Digital Signal Processing

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Lecture - 19

Test for Stability using All Pass Functions

This is the 19th lecture on DSP and the topic for today is Test for stability using all pass functions.

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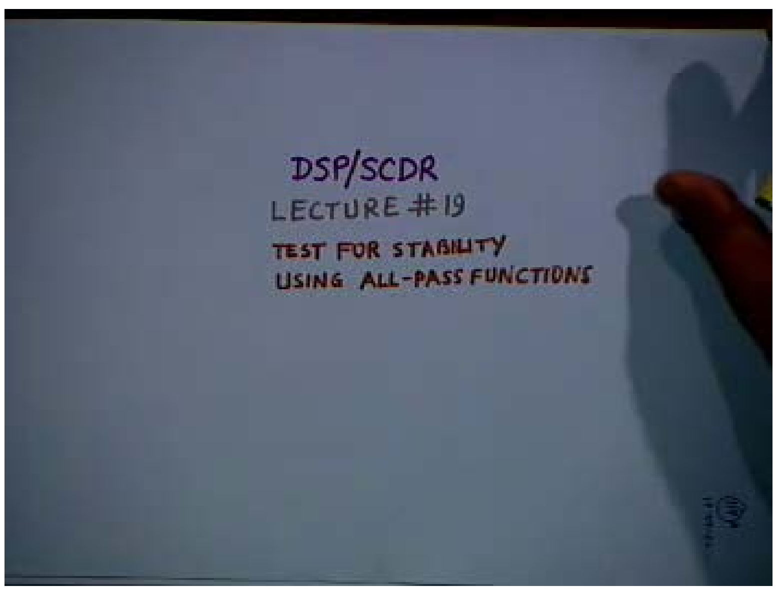

We shall test for stability for an arbitrary IIR transfer function H(z); all pass functions will be a via media. This again shows the importance of all pass functions in DSP. In the last lecture, we showed that the two transfer functions $H_0(z)$ and $H_1(z)$, which are half of the sum and difference of two all pass functions, form an extremely important set of all pass complementary as well as power complementary functions.

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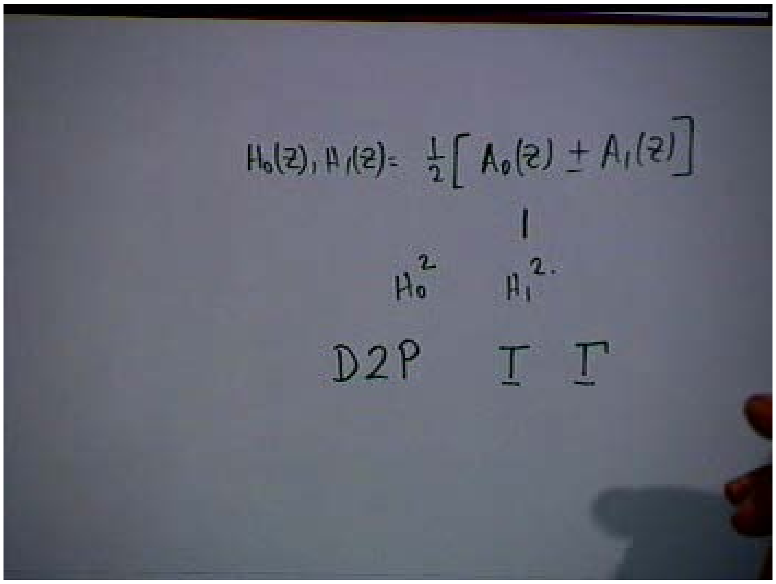

We defined a frequency, called the cross over frequency, at which both H_0 and H_1 have a magnitude equal to $1/\sqrt{2}$. Then we showed that a very simple realization is possible; all that you have to realize are two all pass functions. We took a simple and interesting example of $A_0 = 1$ and A_1 = a first order all pass, which gives a very versatile realization of several kinds of filters and also a magnitude complementary pair by taking H_0^2 and H_1^2. We introduced the concept of digital two pairs and we showed how they can be represented by a transmission matrix $\underline{T}$ and a chain matrix $\underline{\Gamma}$. We showed how cascades can be of two different types: Transmission Cascade and Chain Cascade in which the respective parameters multiply to get the overall transmission or chain parameters. Finally, we started stability testing and we took a second order transfer function $H(z) = N(z)/D(z)$ where $D(z)$ is $1 + d_1z^{-1} + d_2z^{-2}$.

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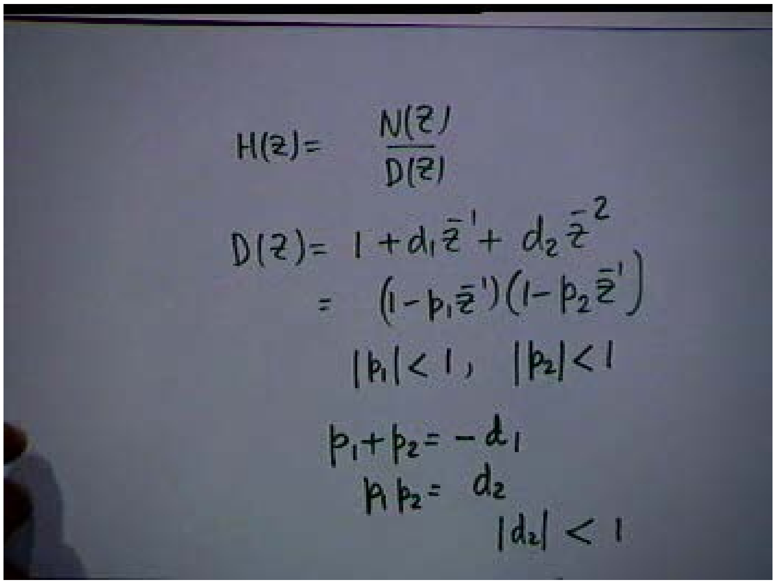

We want to test the stability without determining the roots of the denominator D(z), which are the poles of H(z). Now if I write D(z) as $(1 - p_1z^{-1}) \times (1 - p_2z^{-1})$, then p_1 and p_2 are the poles, and we require, for stability, magnitude $p_1 < 1$ and magnitude $p_2 < 1$. Obviously $p_1 + p_2$ shall be $= -d_1$ and p_1p_2 shall be $= d_2$. Now for stability since p_1 and p_2 both have to be less than 1 in magnitude, therefore the magnitude of d_2 should be less than 1 because the magnitude of d_2 is simply the product of the magnitudes of p_1 and p_2. In other words, in a parameter plane, that is if we plot d_2 versus d_1, then magnitude $d_2 < 1$ means d_2 should be confined between – 1 and + 1.

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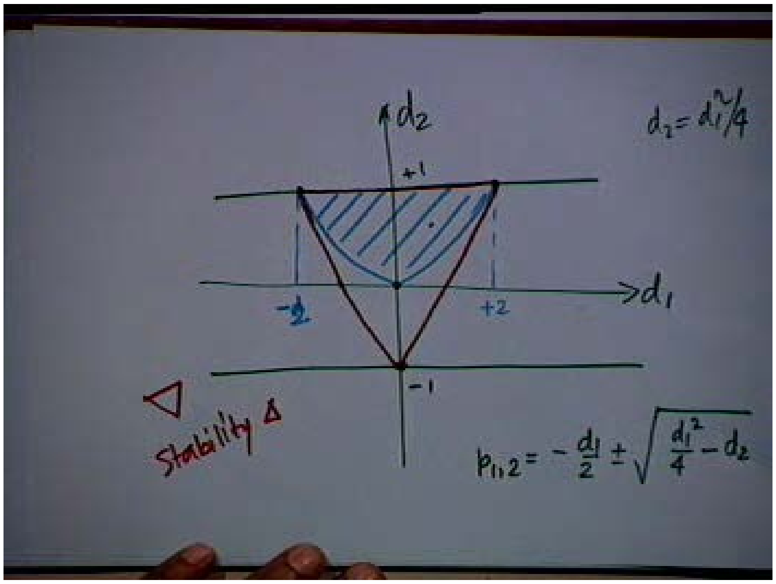

Now there are two possibilities: the poles can be real or poles can be complex. In either case $p_{1\ 2}$ shall be = – $(d_1 / 2) \pm \sqrt{d_1^2\ / 4\ -\ d_2}$. If the roots are complex, then obviously $d_1^2/4$ should be less than d_2. First let us examine complex roots; the condition is that d_2 should be greater than $d_1^2/4$.

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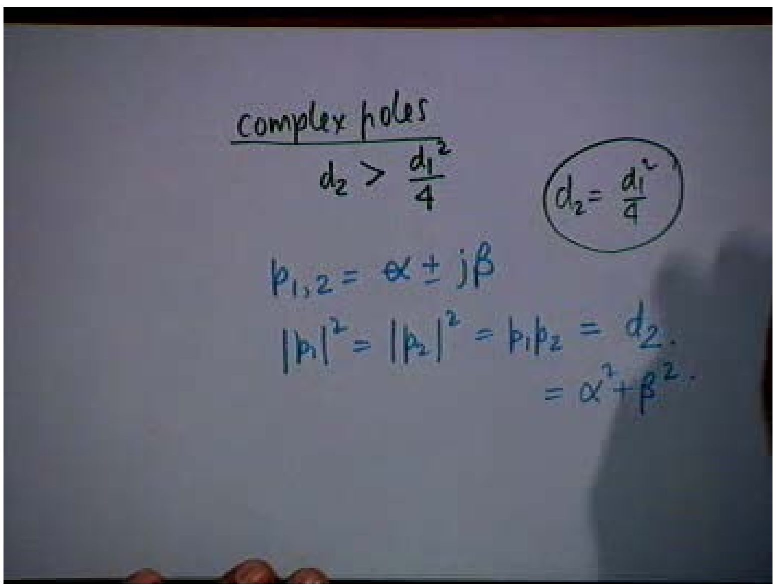

In other words, if I plot the curve with an equality sign that is, $d_2 = d_1^2/4$ then d_2 should be above this curve. Not only that, since it is d_1^2 and d_1 is a real quantity, positive or negative, d_2 being greater than $d_1^2/4$, should therefore be greater than or equal to 0. Now $d_2 = d_1^2/4$ is a parabola. When d_1 is 0 d_2 is 0; when d_1 is + 2 or – 2 it is equal to + 1; these constitute three points, and you can draw the parabola passing through them. Therefore, if the roots are complex, then they should be contained in this cup. The hatched region represents complex poles, that is, if in the parameter plane, the point (d_1, d_2) is within the cup, then you know it is a complex pair of poles. Not only that, if the poles are complex, i.e. $p_{1,2} = \alpha \pm j\beta$, then magnitude p_1^2 shall be equal to magnitude p_2^2; both shall equal $p_1p_2 = d_2 = \alpha^2 + \beta^2$. Now let us examine the case of real poles.

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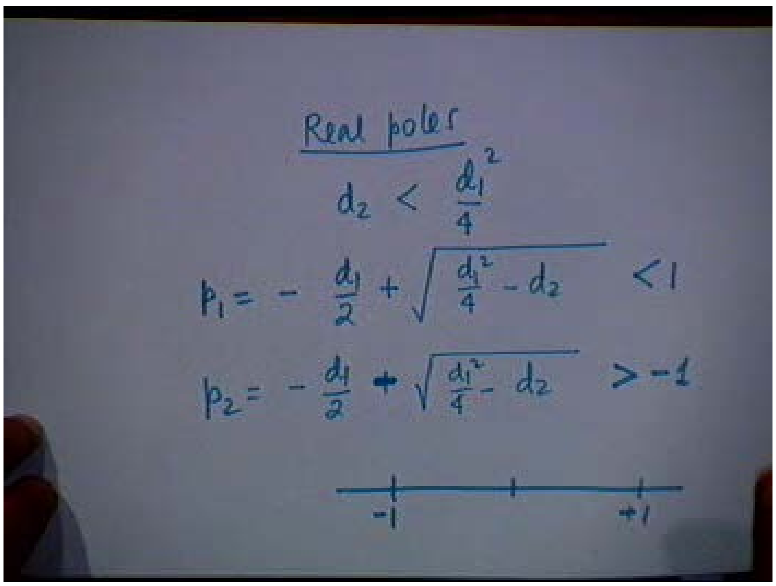

If the poles are real it means that d_2 should be less than $d_1^2/4$. The poles are $p_1 = -d_1/2 + \sqrt{(d_1^2/4 - d_2)}$ and $p_2 = -d_1/2 - \sqrt{(d_1^2/4 - d_2)}$. Since these are real we want their mod to be less than 1 for stability. Which one of these can exceed 1? Obviously, p_1 can exceed 1. In other words, we want p_1 to be less than 1. p_2 can go beyond – 1 because both terms are negative if d_1 is positive. In other words, we want p_2 to be greater than – 1. In any case, p_1 and p_2 should be confined within the region – 1 to + 1 on the d_1 – axis. If we satisfy these two constraints, that is p_1 less than 1 and p_2 greater than – 1, then our job is done. Let us see what these conditions lead to.

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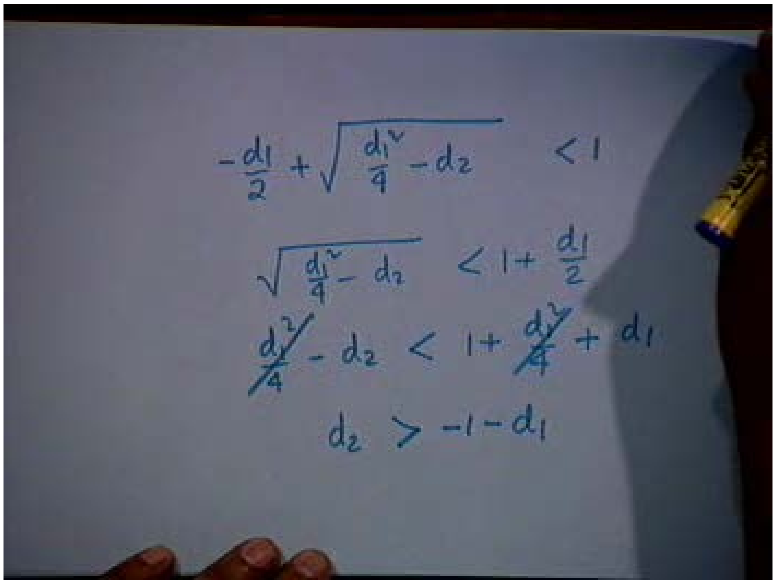

So – $d_1/2$ + $\sqrt{(d_1^2/4 - d_2)}$ should be less than 1 means that $\sqrt{(d_1^2/4 - d_2)}$ should be less than 1 + $d_1/2$. If I square both sides now, then I get $d_1^2/4$ – d_2 should be less than 1 + $d_1^2/4$ + d_1. $d_1^2/4$ cancels from both sides; if I change the sign of d_2 on the left side, we get the condition as d_2 greater than – 1 – d_1. In a similar manner, if I take the other one that is – $d_1/2$ – $\sqrt{(d_1^2/4 - d_2)}$ should be greater than – 1, we get – $\sqrt{(d_1^2/4 - d_2)}$ should be greater than – 1 + $d_1/2$.

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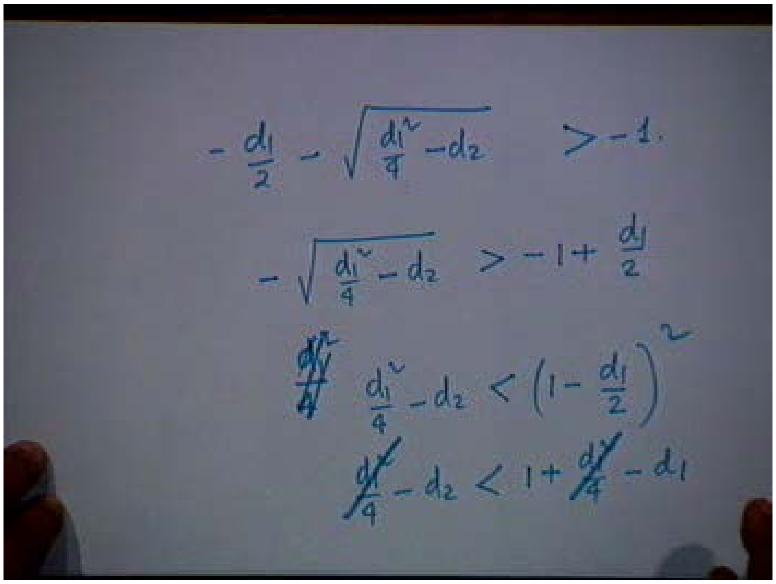

Again, if I square both the sides, then we get the condition that $d_1^2/4 - d_2$ should be less than $(1 - d_1/2)^2$. So I get $d_1^2/4 - d_2$ less than $1 + d_1^2/4 - d_1$ in which $d_1^2/4$ cancels and therefore this condition is that d_2 should be greater than $d_1 - 1$. Now, if I plot this with an equality sign I shall get a straight line. The other condition was d_2 greater than $-1 - d_1$. Once again if I plot this with an equality sign then we shall get a straight line. In this, if d_1 is 0 and then d_2 is -1. If d_1 is -1 then d_2 is 0. Joining these two points will give the straight line $d_2 = -1 - d_1$. In a similar manner, for the other straight line $d_2 = -1 + d_1$, the two points are $(0, -1)$ and $(1, 0)$.

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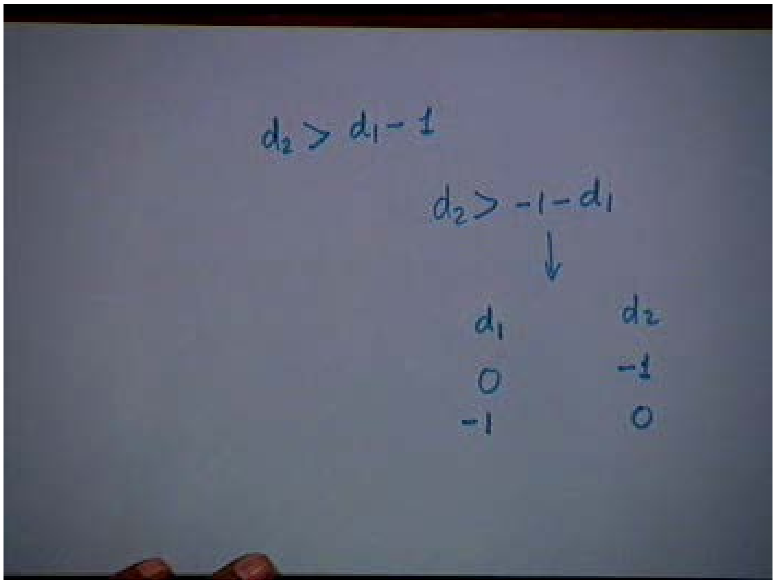

The two points are good enough to draw a straight line. The triangle having the vertices (0, –1), (–2, 1) (+2, 1) therefore defines the parameter plane in which the second order transfer function shall be stable. This is called a stability triangle for a second order IIR transfer function. In addition, if (d_1, d_2) is within the hatched area then the poles will be complex; if not then they shall be real. If (d_1, d_2) point is outside the triangle, the system would be unstable. Does it look like too much of a discussion with a very simple transfer function like the second order?

We can always find the roots and find whether it is stable or unstable, and whether the roots are real or complex. But then this stability triangle is useful in a different context, viz. in studying the quantization errors in a digital signal processor. Therefore I thought we must carry out this discussion at this stage, so that you get some experience. What do we do if we have a general transfer function $H_M(z) = N_M(z)/D_M(z)$ where testing for stability means we test for the roots of $D_M(z)$ and find whether all the roots are inside the unit circle or not. Let $D_M(z)$ be written as $1 + d_1 z^{-1} + d_2 z^{-2} + \dots + d_M z^{-M}$. Instead of finding the roots we go through the via media of an all pass function.

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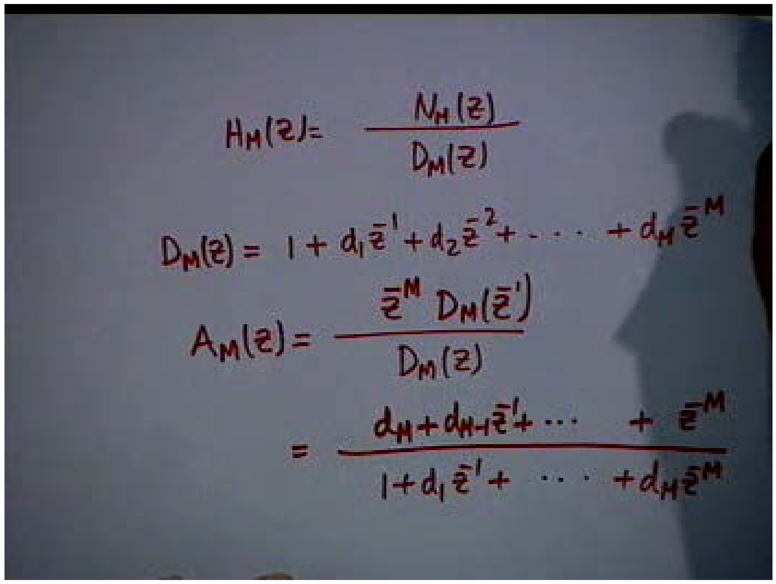

We construct, as the first step, an all pass function $A_M(z)$ with the same denominator $D_M(z)$; then the numerator shall be z^{-M} $D_M(z^{-1})$, that is, the coefficients of the numerator shall be in reverse order. In other words, the numerator shall be $d_M + d_{M-1} z^{-1} + \dots + z^{-M}$. Now let us say p_i; i = 1 to M, are the roots of $D_M(z)$. Then for stability we require |pi| less than 1. d_M is the product of the roots, i.e. d_M = continued product, i = 1 to M, p_i. Since magnitude p_i is less than 1, $|d_M|$ should be less than 1.

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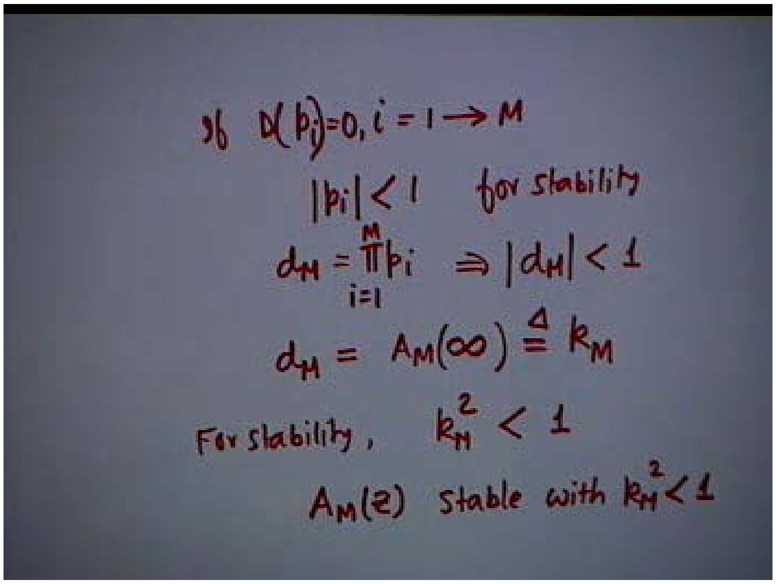

You notice that $d_M = A_M(\infty)$. For reasons to be made clear later, I shall denote it by k_M. They are identical at this stage but they are not so as we proceed ahead. Therefore for stability k_M^2 must be less than 1. This is a necessary condition, but not a sufficient condition. Let us assume that $A_M(z)$ is stable with $k_M^2 < 1$. The next step is to find an all pass with order M – 1 by using the relationship:

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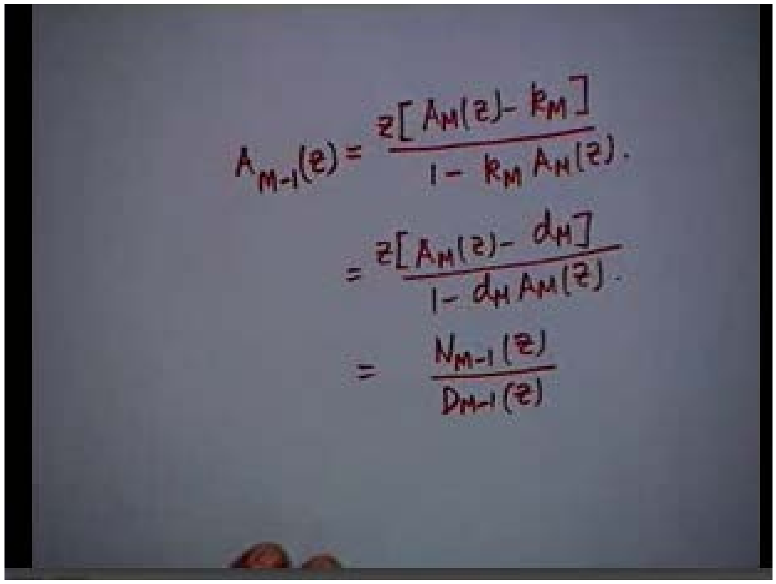

$A_{M-1}(z) = z\,[A_M(z) - k_M]/[1 - k_M\,A_M(z)]$ which is the same as $z\,[A_M(z) - d_M]/[1 - d_M\,A_M(z)]$. Let us see what this transfer function is. We have arbitrarily put the subscript of A on the left hand side as M – 1; we will show that the order is indeed M – 1. Now, if I substitute for A_M, then I can write this as $A_{M-1}(z) = N_{M-1}(z)/D_{M-1}(z)$ (again we shall show that the numerator as well as the denominator are also (M – 1)th degree polynomials) where $N_{M-1}(z) = z\,[(d_M + d_{M-1}\,z^{-1} + \ldots + z^{-M}) - d_M\,(1 + d_1\,z^{-1} + d_M\,z^{-M})]$.

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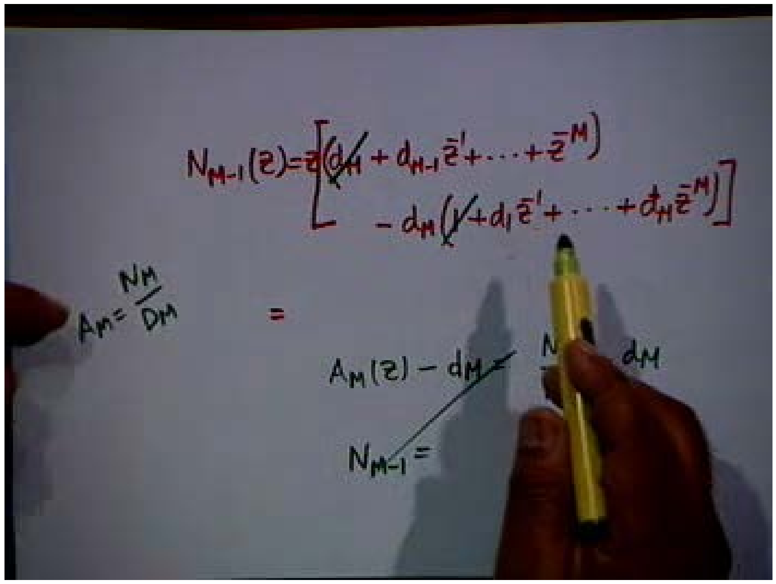

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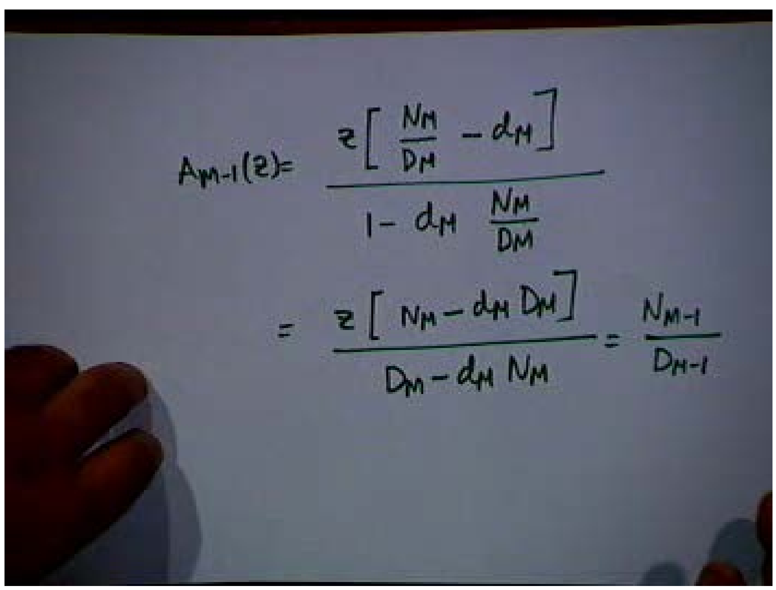

Note that d_M cancels on carrying out the simplification; I get $N_{M-1}(z) = (d_{M-1} - d_M d_1) + (d_{M-2} - d_M d_2) z^{-1} + \ldots + (1 - d_M^2) z^{-(M-1)}$. So the numerator is indeed an $M - 1$th order polynomial in z^{-1}. So our subscript $M - 1$ is justified.

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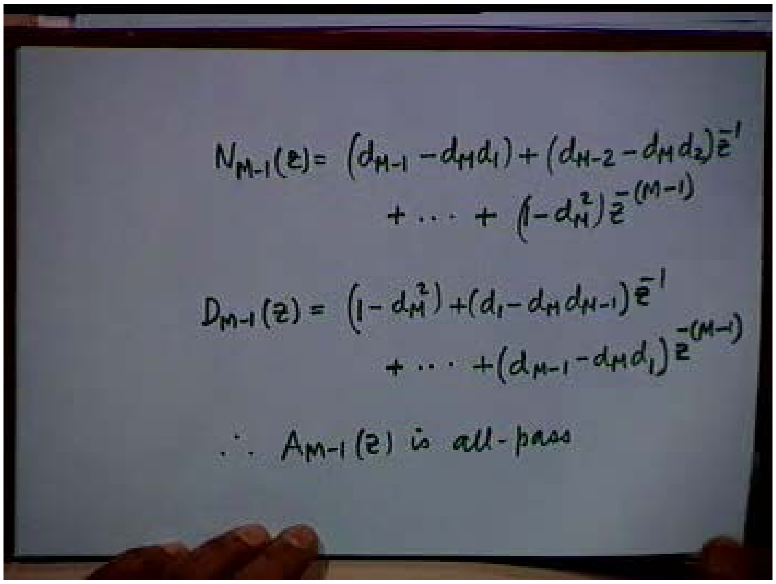

In a similar manner you can show the $D_{M-1}(z) = (1 - d_M^2) + (d_1 - d_M\ d_{M-1})\ z^{-1} + \dots + (d_{M-1} - d_M\ d_1) \times z^{-(M-1)}$. Here $d_M\ z^{-M}$ term cancels. Note that the coefficients of $D_{M-1}(z)$ are exactly those of $N_{M-1}(z)$ but in reverse order. Hence we get an all pass function. In $A_{M-1}(z)$, if we follow the same discipline of having the constant term in the denominator is unity, then we will have to divide both numerator and denominator by $1 - d_M^2$. Therefore $A_{M-1}(z)$ shall be of the form, $A_{M-1}(z) = [d_{M-1}' + d_{M-2}'\ z^{-1} + \dots + z^{-(M-1)}]/[1 + d_1'\ z^{-1} + d_2'\ z^{-2} + \dots + d_{M-1}'\ z^{-(M-1)}]$. It is indeed all pass function.

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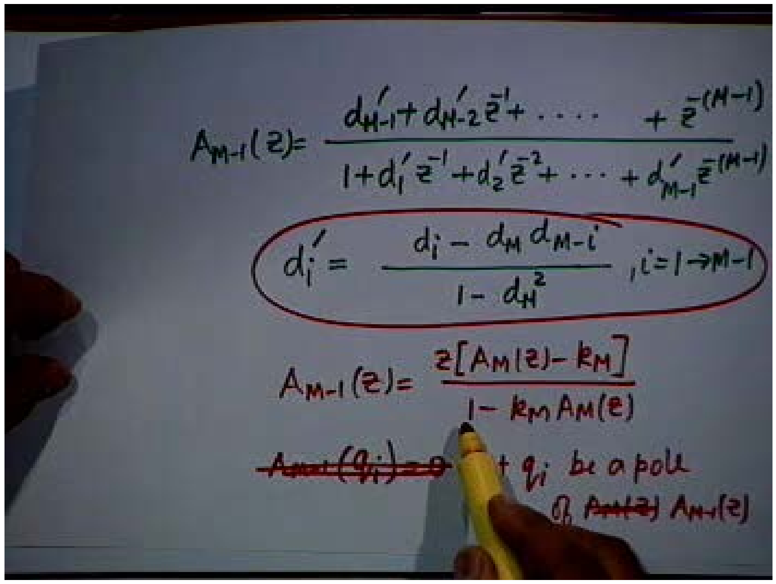

The primed coefficients obey the relationship $d_i' = (d_i - d_M d_{M-i})/(1 - d_M^2)$, i going from 1 to M – 1. But we still have a long way to go before we accept this recurrence relation. Our assumption was that $A_M(z)$ is stable and k_M^2 is less than 1. Now let us see whether $A_{M-1}(z)$ is stable or not. That is the question. Before we proceed, let us take a stock of this situation. We constructed $A_M(z)$ with the same denominator as that of the given transfer function to be tested. We have proved that if $A_M(z)$ is stable, then k_M^2 is less than 1. Then we constructed $A_{M-1}(z)$ which is an all pass function of order M – 1. Now we have to find out whether $A_{M-1}(z)$ is stable or not and for this we have to look at its poles which are the zeros of the denominator. The number of poles is M – 1. Choose an arbitrary pole q_i; at which the denominator must be = 0, that is $1 - k_M A_M(q_i) = 0$ or $A_M(q_i) = 1/k_M$. Therefore $A_M(q_i)$ magnitude = $1/k_M$ magnitude (k_M could be positive or negative).

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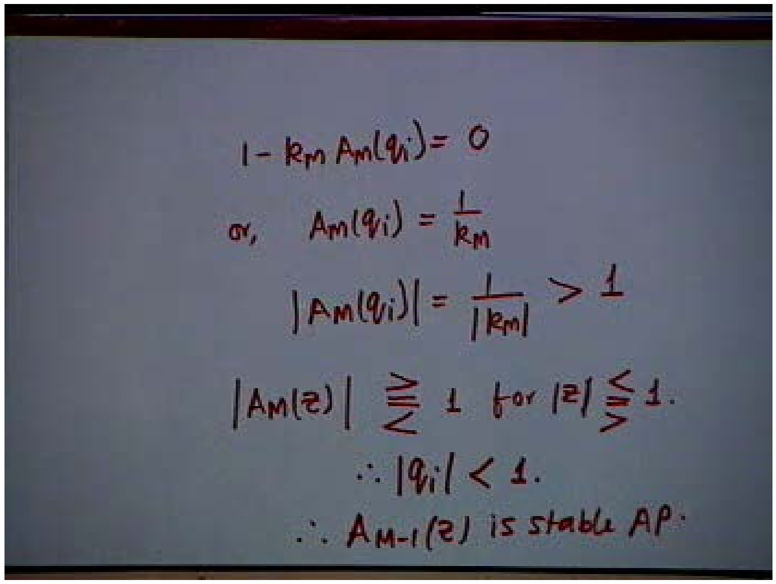

But k_M^2, by hypothesis, is less than 1, therefore $|A_M(qi)|$ must be greater than 1. We have also shown that if $A_M(z)$ is a stable all pass function, then its magnitude is greater than, equal to, or less than 1, for magnitude z less than, equal to, or greater than 1. Therefore if $|A_M(q_i)|$ is greater than 1, then $|q_i|$ must be less than 1. Therefore magnitude q_i is less than 1. What are q_i's? They are poles of $A_{M-1}(z)$ and therefore $A_{M-1}(z)$ is stable, all pass. We have shown that if $A_M(z)$ is stable and k_M^2 is less than 1 then $A_{M-1}(z)$ is a stable all pass function. Now we prove the reverse, that is, let $A_{M-1}(z)$ be a stable all pass and let k_M^2 be less than 1; we have to prove $A_M(z)$ is a stable all pass function. Then these two will constitute necessity as well as sufficiency and that will constitute the core of the test that we are going to learn. What we are going to prove now is that if $A_{M-1}(z)$ is stable all pass and k_M^2 is less than 1, then $A_M(z)$ is stable all pass.

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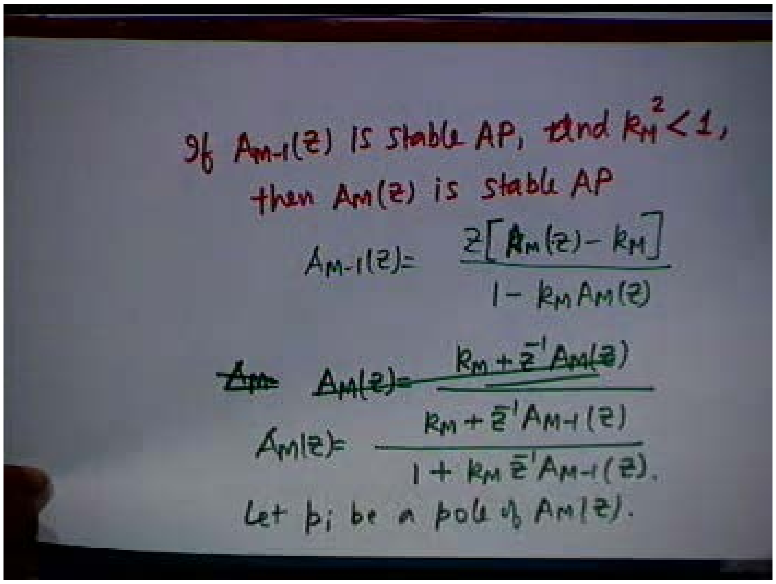

Now our defining relationship was $A_{M-1}(z) = [A_M(z) - k_M]/[1 - k_M A_M(z)]$. Now what we have to do is to find A_M in terms of $A_{M-1}(z)$, therefore cross multiply and simplify. The relationship is $A_M(z) = [k_M + z^{-1} A_{M-1}(z)]/[1 + k_M z^{-1} A_{M-1}(z)]$. We have expressed A_M now in terms of A_{M-1}. Now $A_M(z)$ is all pass, by hypothesis. So all we have to prove is that $A_M(z)$ is stable under these conditions that is $A_{M-1}(z)$ is stable and $k_M^2 < 1$. Let p_i be a pole of $A_M(z)$, then $1 + k_M p_i^{-1} A_{M-1}(p_i) = 0$. Therefore $p_i^{-1} A_{M-1}(p_i) = -1/k_M$. If I take the magnitudes on both sides then $|p_i^{-1}A_{M-1}(p_i)| = |1/k_M|$ which is greater than 1. Therefore magnitude $A_{M-1}(p_i)$ is greater than magnitude p_i. Now $A_{M-1}(z)$ is a stable function; therefore.

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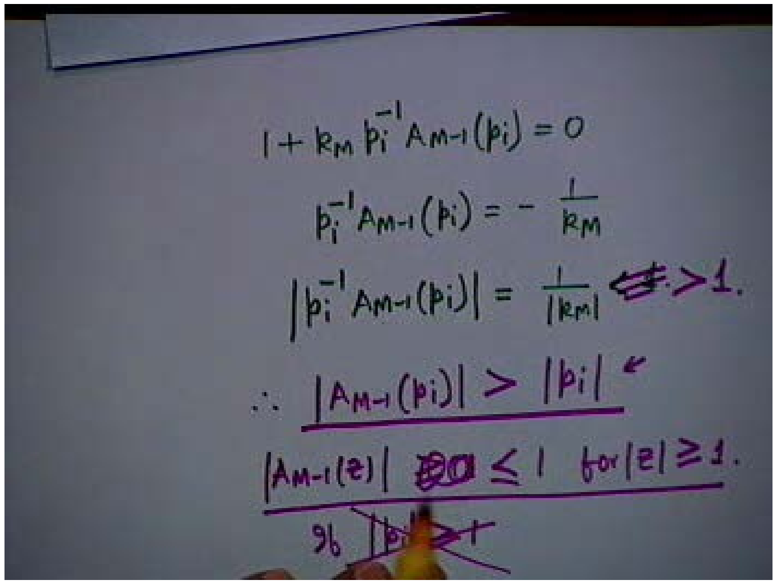

$|A_{M-1}(z)|$ is less than equal to 1 for mod z greater than equal to 1. If p_i magnitude is greater than or equal to 1, then $A_{M-1}(p_i)$ magnitude should be less than equal to 1. But our requirement is that $|A_{M-1}(p_i)|$ should be greater than magnitude p_i which should be greater than 1. These two are contradictory; therefore this cannot be allowed, and p_i must be less than 1. The proof is by contradiction. To repeat, if magnitude p_i is greater than or equal to 1 then A_{M-1} magnitude should be less than equal to 1. On the other hand, this relation shows that it should be greater than magnitude p_i which means it should be greater than 1. These two are contradictory and therefore magnitude p_i must be less than 1.

In other words, we have proved that if A_{M-1}, the derived function, is stable then the original function must also be stable. So we now state the core of the test; it says that; if $A_M(z)$ is stable all pass, then $A_{M-1}(z)$ is also stable all pass. This is necessary as well as sufficient. So the testing simply amounts to the following: if $A_M(z)$ is stable all pass, then k_M^2 must be less than 1. Then we derive the lower order transfer function $A_{M-1}(z)$. Look at $k_{M-1}^2 = d'^2_{M-1}$, this should also be less than 1. Then by recursion, we form A_{M-2} and we test for k_{M-2}^2; that should also be less than 1 and we proceed up to the first order, that is k_1^2 which should also be less than 1.

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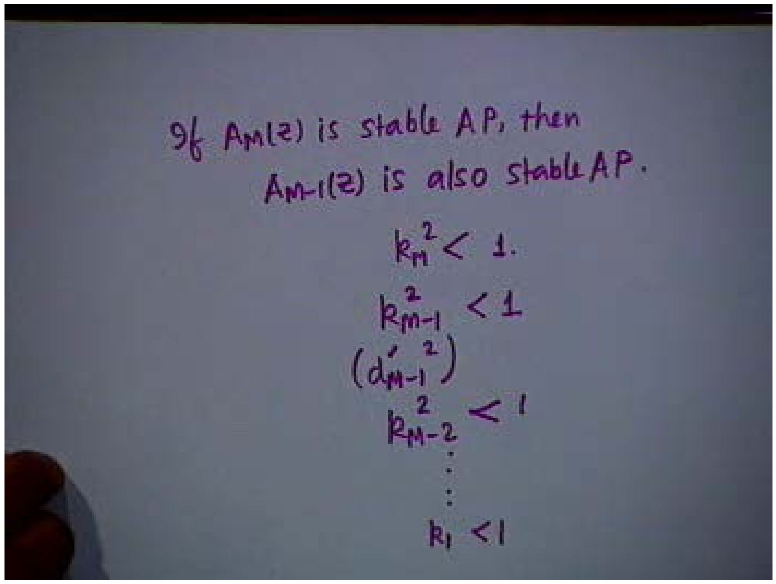

This is the stability test. It is very important that you understand the logic. It is one of the most beautiful theorems in DSP which shows how to avoid root finding. It is a little bit of exercise but with experience, you do not have to find all the transfer functions. All you have to find are the denominator polynomials. The coefficients are found by the relationship between d_i' and d_i for the general order m – 1 from the previous one of order m (m = M to 2). So you find k_M, k_{M-1}….k_1 and test whether $k_i^2 < 1$, 1 = M to 1. We take an example. Let $H(z) = 1/(6 + 5z^{-1} + 4z^{-2} + 3z^{-3} + 2z^{-4} + z^{-5})$; you are required to find whether this is stable or not.

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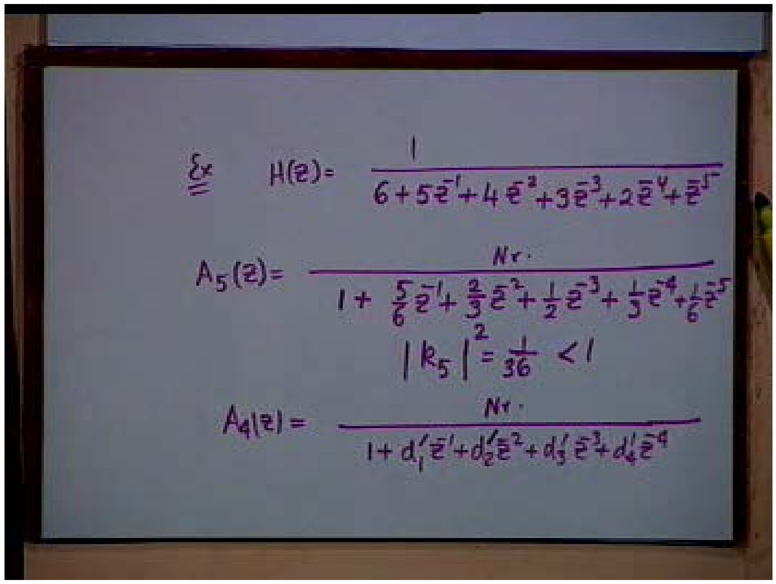

The numerator is irrelevant; it can be taken as 1 or any other function. The first thing you do is form an $A_5(z)$. Now you must be careful; intentionally I have taken the constant term in the denominator as 6; you must make this 1; so your numerator will be 1/6 but this need not be written because all we are concerned with are the denominator coefficients. You see that magnitude $k_5 = 1/6$, so k_5^2 is 1/36, less than 1. We take square because the last coefficient may come positive or negative. First condition is satisfied. If this was not so, then you need not carry out the test further. Then to find $A_4(z)$; let $A_4(z)$ denominator be $1 + d_1' z^{-1} + d_2' z^{-2} + d_3' z^{-3} + d_4' z^{-4}$; the numerator need not be constructed. Here our interest is only d_4' but to find the d_4', you have to go through finding all of them because the next one shall require the values of d_1', d_2' and d_3'. So we use the formula $d_i' = (d_i - \frac{1}{6}d_{5-i})/(1 - \frac{1}{36}) = 36/35\,[d_i - (d_{5-i}/6)]$.

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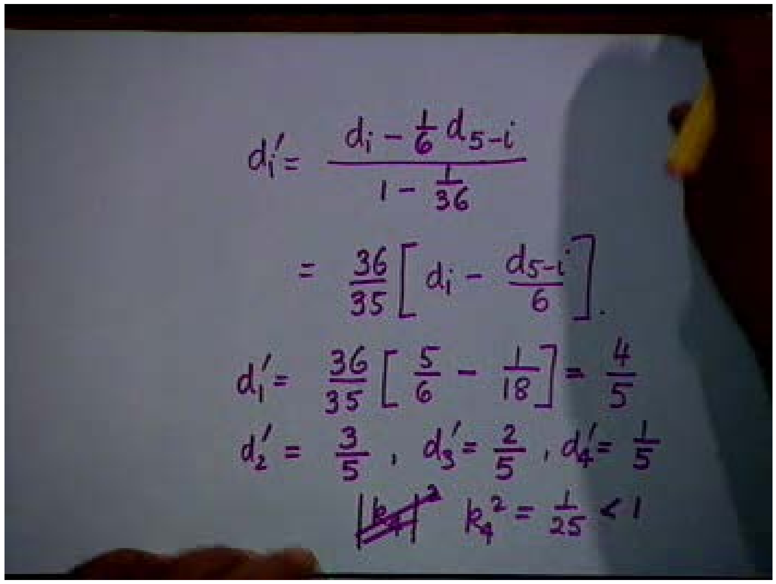

Using this, I get d_1’ = 4/5, and similarly d_2' = 3/5, d_3' = 2/5 and d_4' = 1/5. Now therefore k_4^2 =1/25 which is less than 1. The test has not failed yet, so we are encouraged to go to the next step. That is we have to find $A_3(z)$; write $D_3(z) = 1 + d_1''z^{-1} + d_2'' z^{-2} + d_3'' z^{-3}$. You are only concerned with the denominator, so you can ignore the numerator (which, if necessary, you can easily write by reversing the order of the coefficient d_i”).

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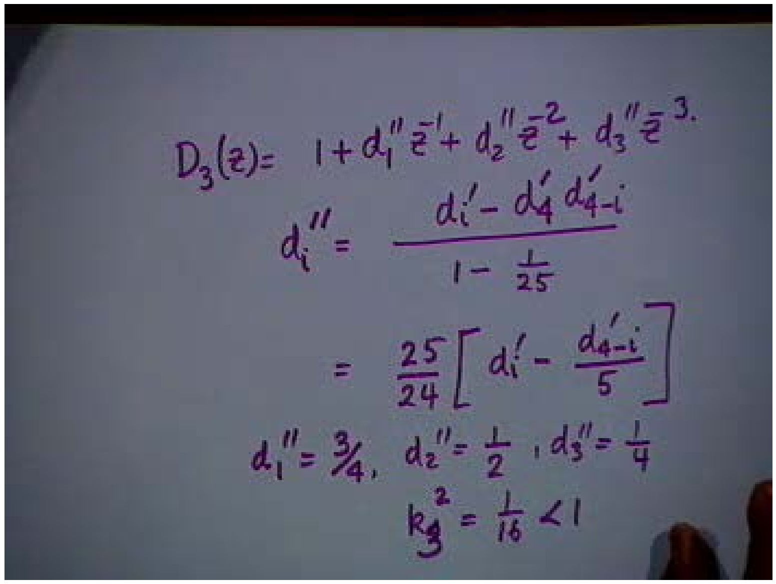

The formula now shall be $d_i'' = (d_i' - d_4' d_{4-i}')/(1 - 1/25) = [(25/24)[di' - d_{4-i}'/5]$. Calculate the coefficients and it gives $d_1'' = 3/4$, $d_2'' = ½$, $d_3'' = ¼$. Therefore $k_3^2 = 1/16$, which is less than 1. Now, if we continue the test, then you get $k_2^2 = 1/9$, less than one, and $k_1^2 = ¼$, which is also less than 1. Hence the given H(z) is stable. If at any stage, k_i^2 is equal to or greater than 1, no further testing is needed; you conclude that the given function is unstable.

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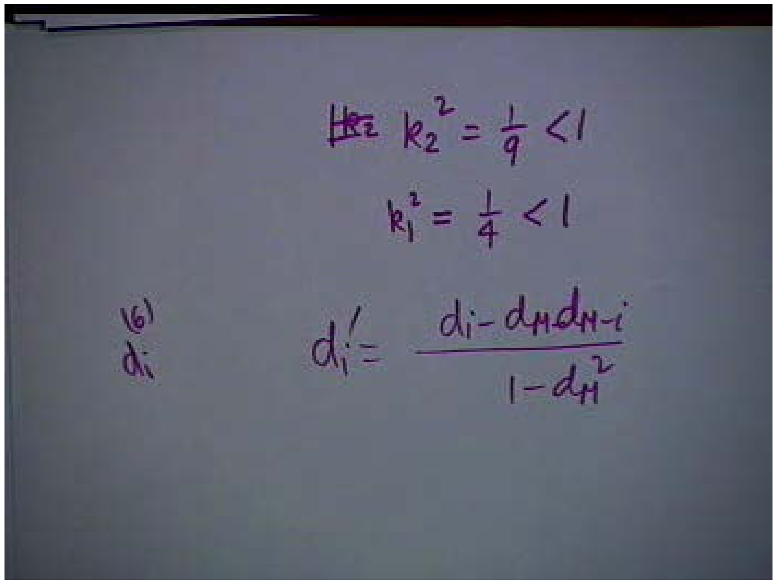

If you are designing a filter, then you better do the test twice or thrice so that you are absolutely sure that the given function is stable. The recurrence relation $d_i' = (d_i - d_M d_{M-i})/(1 - d_M^2)$ can be easily programmed to compute k_is but you have to keep a track of the primes. If the order is high, you should not use many primes; instead use the subscripts a b c d and so on. If you have exhausted the alphabets, then you go to alpha, beta, gamma, delta etc. What the book says is; $d_i^{(M)}$ but this can also be confusing sometimes; it may be taken as the power. We will stop here today.