Digital Signal Processing

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Lecture - 22

Problem Solving Session: FT, DFT and Z-Transforms

This is the 22nd lecture and we continue our problem solving session on Fourier Transforms, Discrete Fourier Transforms and z Transform.

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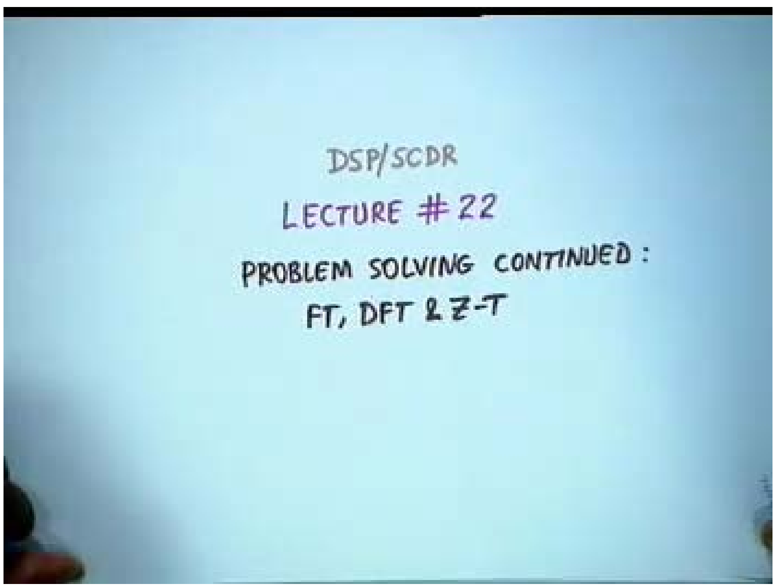

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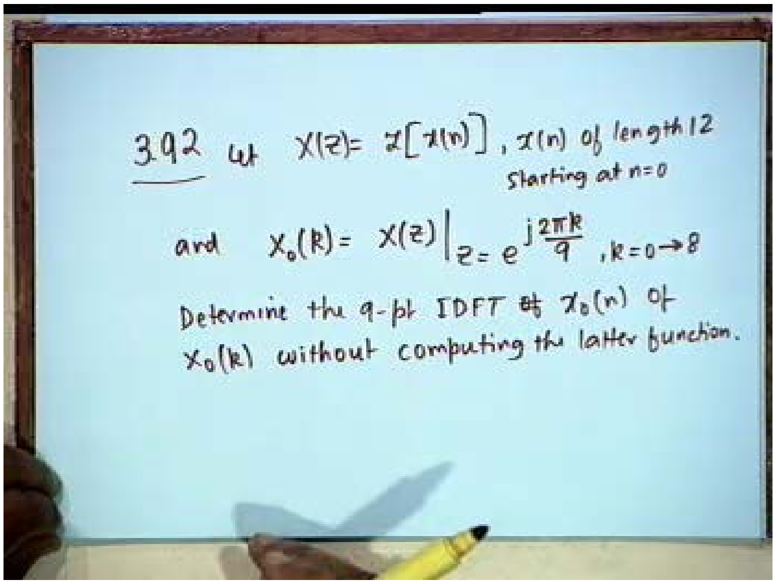

The problem that we start with is 3.92 and the statement of the problem is: let X(z) = z transform of x(n) where x(n) is of length 12 starting at n = 0 that is we have $x_{0,}$ $x_{1,}$ up to x_{11}. Let $X_0(k)$ = X(z) with z = $e^{j2\pi k/9}$ where k = 0 → 8; so what does this mean? This means a 9 point DFT of a sequence whose length is 12. First you have taken the z transform, then the z transform is sampled on the unit circle at 9 points.

The problem is to determine the 9 point IDFT $x_0(n)$ of $X_0(k)$ that is, invert this DFT but the constraint is you must not compute $X_0(k)$. Why we did not write X(k) here? Why did we use a subscript 0? It is because X(k) would be the same as X(z) with z replaced by k whereas $X_0(k)$ is the DFT, a different function altogether. You must distinguish between them. Many textbooks do not distinguish this and they make a fundamental mistake. To understand the problem, we have taken X(z) as the z transform of a finite length sequence and then we have taken DFT for the same sequence but we have not computed the DFT. What we have done is that we have sampled X(z) on the unit circle at 9 points; so this is a 9 point DFT. You now have to invert the DFT. Naturally you shall not get x(n) because this was 12 point and this DFT is 9 point. So you get a different $x_0(n)$ but you must do this without computing $X_0(k)$ which means that we have to appeal to the basics. The basics are X(z) = summation x(n) z^{-n} and n = 0 to 11. So $X_0(k)$ = summation (n = 0 to 11) x(n) $e^{-j2\pi kn/9}$.

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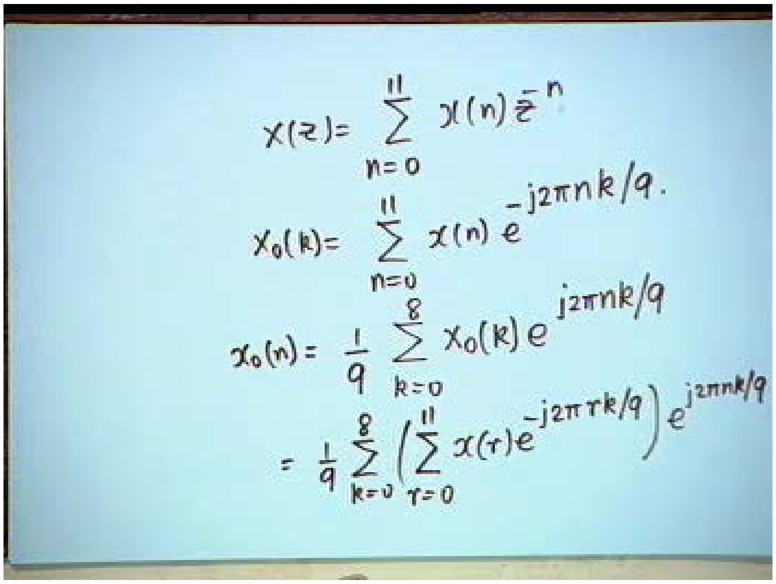

The limits remain the same because x(n) remains the same and only z changes. Here z^{-n} becomes $e^{-j2\pi nk/9}$. I have to compute $x_0(n)$. I am not allowed to compute $X_0(k)$ so I will not compute it. Let us see what is IDFT? IDFT = (1/9) $\sum_{k=0}^{8} X_0(k)\, e^{j2\pi nk/9}$. I have not computed $X_0(k)$ but I can substitute this relationship. So I substitute but then in substitution you must change the dummy variable otherwise you run into problems. So I write this as (1/9) $\sum_{k=0}^{8}$ [∑ (r = 0 to 11) x(r) $e^{-j2\pi rk/9}$] × $e^{j2\pi nk/9}$. Then we interchange the two summations.

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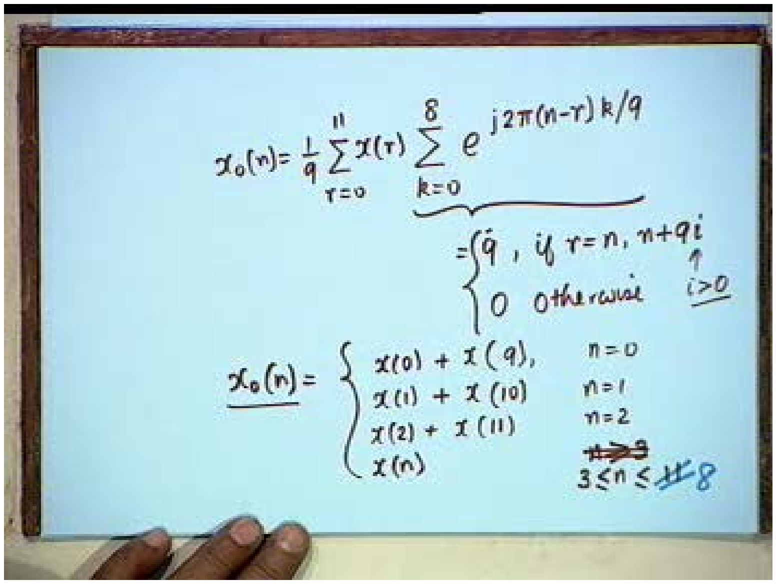

So we write $x_0(n) = (1/9) \sum_{r=0}^{11} x(r) \sum_{k=0}^{8} e^{j2\pi(n-r)k/9.}$ And the last summation, as you know, is equal to 9 if r = n. Not only that, if n is increased by 9, it should not matter; so n can be replaced by n + 9i where i is an integer. And if you ignore this, you will run into problems; you cannot compute all the $x_0(n)$. In this summation, I shall have only r = n and r = n + 9i; other terms will vanish. If I put n = 0 then r = 0 gives x(0) and I shall have another term x(9). For n = 1, I shall have x(1) + x(10), then x(2) + x(11) for n = 2. I do not have x(12), so $x_0(n)$ will be x(n) for $3 \leq n \leq 8$. This is the total solution.

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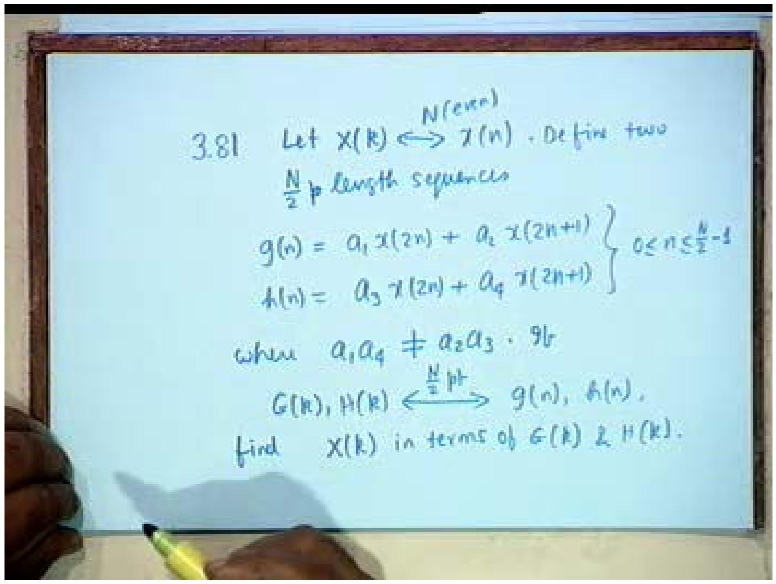

The next problem is 3.81. It says: let X(k) be the N point DFT of x(n) where N is even. Then it says define two N/2 length sequences: g(n) = a_1x(2n) + a_2x(2n + 1); h(n) = a_3x(2n) + a_4x(2n + 1); where obviously n lies between 0 and (N/2) – 1. It is given that a_1 a_4 ≠ a_2 a_3. So given a sequence x(n), you have taken the even numbered samples to form a sequence x(2n) and odd numbered samples to form a sequence x(2n + 1) and then you make the linear combination in two ways with constants a_1, a_2, a_3 and a_4. So how many points are there in g(n)? There are N/2 points; h(n) also has N/2 points.

If you have understood this, then you will understand next part also. If G(k) and H(k) are the N/2 point DFTs of g(n) and h(n), find X(k) in terms of G(k) and H(k). Now, is the problem understood? To say it again, N is always even, we have defined two N/2 point sequences by linear combinations of even numbered samples and odd numbered samples in two different ways and we have taken the DFT of each. Can we find out the DFT of the original sequence, the mother sequence, from which these 2 children are derived, in terms of these two N/2 point DFTs? There are many ways of solving this problem. First thing we do is to express x(2n) and x(2n + 1) in terms of g and h and that is where this inequality is useful. It is a set of two linear equations so in the denominator of the solutions, we shall have a_1 a_4 – a_2 a_3 that cannot be 0. If it is 0 then the solution breaks down. So, we solve for them. We will not compute DFTs, but we will compute the z transforms. Finally we shall compute them on the unit circle with the required number of points. This is one way of solving the problem by keeping the algebra a little more tractable. We will work in terms of z transform.

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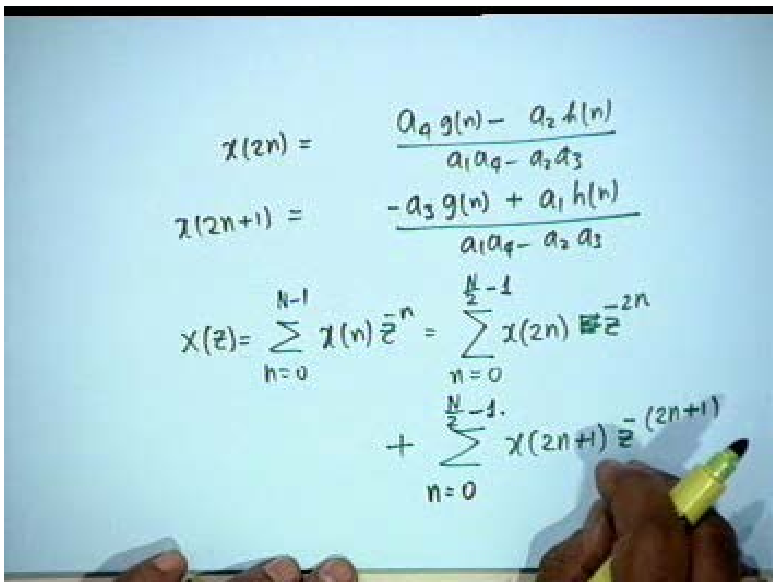

The solutions for x(2n) and x(2n + 1) from these two derived sequences become $[a_4 g(n) – a_2 h(n)]/(a_1\ a_4 – a_2\ a_3)$ and $[– a_3\ g(n) + a_1 h(n)]/(a_1\ a_4 – a_2\ a_3)$. Using the definition $X(z) = \sum^{N-1}_{n=0} x(n)\ z^{-n}$ we have X(z) = Z[x(2n)] + Z[x(2n+1)], where $Z[x(2n)] = \sum^{(N/2)–1}_{n=0} x(2n)\ z^{–2n}$ and $Z[x(2n+1)] = \Sigma\ x(2n + 1)\ z^{–(2n+1)}$ (n = 0 to ((N/2) – 1)). In this relation we substitute for x(2n) and x(2n + 1) in terms of g and h. Thus X(z) is split into two summations, one containing x(2n) and another containing x(2n + 1). The result for the first summation is $\sum^{(N/2)-1}_{n=0} \{[a_4\ g(n) – a_2\ h(n)]/(a_1\ a_4 – a_2\ a_3)\}\ z^{–2n}$.

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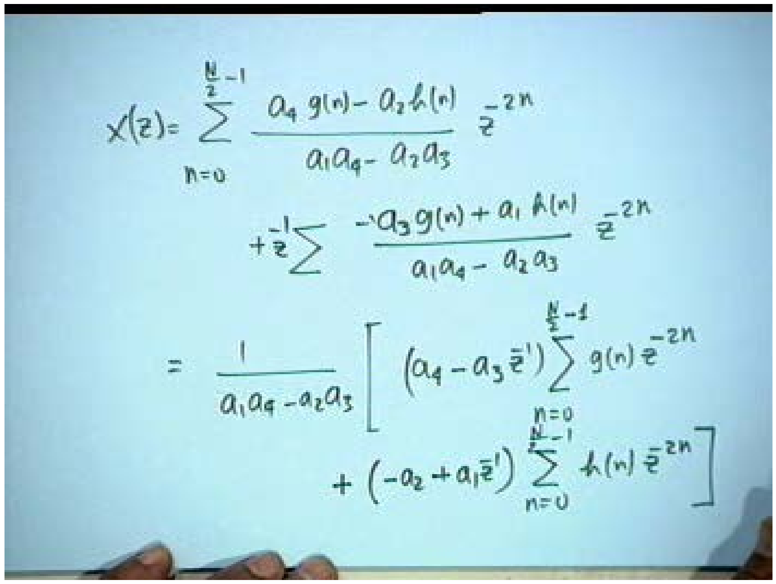

In the second summation, I can take z^{-1} out and write $\Sigma(n = 0 \text{ to } ((N/2) - 1))\ \{(-a_3 g(n) + a_1 h(n))/(a_1 a_4 - a_2 a_3)\}\ z^{-2n}$.

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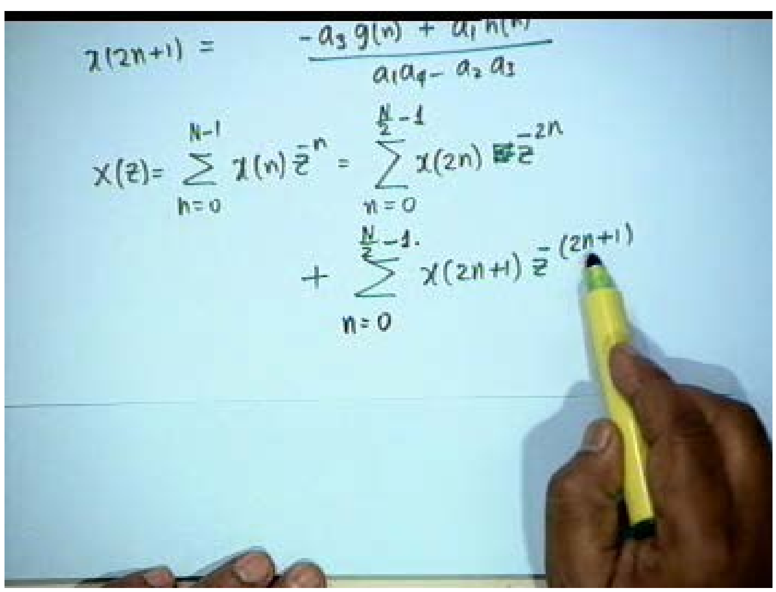

Now I put g's together and h's together and then put $z = e^{j2\pi k/N}$. When you have written up to this, the solution is clear and the rest is simply algebra. This is what we get: we take out the factor $1/(a_1 a_4 - a_2 a_3)$ and then I get $(a_4 - a_3 z^{-1}) \sum_{n=0}^{(N/2)-1} g(n) z^{-2n} + (-a_2 + a_1 z^{-1}) \sum_{n=0}^{(N/2)-1} h(n) z^{-2n}$. Now we put $z = e^{j2\pi k/N}$ because X has to be computed with N number of points.

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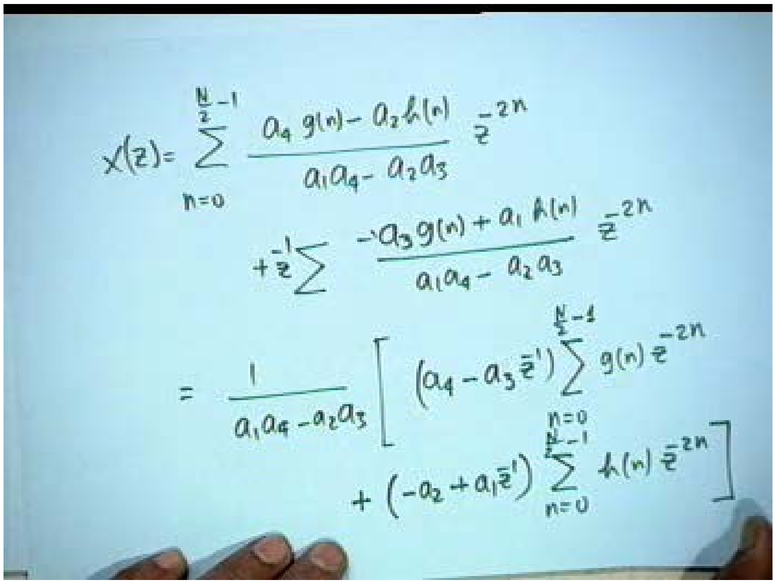

Then $z^2 = e^{j2\pi k/(N/2)}$ which is equal to $(W_{N/2})^{-k}$. Thus $X(z) = [1/(a_1a_4 - a_2a_3)]\ [(a_4 - a_3 W_N^k]$, summation $g(n)\ (W_{N/2})^{nk}$ (where n goes from 0 to ((N/2) – 1) $+(- a_2 + a_1\ W_N^k) \sum_{n=0}^{(N/2)-1} h(n) (W_{N/2})^{nk}$. It is $[1/(a_1 a_4 - a_2a_3)]\ [(a_4 - a_3\ W_N^k)\ G(k) + (- a_2 + a_1\ W_N^k)\ H(k)]$.

Now there is a problem here. In the formulation of the problem itself, we should have distinguished between them. Let us change X(z) to $X_1(z)$, and let $X_1(e^{j2\pi k/N}) = X(k)$. This was a notational mistake. Still more on this, we can compute G(k) and H(k) only from 0 to (N/2) – 1 whereas X(k) has to be computed from 0 to N – 1; so how do we do that? When k exceeds N/2, since G(k) and H(k) are periodic, we must put here G(k) modulo N/2. Similarly for H(k). Then you can go from k = 0 to N – 1.

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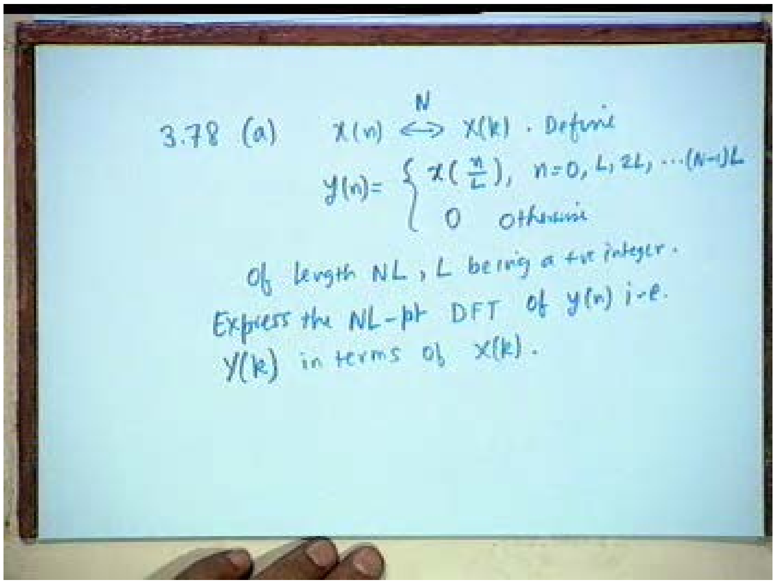

The next problem also has similar complication. This is problem 3.78: I will only solve part (a). It says: consider a sequence x(n) of length N which has the DFT X(k). Then define a sequence y(n) which is the up sampled version of x(n) that is x(n/L), n = 0, L, 2L,… (N – 1)L. The reason we did not take ± in values of n is that x(n) is a finite sequence, one sided, starts at 0 and ends at N – 1 and is 0 otherwise. The up sampled version y(n) is now of length N × L where L is a positive integer. We had N number of points in x(n) so when we up sample it we are adding N – 1 zeros in between, so total length of y(n) becomes N × L. The question is to express the NL point DFT of y(n), that is Y(k), in terms of X(k). We start from the definition of Y(k).

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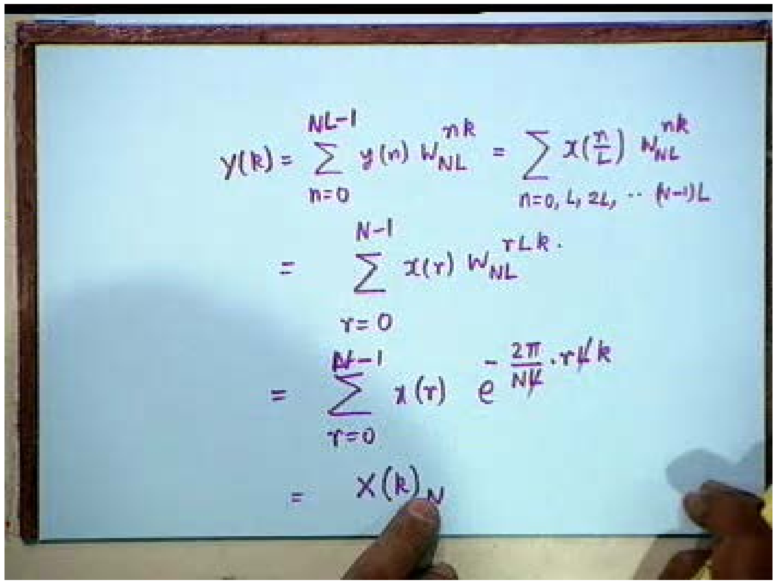

So $Y(k) = \Sigma^{NL-1}_{n=0}\ y(n)\ W_{NL}^{nk}$. Now put y(n) = x(n/L) where now in the summation n = 0, L, 2L and so on up to (N – 1)L. It is important to recognize that W_{NL}^{nk} does not change as we are only substituting for y(n). Now let n/L = r; then r should go from 0 to N – 1. The summation becomes that of $x(r)\ (W)_{NL}^{rLK}$ with limits 0 and N – 1 i.e. $Y(k) = \Sigma^{N-1}_{r=0}\ x(r)\ W_{NL}^{rLk}$. Now $W_{NL}^{rLk} = e^{-[(j2\pi/(NL)]rLK}$; so L and L cancel. Recognize that this is precisely X(k), but there is a problem. Do not leave it here because Y(k) exists from k = 0 to NL – 1 whereas X(k) exists only from 0 to N – 1. Now X(k) will repeat, the DFT of Y(k) up to N points is identical with X(k); then it shall repeat L times.

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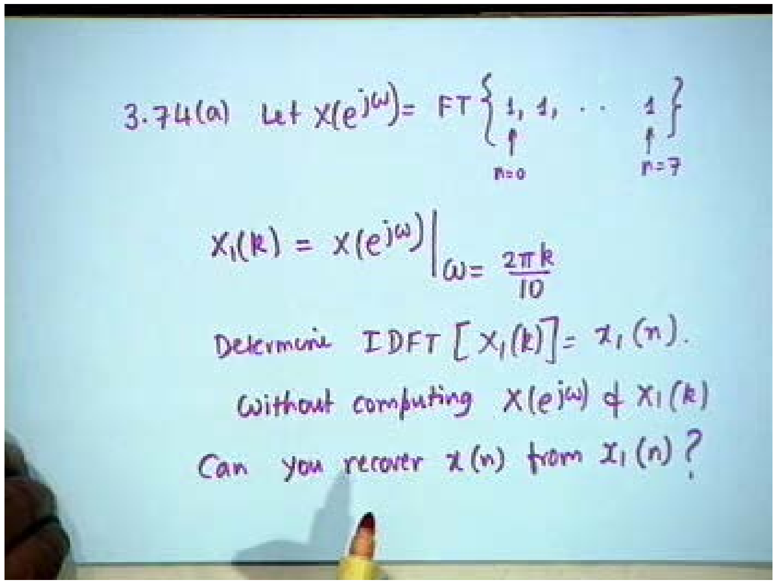

The next problem is 3.74 (a). Once again I am solving part (a) and part (b) is the negative of this. It says let X $(e^{j\omega})$ be equal to the Fourier Transform of the sequence {1, 1, …1} where the first sample is at n = 0 and the last one is at n = 7. In other words, we have 8 points in x(n) and its Fourier Transform is $X(e^{j\omega})$. Now $X(e^{j\omega})$ is taken and sampled at 10 points. In other words, $X_1(k)$ is obtained from $X(e^{j\omega})$ by putting $\omega = 2\pi k/10$. The number of points in x(n) is 8 and the number of points at which the DFT is computed is 10. If they are identical there is no problem but they are different. We want to see the effect of this.

$X_1(k)$ is obtained by sampling $X(e^{j\omega})$ at intervals of $\pi/5$; so the number of points is $2\pi/(\pi/5) = 10$, starting from $\omega = 0$. Determine the IDFT of $X_1(k)$, call this $x_1(n)$ which we have to find without computing $X(e^{j\omega})$ and $X_1(k)$; that is the restriction. In other words, you are required to find $x_1(n)$ from x(n) without explicitly computing the Fourier Transform or the Discrete Fourier Transform. And then there is an appendage: can you recover x(n) from $x_1(n)$? The answer in this case is yes but the answer in part (b) is no. How do we proceed? We start from $X(e^{j\omega}) = \Sigma\, x(n)\, e^{-j\omega n}$; n = 0 to 7. Also, $X_1(k) = \sum^{7}_{n=0} x(n)\, e^{-j2\pi nk/10}$: You are not allowed to compute this, so we will invert this. So we get: $x_1(n) = (1/10) \sum^{9}_{k=0} X_1(k)\, e^{j2\pi nk/10}$. Whenever there is the question of double summation, you must change one of the variables. $x_1(n)$ is therefore $(1/10) \sum^{9}_{k=0}\sum^{7}_{r=0} x(r)\, e^{-}$

$^{j2\pi rk/10} \times e^{j2\pi nk/10}$. Now you interchange the two summations. k and r are two independent dummy variables so you can interchange the summations. And you get $x_1(n) = (1/10) \sum_{r=0}^{7} x(r) \sum_{k=0}^{9} e^{j2\pi(n-r)k/10.}$ The last summation would be = 10 for r = n or n + 10i, i being a positive integer; it is zero otherwise. Thus $x_1(n) = x(n)$, n = 0 to 7; beyond n = 7, x(n) does not exist, hence $x_1(n) = 0$, n = 8 and 9.

There is no aliasing here. Aliasing will occur if the number of points at which the DFT is computed is less than the number of points in the original sample. So you know the answer, that is x(n) can be recovered from $x_1(n)$. From the z transform if you go to the DFT with a larger number of points then there is no problem as 0 padding has occurred. If you go to a smaller number of points then aliasing shall occur in the time domain.

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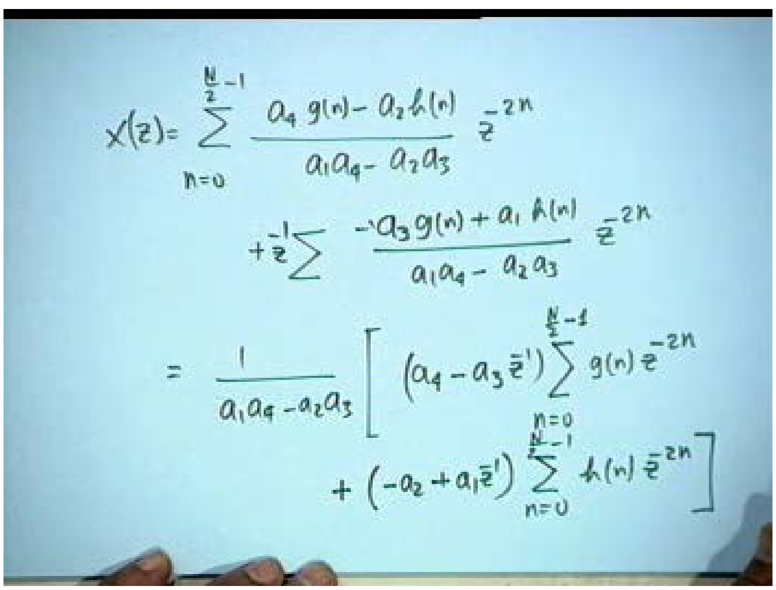

Now let us work out problem 3.68. It is a problem in similar lines and illustrates the fact that DFT is a versatile instrument. DSP designers use DFT wherever possible and therefore all kinds of problems arise. Now look at this: let x(n) and X(k) be N point pairs in DFT. With N divisible by 4, let y(n) = x(4n) which is a decimator that is you ignore the intermediate three samples, you take x(0), x(4) etc and you construct a new sequence x(4n) where obviously the length now shall be reduced by a factor of 4. So $0 \le n \le = (N/4) - 1$ and that is why N was given as divisible by 4. The question is to express the N/4 point DFT Y(k) in terms of X(k). The number of points at which the DFT is taken is less than the number of points in x(n) so there shall be aliasing. Let us see how to solve this. So y(n) can always be computed from Y(k). Can you compute x(n) from Y(k)? This cannot be done.

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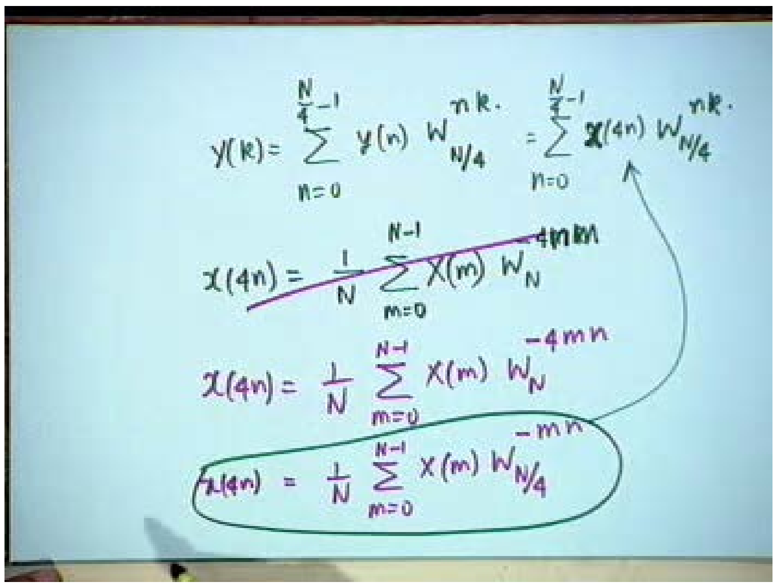

Y(k) by definition is summation (n = 0 to ((N/4) – 1)) y(n) $W_{N/4}^{nk}$ and I substitute for y(n), which is x(4n). Unlike up sampler where there was a division there is no problem here as n is an integer and 4n is also an integer. Now we will compute x(4n) in terms of X(k). I compute by inverting X(k); x(4n) would be (1/N) ∑ (m = 0 to N – 1) X(m)W_N^{-4mk}. I can rewrite this as (1/N) $\sum_{m=0}^{N-1}$ X(m)$W_{N/4}^{-mk}$ and I now substitute for X(m) and I get a double summation; we apply the same trick, that is, interchange the summations.

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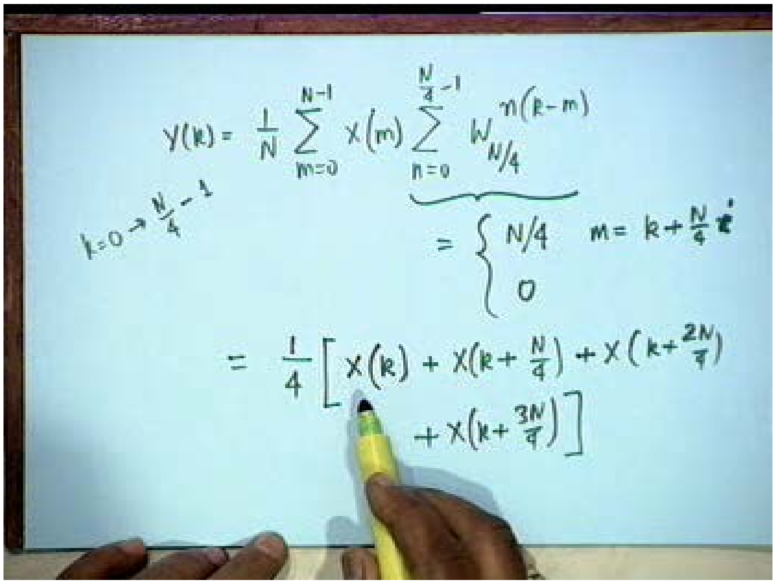

After the interchange, I therefore have Y(k) = (1/N) $\sum_{m=0}^{N-1}$ X(m) $\sum_{n=0}^{(N/4)-1} W_{N/4}^{n(k-m)}$. The number of terms in the second summation is N/4; so for m = k + N/4 × some integer i, the summation equals N/4; otherwise it is zero. So Y(k) would be (1/4) multiplied by the sum X(k + N/4) + X(k + 2N/4) + X(k + 3N/4). We end here because the next one is simply X(k + N) = X(k), X(k) being periodic. Therefore Y(k) becomes a sum of several versions of X(k).

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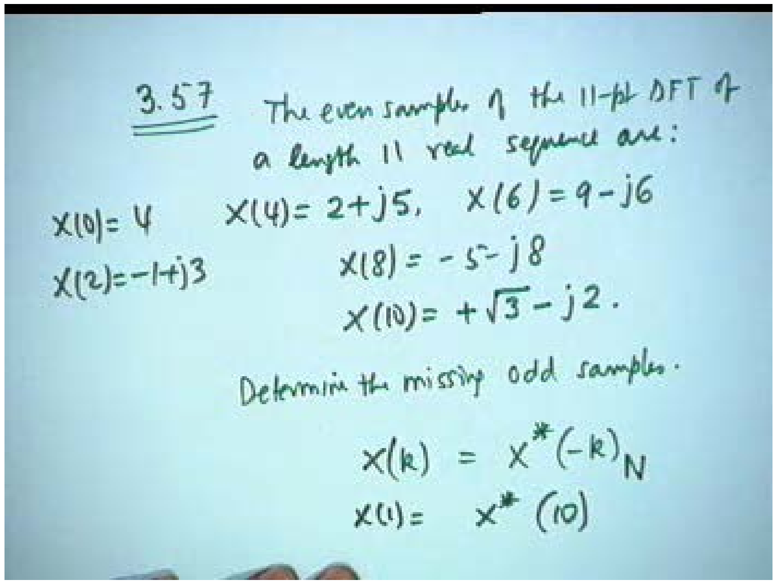

The last problem is 3.57. It is a tricky problem but the solution is very simple. It says the even samples of the 11 point DFT of a length 11 real sequence are X(0) = 4; X(2) = – 1 + j3; X(4) = 2 + j5; X(6) = 9 – j6; X(8) = – 5 – j8; and X(10) = [$\sqrt{3}$ – j2]. It simply asks you to determine the missing odd samples. Nothing else is given. Only the even numbered samples are given. How many? Six of them are given. The remaining five have to be found. We know that for a real sequence x(n), X(k) is X*(– k) modulo N. So X(1) for example would be X*(10) which would be $\sqrt{3}$ + j2. Similarly for others.