Digital Signal Processing

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Lecture - 23

Analog Filter Design

This is the 23rd lecture on DSP and our topic for today and a few more lectures to come will be analog filter design.

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The last two sessions, namely lectures 22 and 21 were on problem solving on the topics of Fourier Transform, Discrete Fourier Transform, and Z transform. The previous lecture 20 was concerned with digital processing of analog signals. In particular, we examined the sampling theorem very closely, and also made some comments about band pass sampling. Band pass sampling requires a minimum sampling frequency equal to twice the bandwidth, not twice the highest frequency. The motivation for analog filter design are two–fold: one is that, in digital processing of analog signal, we do require low pass filters at the front end and also at the receiving end. At the front end, we constrain the bandwidth so that the requirements of the sampling theorem are satisfied; at the far end, we get rid of high frequencies from digital to analog converter– that is the first motivation.

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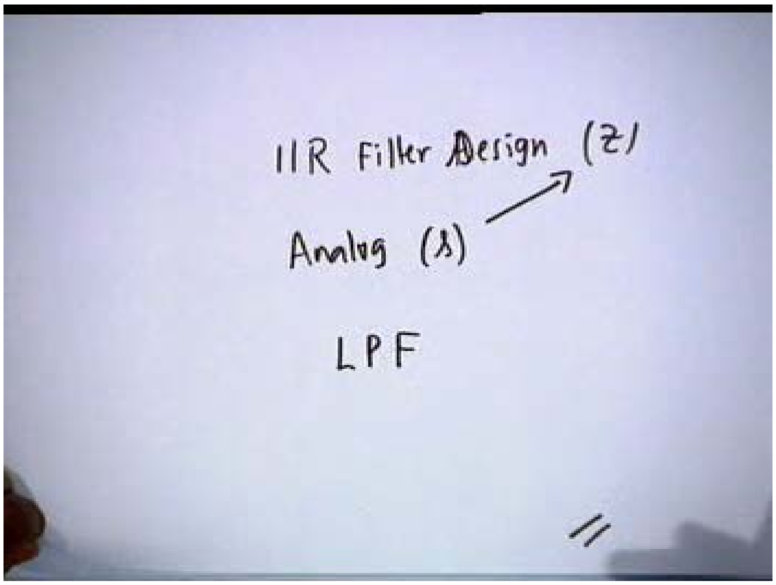

The second motivation is that our IIR design, one of the most important and useful filters in practice, is based on analog filter design using a transformation from s plane to z plane. The procedure is like this: in the first step we get the specifications of the IIR filter. You have to convert these into digital normalized frequency specs, convert them to an analog filter design specs, design the analog filter, and then go back to the z plane through the s to z transformation. So the motivation for analog filter design discussion in a DSP course is two–fold: one is we do require analog filters and the second is that a major part of digital filter design is based on transformation of analog filters. It is also a fact that a discussion of low pass filters is adequate because any low pass filter can be converted to any other kind of filter. As far as low pass filters are concerned, we shall concentrate on two types: Butterworth and Chebyshev.

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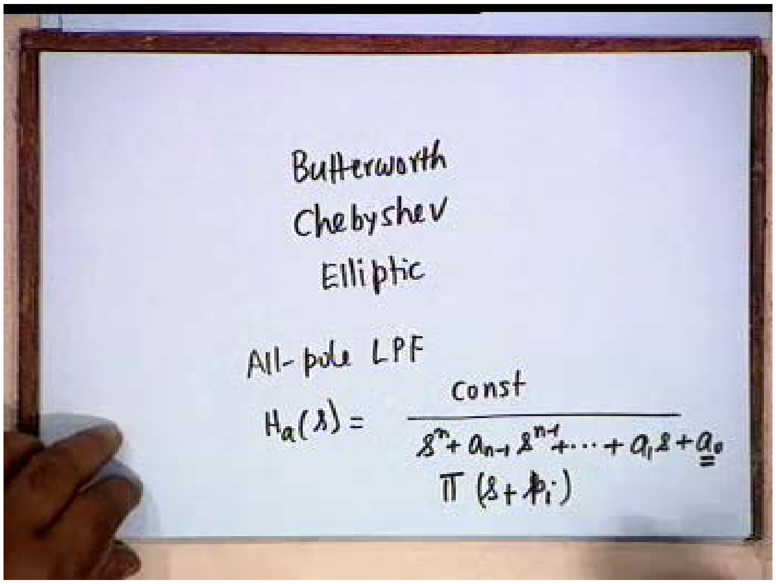

In most of the practical cases, these two types suffice. The most optimum filter is the elliptic filter. But the problem in elliptic filter is that the design requires numerical computation in almost every case. It is very difficult to tabulate elliptic filters, and no elegant mathematical formulas are available. Therefore, in most cases digital signal processing engineers appeal either to the Butterworth or to the Chebyshev. Butterworth is the simplest but it may involve a little more cost than a Chebyshev filter; we shall discuss both of them. In LPF design, we also concentrate on only one kind of filter, viz. the so called all pole filters. That means the filter only has poles in the finite region of the s plane. Now, there cannot be a filter without zeros. The number of zeros and number of poles should be the same for any transfer function, so there are zeros but these zeros are all at infinity. So we shall consider the analog transfer function $H_a(s)$ = constant divided by ($s^n + a_{n-1} + s^{n-1} \ldots + a_1 s + a_0$). In digital filter transfer function, we always write the constant term as unity; here, we shall write the coefficient of the highest power term as unity, because we shall write the pole factors as ($s - p_i$). The continued product of such factors shall ensure that the highest power has a coefficient of 1. If we take the pole factors as $s + p_i$ then a_0 will be the continued product of p_i. Similarly what is a_1? What shall be a_{n-1}?

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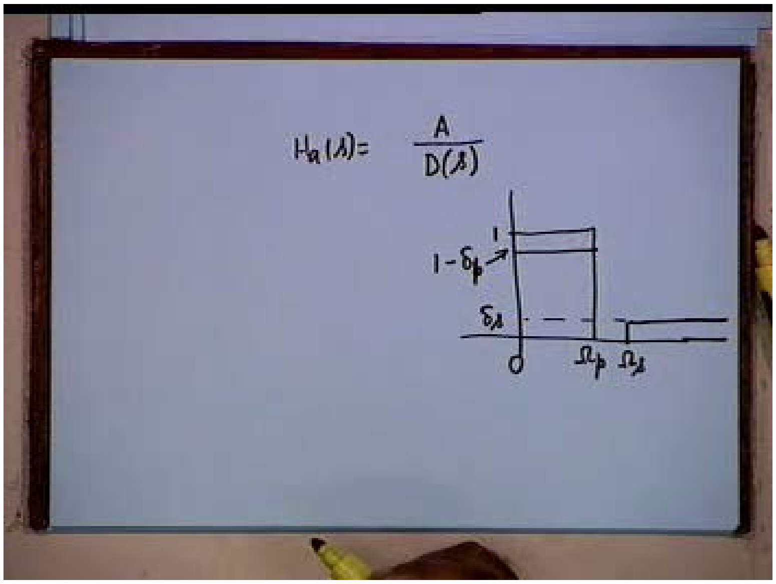

Our aim therefore, is to find $H_a(s) = a/D(s)$; where $D(s)$ is of the form $s^n + a_{n-1}\, s^{n-1} +…..+ a_0$ where a_i’s are chosen in such a manner that the given specifications are satisfied. Given specifications for a low pass filter are typically in terms of magnitude, whose maximum is normalized to 1. The specifications are: a pass band from 0 to Ω_p (we use capital Ω for analog radian frequency); a stop band from Ω_s to infinity (the difference between Ω_p and Ω_s is the transition band); a tolerance Δp in the pass band; and a tolerance Δs in the stop band. Typically, what will be specified are: Ω_p, Ω_s, Δp, and Δs. From these four specifications, you shall have to determine $H_a(s)$ which satisfies these specifications. This satisfaction of specifications also has to be qualified. You may not be able to exactly satisfy these specifications. You can take liberties with the stop band but not with the pass band. Many of the textbooks say the contrary but the practical design experience is that if you take liberties with the pass band, you spend more man hours in designing the filter. The pass band is sacred; with the stop band you can take liberties. Liberties means you can over satisfy. For example, if the stop band starts earlier, you should be happy, because you over satisfy the stop band. So this is what our job is.

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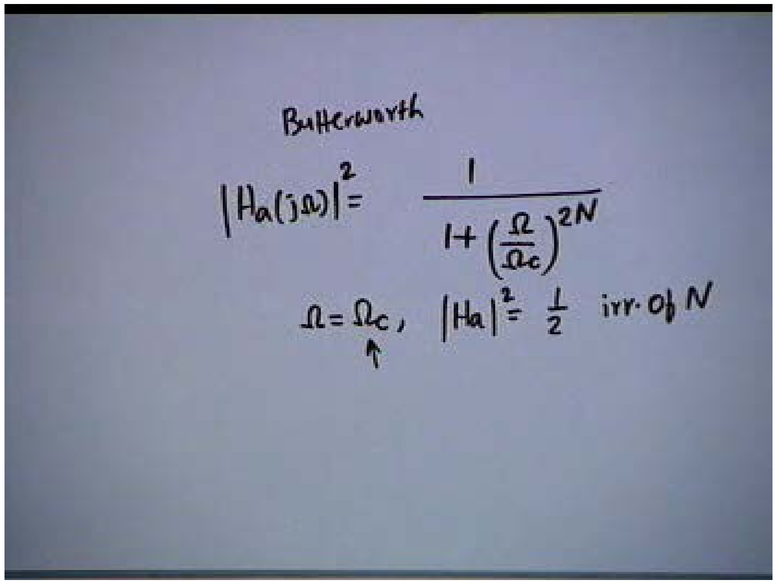

To this end, the Butterworth filter simply takes $|H_a(j\Omega)|^2 = 1/[1 + (\Omega/\Omega_c)^{2N}]$. The Nth order Butterworth filter has a magnitude squared function equal to this. Notice that at $\Omega = \Omega_c$, $|H_a|^2 = 1/2$ irrespective of N. And wherever $(\text{magnitude})^2 = 1/2$ is the frequency at which 3dB attenuation occurs; therefore Ω_c is called the 3dB bandwidth of the Butterworth filter. Notice that the $(\text{magnitude})^2$ plot versus Ω goes down monotonically. There are no maxima or minima. If we plot $|H_a(j\Omega)|$, then the value at dc is 1 and the value at $\Omega_c = 1/\sqrt{2}$ irrespective of the value of N. And therefore if we plot for various values of N, we will get plots like those shown on the slide where N is increasing in the left direction.

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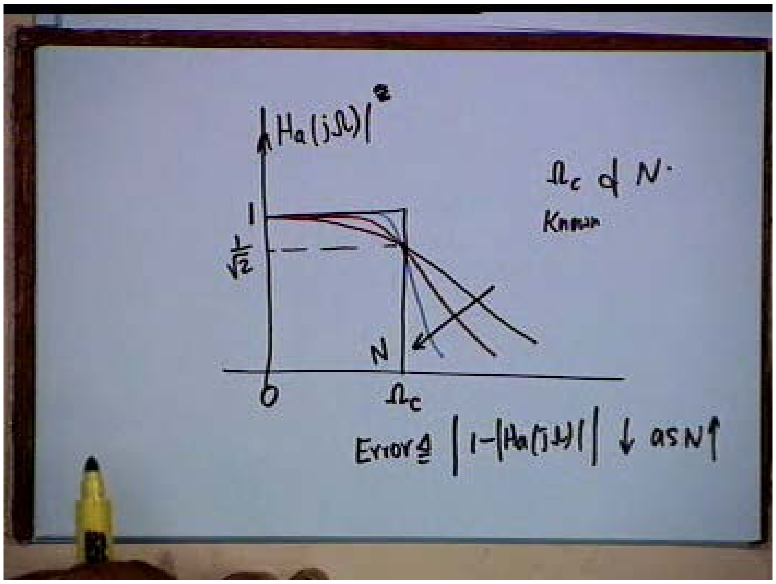

That is, as you increase the order of the filter, all of them have the same originating point Ω = 0 where the value is 1; all of them have the same end of the pass band at the 3dB frequency, Ω_c, where the value is $1/\sqrt{2}$. Obviously, the error defined as $1-|H_a(j\Omega)|$ decreases as the order increases. By error, we mean error in the pass band as well as in the stop band. As N increases, the cutoff slope increases and therefore the edge of the stop band, Ω_s, decreases. What you do is to determine the order N required for meeting the given specifications. If Ω_c is known, then all you have to do is to find the order that is required to satisfy the given stop band specifications. You should not unnecessarily over satisfy the specs by choosing a higher order. You should choose only the minimum possible order because if you make a mass manufacture, as most of the industries do, every increase in order requires an increase in cost. The industry usually sticks to the minimum possible, the guiding principle being how much less can I spend to be able to earn how much more.

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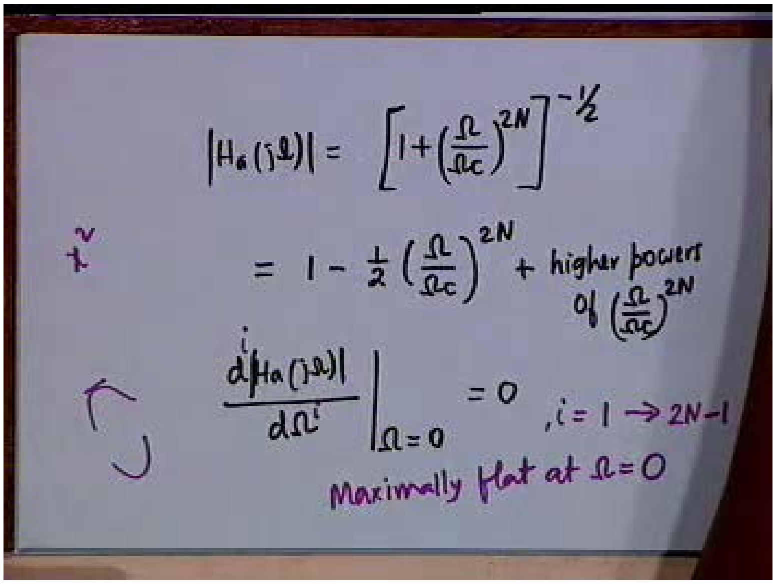

You also notice that, this monotonic $|H_a(j\Omega)|$ can be written as, $[1 + (\Omega/\Omega_c)^{2N}]^{-1/2}$ and if we expand this then you get $1- (1/2) (\Omega/\Omega_c)^{2N}$ + higher powers of $(\Omega/\Omega_c)^{2N}$. And you see that if you differentiate the magnitude with respect to Ω and put Ω = 0, obviously the result will be equal to 0. The next differential coefficient shall also be equal to 0 and this will continue for the i–th differential coefficient where i goes from 1 to 2N – 1. The 2N–th differentiation will give you a constant; you cannot think of any other function which will have 2N – 1 differentiation vanishing at Ω = 0. If one differential coefficient vanishes for a function at a particular point, we say the function is flat at that point. It is either a maximum or a minimum and we say that the function is flat at this point. The Butterworth function has the largest possible number of derivatives vanishing at Ω = 0 and therefore the function is called "maximally flat". The order of flatness cannot be more than 2N – 1 at Ω = 0. The next question is that of the asymptotic behavior of the Butterworth function.

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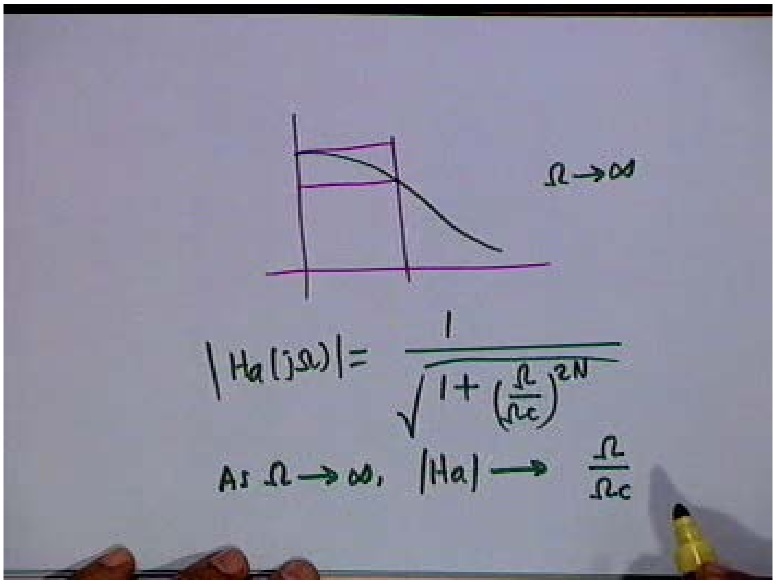

Asymptotic behavior is related to the characteristic when Ω goes to ∞. Now $|H_a (j\ \Omega)| = 1/\sqrt{(1 + (\Omega/\Omega_c)^{2N})}$. If we take the magnitude when Ω goes to ∞, $|H_a|$ becomes $(\Omega/\Omega_c)^{-N}$. If we take the log of this, that is, we want to express this in decibels then you get 20 log of $|H_a|$ to the base 10, as the frequency goes to ∞, going to – 20N log of Ω/Ω_c to the base 10.

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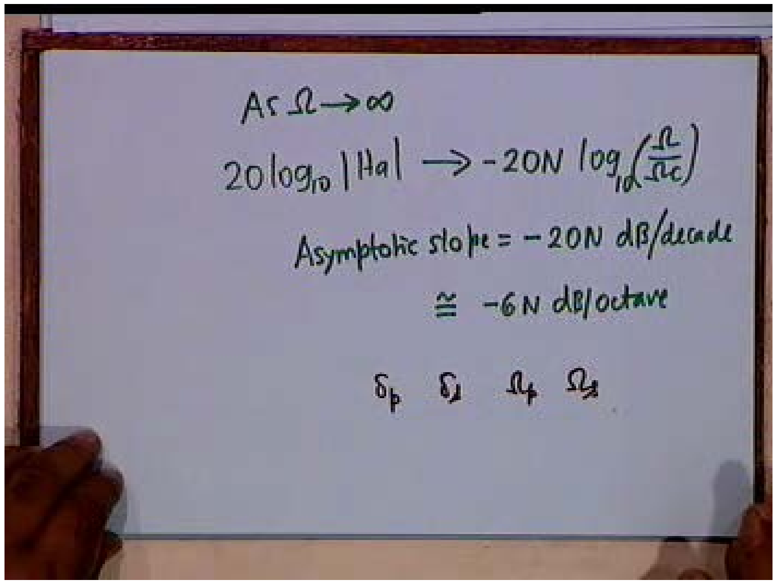

In other words, as Ω increases by a factor of 10 the characteristic goes down by 20N dB. It is expressed in decibels and therefore the asymptotic slope = − 20 N dB/decade. Ω increasing or decreasing by a factor of 2 means an octave, and the characteristic goes down approximately by 6N decibel/octave. This should have been 6.0206 because log2 = 0.30103. But for engineering purposes, we take it as the round figure 6N. It does not make much of a difference. The slope is 6N dB/octave where octave is doubling or halving the frequency. This makes it clear that if we know the asymptotic slope, then you know the order. If you are required to design a Butterworth filter with 18dB/octave asymptotic slope, then you require a third order filter. But this is not the way to specify a Butterworth filter. As I said, specifications will always be in the form of Δp, Δs, Ω_p and Ω_s. This is how a required filter is specified in practice.

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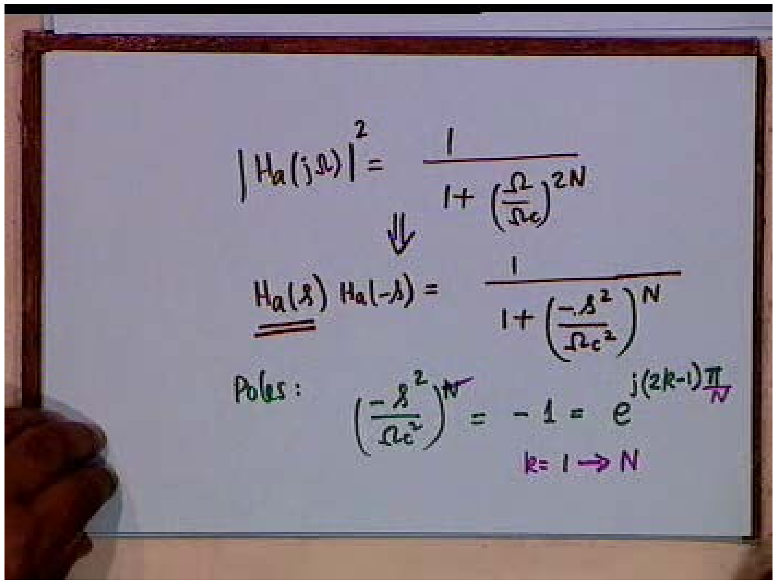

Knowing $|H_a(j\Omega)|^2$ as $1/(1 + (\Omega/\Omega_c)^{2N})$ is not adequate for designing the filter; we require the transfer function $H_a(s)$ in the s plane. The process of analytic continuation says that $H_a(s)\ H_a(-s) = 1/1 + (-s^2/\Omega_c^{\ 2})^N$. It is obvious that I can go from s to $j\Omega$ by putting $s = j\Omega$. The converse is also possible only if the function is analytic, that is, we are looking for $H_a(s)$ which has a finite number of singularities and the function is differentiable in the total s plane except at the singularities; that is the definition of an analytic function, and $H_a(s)$ satisfies the conditions of analyticity. Therefore what we require now, is to find the function $H_a(s)$. This requires that we find all the factors of $1 + (-s^2/\Omega_c^{\ 2})^N$, and assign half of them to $H_a(s)$ and the other half to the $H_a(-s)$.

The factors that we assign to $H_a(s)$ must be such that their zeros are in the left half of the s plane. In the z plane they were inside the unit circle; in the s plane they must be in the left half, for $H_a(s)$ to become a stable transfer function. And therefore we must find all the roots and then make assignments. Well, the poles obviously satisfy $(s^2/\Omega_c^{\ 2})^N = -1$. At the poles, the denominator should be equal to 0; -1 you can write as $e^{j(2k-1)\pi}$ in general. k will go from 1 to N because now in this form we are finding the values of s^2 and the total function shall have 2N number of roots.

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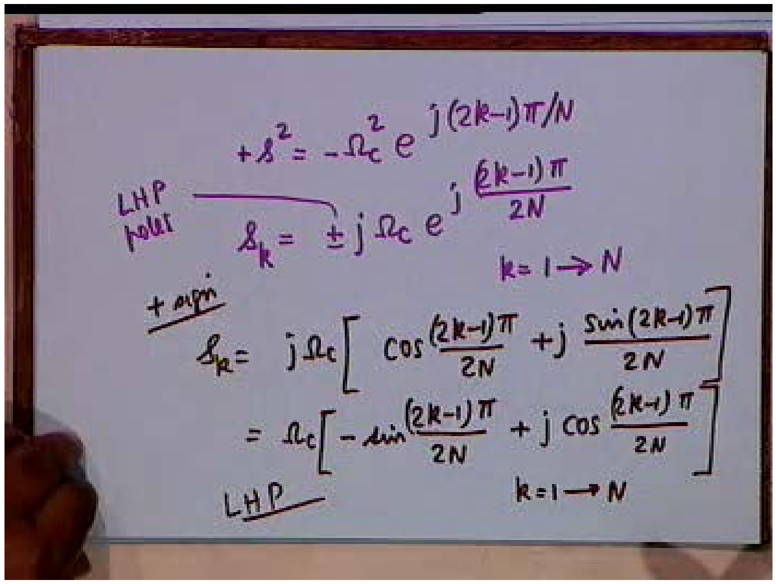

So $s^2 = -\Omega_c^2 e^{j(2k-1)\pi/N}$. And if I take the square root of this then $s = \pm j \Omega_c e^{j(2k-1)\pi/(2N)}$ where k goes from 1 to N because plus sign takes care of N roots; minus sign takes care of the other N roots. It is now very easy to show that the plus sign corresponds to left half plane poles. Let us call these as s_k; then $s_k = j \Omega_c [\cos (2k - 1)\pi/(2N) + j \sin (2k - 1)\pi/(2N)]$ and if I multiply by j then I get $\Omega_c [-\sin (2k - 1)\pi/(2N) + j \cos (2k - 1)(\pi/2N)]$. Obviously one observation that can be made from here itself is that all poles are on a circle of radius Ω_c because magnitude $s_k = \Omega_c$. Second, you look at the real part with the positive sign; the real part is at $-\Omega_c [\sin (2k - 1)\pi/(2N)]$ and k goes from 1 to N. If k goes from 1 to N we start from $\pi/2N$ and end up in $(2N - 1) \pi/(2N)$ that is less than π by an angle π by 2N. So we remain in the upper semi circle, first and the second quadrants; in the first and second quadrants, sine is positive. Therefore the real part of s_k with k = 1 to N is always negative which means that these are the poles in the left half plane. So these are our required poles.

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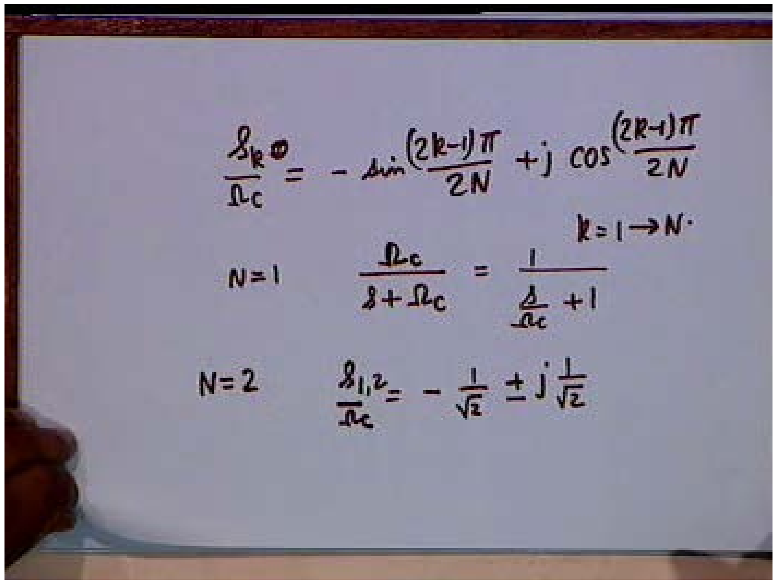

Let us take a few examples. Let us normalize the pole as s_k/Ω_c. Now I shall have $s_k/\Omega_c = -\sin(2k-1)\pi/(2N) + j\cos(2k-1)\pi/(2N)$, k = 1 to N. Suppose I have the first order, N = 1; then since $\sin(\pi/2)$ is 1, and $\text{cosine}(\pi/2) = 0$, the root must be at $-\Omega_c$ and therefore $H_a(s)$, the transfer function is simply $\Omega_c/(s + \Omega_c)$. You remember, for the transfer function the magnitude is 1 at $\Omega = 0$. This is why we have Ω_c in the numerator. Take N = 2; then $s_1/\Omega_c = -\sin\pi/4 + j\cos\pi/4$. If I know that a pole is complex, its conjugate must also be present. Since we are only taking about second order transfer function, everything is found. Our $H_a(s) = 1/\{[(s/\Omega_c) + (1/\sqrt{2}) + (j/\sqrt{2})][(s/\Omega_c) + (1/\sqrt{2}) - (j/\sqrt{2})]\}$.

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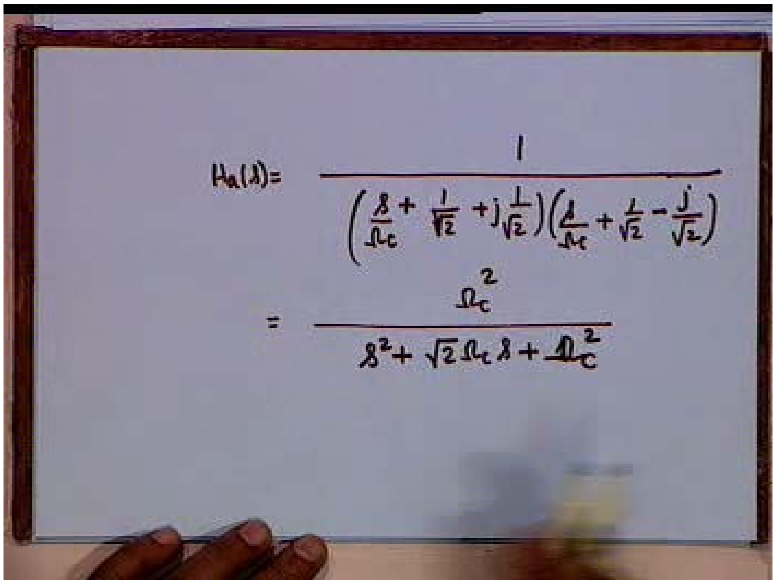

We have written the transfer function in terms of s/Ω_c; multiplying both numerator and denominator we shall have $H_a(s) = \Omega_c^2/(s^2 + \sqrt{2}\,\Omega_c s + \Omega_c^2)$. $H_a(s)$ becomes $1/(s^2 + \sqrt{2}s + 1)$ if Ω_c is normalized to 1. The denominator is called the second order Butterworth polynomial. Now, if we go to the third order, we have $N = 3$ and it is convenient to normalize Ω_c to unity. Later on, if we require $\Omega_c \neq 1$ then we shall simply replace s by s/Ω_c.

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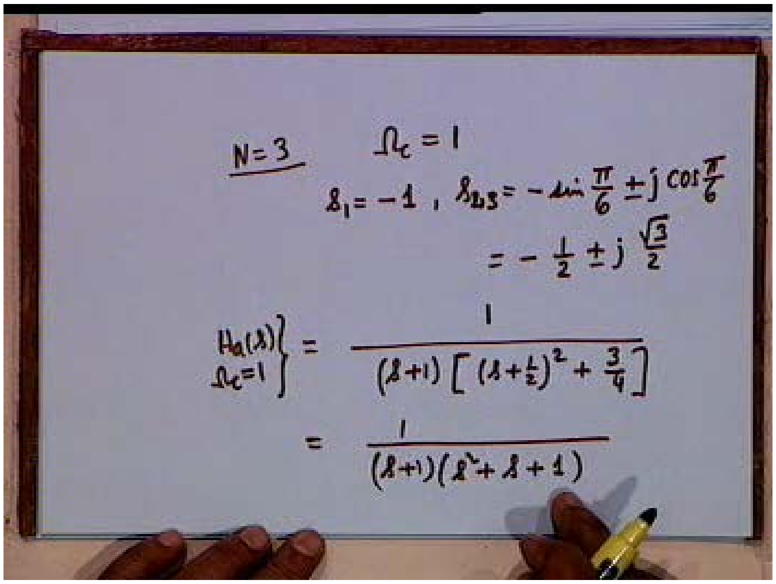

If $\Omega_c = 1$ and N = 3, where would be the poles? Obviously, one is at − 1 and the other two are $s_{2,3} = -\sin(\pi/6) \pm j\cos(\pi/6)$, that is at $-1/2 \pm j\sqrt{3}/2$ and therefore $H_a(s)$ shall be equal to $1/[(s + 1)((s + ½)^2 + ¾)]$. There are two factors: $s + (1/2) + j(\sqrt{3}/2)$ and the other factor is $s + (½) - j(\sqrt{3}/2)$, the product of which is $s^2 + s + 1$. The third order Butterworth transfer function is therefore $1/[(s + 1)(s^2 + s + 1)] = 1/(s^3 + 2s^2 + 2s + 1)$.

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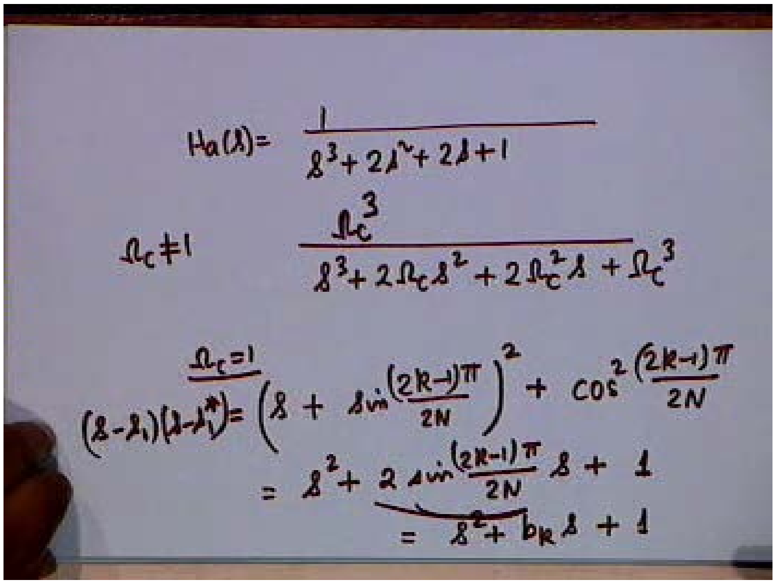

If $\Omega_c \neq 1$ then we have to modify the denominator of $H_a(s)$ as $(s^3 + 2\Omega_c\, s^2 + 2\Omega_c^{\,2}\, s + \Omega_c^{\,3})$ while its numerator would be $(\Omega_c)^3$, that is, we have to replace s by s/Ω_c. Now this should convince you that if the order of the Butterworth filter is odd, then there must a real pole at $s = -\,\Omega_c$. All other poles are complex. On the other hand, if N is even, then all poles are complex conjugates, there are no real poles. In case the poles are complex conjugate and let us say for simplicity $\Omega_c = 1$, then two complex poles shall give rise to a quadratic, as the example of the third order case shows. In general, the quadratic polynomial would be $\{s + \sin [(2k - 1)\pi/(2N)]\}^2 + \cos^2[2(k - 1)\pi/(2N)] = s^2 + 2\sin[(2k - 1)\pi/(2N)]\, s + 1$. Therefore one need not even find the location of the poles; if one knows the order and Ω_c, one may write the transfer function directly. All that you have to compute is this factor $2 \sin [(2k - 1)\pi/(2N)]$. Let us give the name b_k to this; then the quadratic factor is $s^2 + b_k\, s + 1$.

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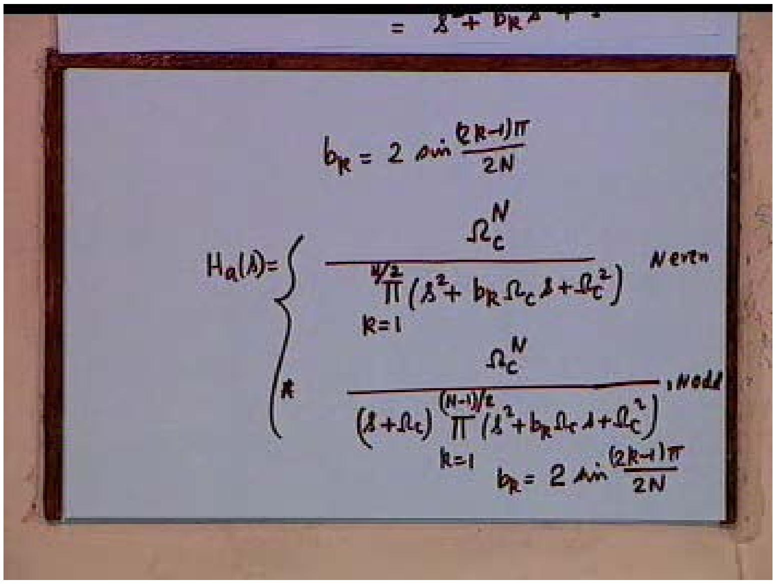

We must not lose sight of Ω_c, as Ω_c in all practical cases would not be one radian per second; it will be something else, may be 500 π or 100 π radians/sec, depending on your 3dB frequency. So, in general $H_a(s)$ would be of the form Ω_c^N divided by the continued product of $(s^2 + b_k \Omega_c s + \Omega_c^2)$, where k shall go from 1 to N/2, if N is even. There should be N number of complex poles and complex poles occur in conjugate pairs. There are only N/2 conjugate pairs, so the continued product is from k = 1 to N/2. On the other hand, if N is odd, then the denominator shall be $(s + \Omega_c)$ multiplied by the continued product of $(s^2 + b_k \Omega_c s + \Omega_c^2)$, where now k will go from 1 to (N – 1)/2. The numerator shall still be Ω_c^N. You see how simple it is to design a Butterworth filter, once you know Ω_c. You just have to compute the b_ks; then you get the transfer function directly. You do not have to compute the poles, only the sine factors.

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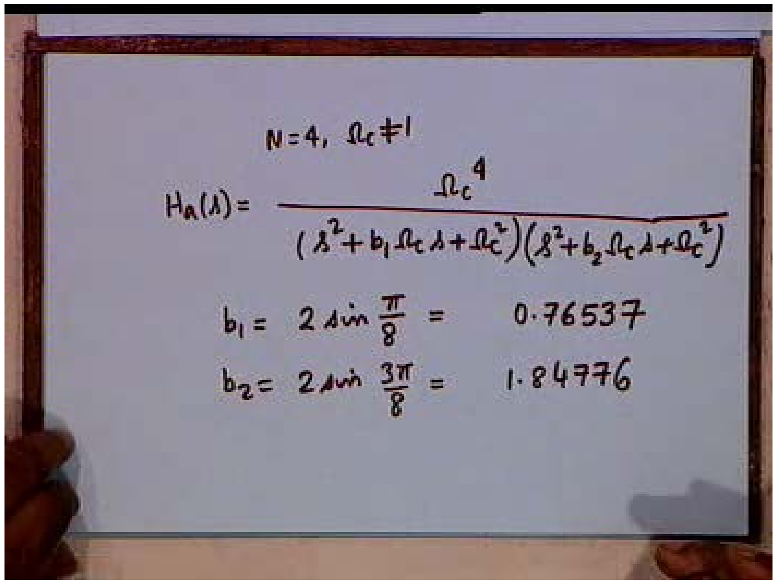

Let us take an example of $N = 4$; also let $\Omega_c \neq 1$. We can directly write $H_a(s)$ as Ω_c^4 divided by the two factors $(s^2 + b_1 \Omega_c s + \Omega_c^2)(s^2 + b_2 \Omega_c s + \Omega_c^2)$ where $b_1 = 2\sin \pi/8$ and this comes out as 0.76537. If this filter is going to be used for design an IIR digital filter, then you better take care of these constants to as many decimal places as you can. Do not make truncations unnecessarily. Quantization error adds further to the errors and may make the filter unacceptable. b_2 shall be twice sine $3\pi/8$, and this comes as 1.8476. So you know the total transfer function.

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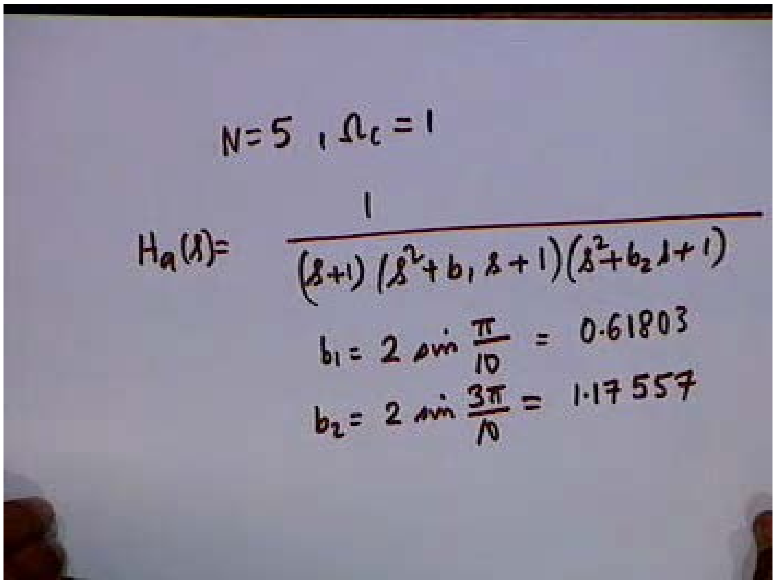

If N = 5 and $\Omega_c = 1$ then you can write the transfer function directly as $1/[(s + 1) (s^2 + b_1 s + 1) (s^2 + b_2 s + 1)]$. I am going back and forth between Ω_c and 1 to familiarize you with the actual procedure. Here $b_1 = 2 \sin (\pi/10)$. Once you get some experience with this you can even forget $2(k - 1) \pi/(2N)$. b_1 comes out as 0.61803 and b_2 is twice sine $(3\pi/10)$, and this comes as 1.17557. How does one determine the order N? The two things that are needed for Butterworth filter are Ω_c and N.

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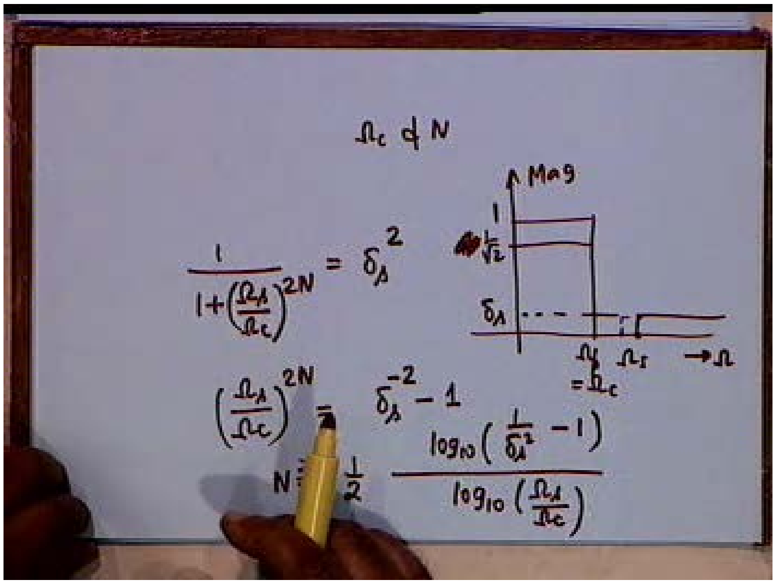

Suppose Ω_c is given, you have to find the order. How do you do that? You look at Ω_s and the stop band tolerance Δs. Then I get $(\Omega_s/\Omega_c)^{2N} = \Delta s^{-2} - 1$. This gives N = half $\log_{10}((1/\Delta s^{-2}) - 1)/\log_{10}(\Omega_s/\Omega_c)$.

Now there is no guarantee that this quantity on the right shall be an integer whereas N is required to be an integer and therefore we shall choose an integer which is higher than this, but closest to this. If this comes out as 6.7 then you have to use 7. Do not go for 8 because that unnecessarily increases the cost of the filter. So you choose N greater than or equal to this. If N is greater than the right hand side, then obviously the actual Ω_s that you shall realize will be less than the specified Ω_s. In other words, the stop band is shifted to the left and you have over–satisfied the specs, which is a welcome feature. But you must not play with the pass band; pass band is sacred and it cannot be touched. Now this is a situation where Ω_c was specified.

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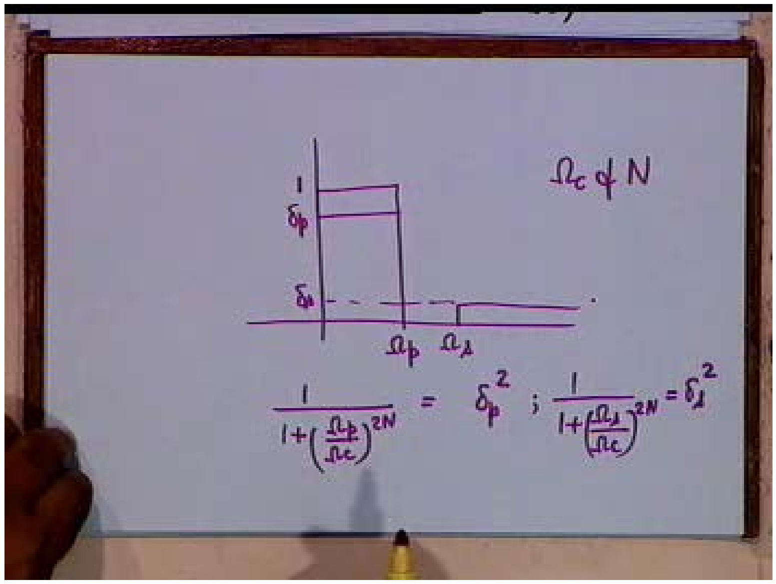

In general Ω_c shall not be specified; what shall be specified is Δp, the minimum possible value in the pass band extending from 0 to Ω_p; Δp is not the tolerance in the pass band; tolerance in the pass band is $1 - \Delta p$. So how do you determine N? Now you have to determine two things Ω_c and N. If the tolerance is 3dB, then you know Δp is 0.707 and Ω_p is the same as Ω_c. But, for general specification, you write the two conditions that is $1 + (\Omega_p/\Omega_c)^{2N}$ is equal to Δp^{-2} and $1 + (\Omega_s/\Omega_c)^{2N} = \Delta s^{-2}$.

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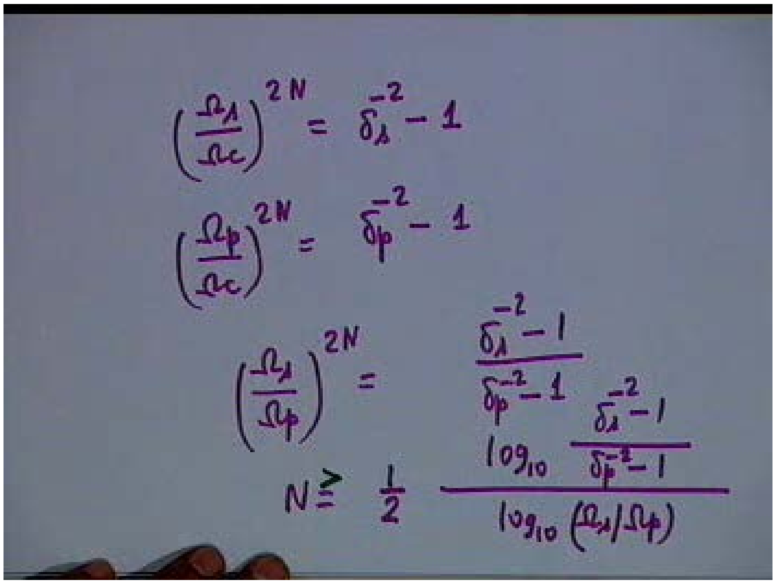

Now, $(\Omega_s/\Omega_c)^{2N}$ is $\Delta s^{-2} - 1$. In a similar manner $(\Omega_p/\Omega_c)^{2N}$ shall be $\Delta p^{-2} - 1$. And now we take the ratio and get $(\Omega_s/\Omega_p)^{2N} = (\Delta s^{-2} - 1)/(\Delta p^{-2} - 1)$. And therefore N shall be $1/2 \log_{10} [(\Delta s^{-2} - 1)/(\Delta p^{-2} - 1)]/\log_{10} (\Omega_s/\Omega_p)$. This is the general formula for determining the order of the Butterworth filter. And in order to emphasize that N has to be an integer, N should be greater than or equal to this. Once you obtain N, the next step is to obtain Ω_c. There are two equations which compete with each other for determining Ω_c; which one should we use? We should use the formula with Ω_p because the pass band is sacred. I must satisfy the pass band tolerance exactly. I can over satisfy the stop band tolerance. So whatever value of N you have obtained, you substitute this in $1/[1 + (\Omega_p/\Omega_c)]^{2N} = \Delta p^2$ and from this you find Ω_c.

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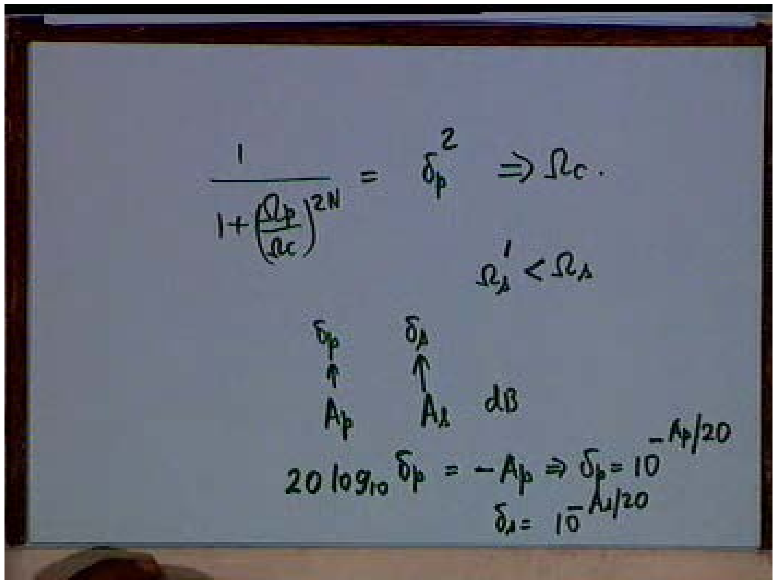

Then if N is greater than half of ratio of two logs, then the realized Ω_s shall be less than the specified Ω_s. In other words, you are bringing the stop band in the transition band and that is a welcome feature. Now Δp and Δs are usually specified in decibels. If they are specified in decibels, then you have to convert them into ratios. Suppose Δp corresponds to A_p decibels, and Δs corresponds to A_s in decibels, then 20 $\log_{10} \Delta p = - A_p$ because 1 corresponds to 0 decibel, anything below that must be negative and therefore $\Delta p = 10^{-A_p/20}$. In a similar manner $\Delta s = 10^{-A_s/20}$. So you shall have to convert the decibel specifications into ratios and then find out N.