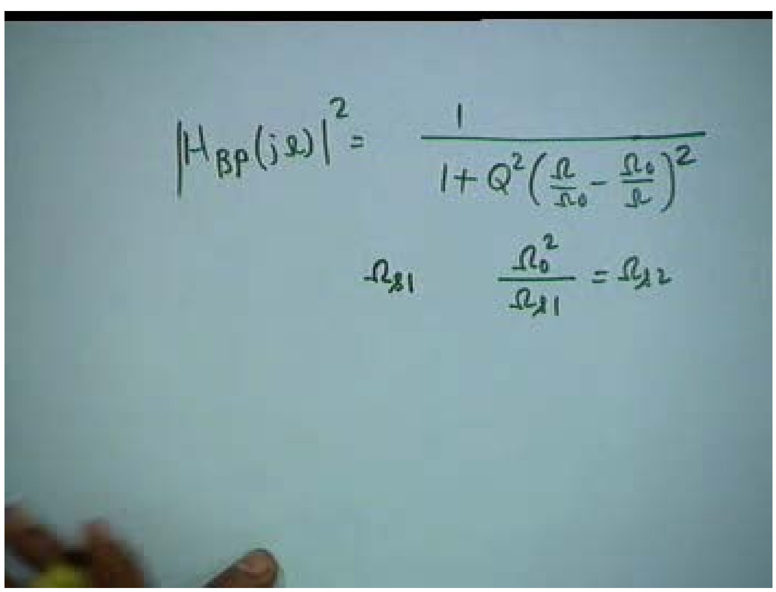

Using $\Omega_o/B = Q$, we get $|H_{BP}(j\Omega)|^2 = 1/[1 + Q^2((\Omega/\Omega_o) - (\Omega_o/\Omega))^2]$. If you take any frequencies Ω_{s1} and another frequency $\Omega_{s2} = \Omega_o^2/\Omega_{s1}$, then you see that the magnitudes at these frequencies should be equal, because $\Omega_{s2}/\Omega_o = \Omega_o/\Omega_{s1}$. At the two frequencies which are related by the relation $\Omega_{s1}\,\Omega_{s2} = \Omega_o^2$, the magnitude is the same, which demonstrates geometric symmetry.

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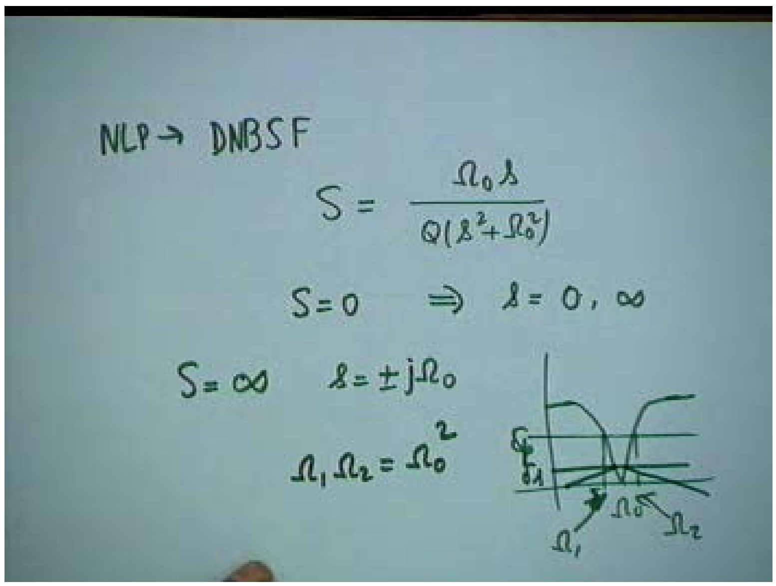

This is another way of saying that at the same attenuation the two frequencies are geometrically symmetrical with respect to Ω_o.

Let us finally consider the transformation of a normalized low pass to de-normalized band stop filter. The transformation here is simply the reciprocal of the low pass to band pass transformation. In other words, we shall simply have $S = \Omega_o s/[Q(s^2 + \Omega_o^2)]$. Here $S = 0$ corresponds to $s = 0$ or $s = \infty$. Therefore if it is a Butterworth filter of unity magnitude at $S = 0$, the feature is transferred to the de-normalized filter at zero and infinite frequencies. Therefore our job is done; in between there must be a minimum. That minimum you can see by recognizing that when $S = \infty$, then $s = \pm j\Omega_o$. When $S = \infty$ in the low pass filter, the transmission should be 0 or negligible and therefore at Ω_o, the band stop response must come down to the same value. Let Ω_1 and Ω_2 be two frequencies at which the transmission is the same. It can be shown that here also, $\Omega_1 \, \Omega_2 = \Omega_0^2$. The band stop filter is just the complement of the band pass filter. There are two pass bands and one stop band.

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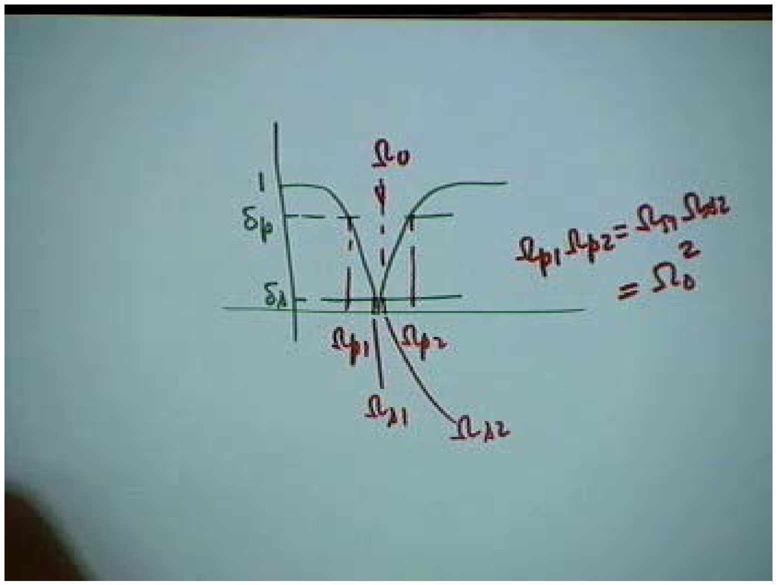

Let the stop band, defined by magnitude $\leq \delta s$, extend from Ω_{s1} to Ω_{s2}. Also let the pass band, defined by $1 \leq$ magnitude $\leq \delta p$ extend from 0 to Ω_{p1}, and Ω_{p2} to infinity. The centre frequency, or the notch frequency is Ω_o. The centre frequency here means the frequency of null or notch. Ω_o is the centre for geometric symmetry. Here also by the same procedure, you can show that $\Omega_o^2 = \Omega_{p1} \, \Omega_{p2} = \Omega_{s1} \, \Omega_{s2}$.

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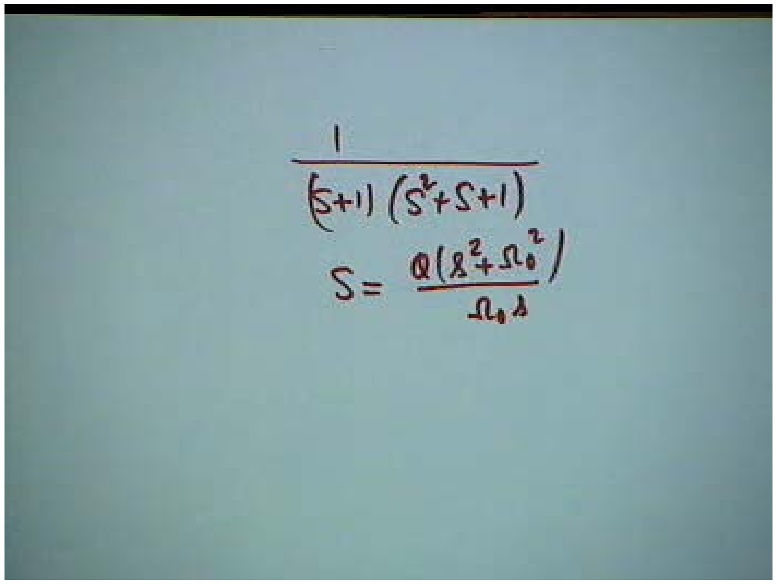

As an example, if you want to design a 6th order Butterworth band-pass filter then you will start with 3rd order Butterworth normalized low-pass filter. The transfer function is $1/[(S + 1)(S^2 + S + 1)]$. All that you do now is to put $S = Q(s^2 + \Omega_o^2)/(\Omega_o s)$. Given the normalized low-pass filter, then you can always go to the band pass or band stop or high pass filter. In all these transformations, the pass band is to be strictly satisfied; whatever frequencies come for the stop band edges, we shall have to accept. But the practical problem is not that; the practical problem is that the specifications of the desired filter are given to you for band-pass and band-stop filters, where $\Omega_{s1}\,\Omega_{s2}$ may not be equal to $\Omega_{p1}\,\Omega_{p2}$. So what you have to do is to predistort the stop band edges so that $\Omega_{s1}\,\Omega_{s2}$ becomes equal to $\Omega_{p1}\,\Omega_{p2}$ with over satisfied stop band specs. Then convert these specs to that of a normalized low-pass filter. Design the normalized low-pass filter and then by frequency transformation, you go to the band-pass, band-stop or high-pass filter.

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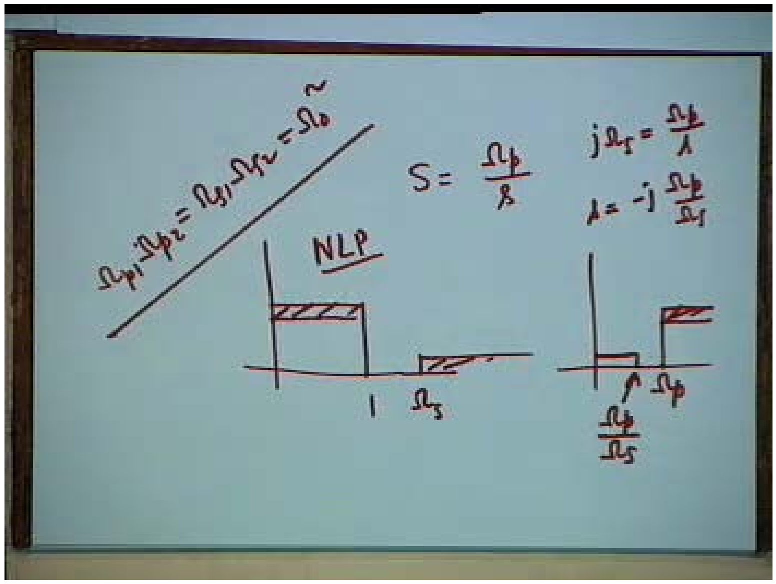

Do you understand the steps? The steps are: from the specifications of the desired filter, you go to specs of normalized low-pass filter (NLPF); then design the (NLPF) and use transformation to get the required H(s).