Digital Signal Processing

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Lecture - 17

Linear Phase Filters, Complementary Transfer Functions

This is the 17th lecture on DSP and we continue the discussion of linear phase filters.

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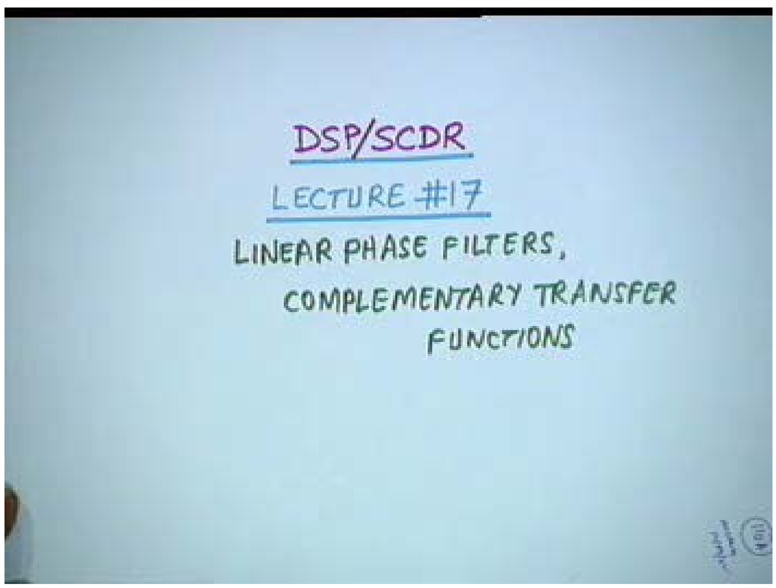

We shall also talk about a very interesting class of filters which comprises a set, and they are known as complementary filters. This is where the importance of all pass filters shall be demonstrated. In the last lecture, we talked about all pass filters and showed that the poles and zeros have mirror image symmetry. That is, poles and zeros occur in reciprocal pairs and as a result, all pass filters are necessarily maximum phase filters. None of their zeros can be inside the unit circle.

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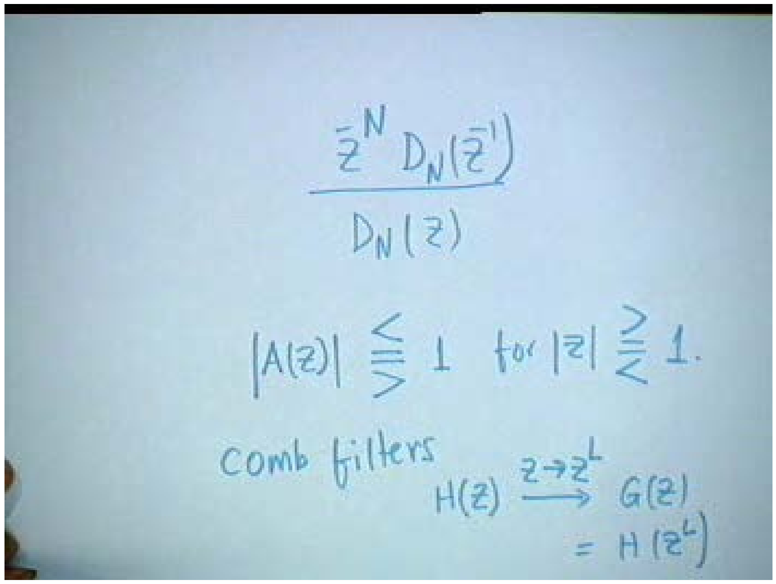

We also said that they are lossless bounded real functions, provided we take the form $A(z) = z^{-N} D_N(z^{-1}) / D_N(z)$ in which the magnitude is normalized to unity. We also stated a theorem that the magnitude of an all pass filter is bounded by the inequality: $|A(z)|$ is less than, equal to or greater than 1 for mod z greater than, equal to or less than 1. I asked you to prove it. The proof is not very simple. We also introduced the Comb filters, so called because the shape of the frequency response resembles a comb. Comb filters can be obtained from any filter which has one pass band or one stop band or both, that is one pass band and one stop band, by changing z to z^L and then we get $G(z) = H(z^L)$,which has multiple pass bands and multiple stop bands. And one of the most important uses of such filters is in the elimination of periodic interference like the 50 Hertz signal in a biomedical signal processing situation.

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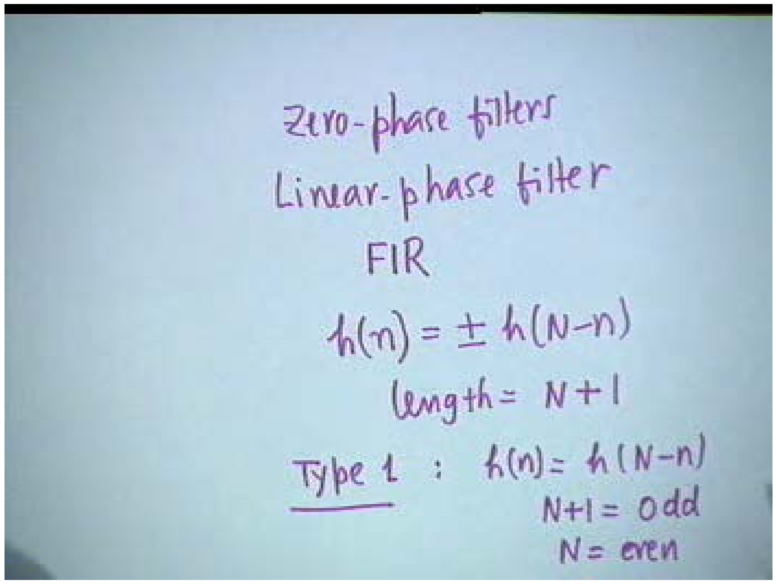

We talked about zero phase filters and also commented that they are necessarily non- causal and therefore cannot be realized in real time. What can be realized in the real time are the Linear phase filters, which play an important role in any system, transmission or otherwise, and we said that only FIR filters can realize exact linear phase. We also stated that the condition for linear phase is that the impulse response should be either symmetric or anti symmetric i.e. $h(n)=\pm h(N-n)$, the filter being of length = N + 1, and N being the order of the filter. Now, depending on even or odd length, and symmetry or anti symmetry, we stated that we have four types of filters. We discussed types 1 and 2 in the last lecture.

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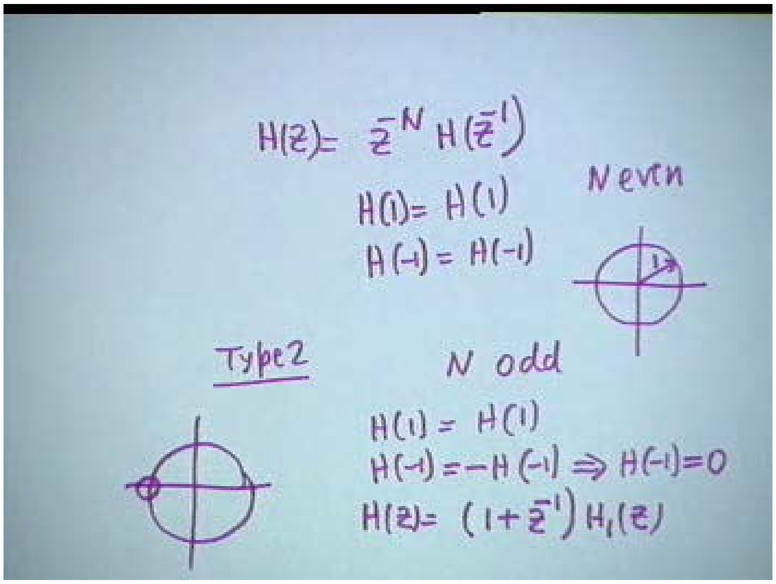

Type 1 was symmetric: h(n) = h(N – n), and N + 1 = is odd. So N, the order of the filter is even. Under this condition, we showed that H(z) is the same as z^{-N} $H(z^{-1})$, which shows, that if z_0 is a zero then $1/z_0$ is also a zero. This property is common to all the types: Type 1, Type 2, Type 3 and Type 4, where for the latter two types, H(z) is the negative of z^{-N} $H(z^{-1})$. So zeros occur in reciprocal pairs. We also showed that since N is even in type 1, H (1) = H (1) and H (– 1) = H (– 1) and therefore there are no restrictions on the kinds of filters which can be realized. In other words, type 1 filter does not have a zero on the unit circle, unless you introduce it intentionally. And in type two, the order N is odd and therefore if I substitute in this relation z =1, then H (1) = H (1) but H (– 1), because N is odd, is – H (– 1) which means that H (–1) = 0. This means H (z) must have a zero at z=-1. In other words, I must have a factor $(1 + z^{-1})$ in H (z) and the consequence of this is that high pass filter is not possible because high pass filter requires unity magnitude at z = – 1 or omega = pi. If we cannot have a high pass filter, can we have a band stop filter? A band stop filter also requires unity magnitude at omega = 0 as well as omega = pi. Therefore band stop is also not possible. HPF and BSF of the type that we have been discussing are not possible. Let us go to type 3, we shall have more fun in type 3 filters.

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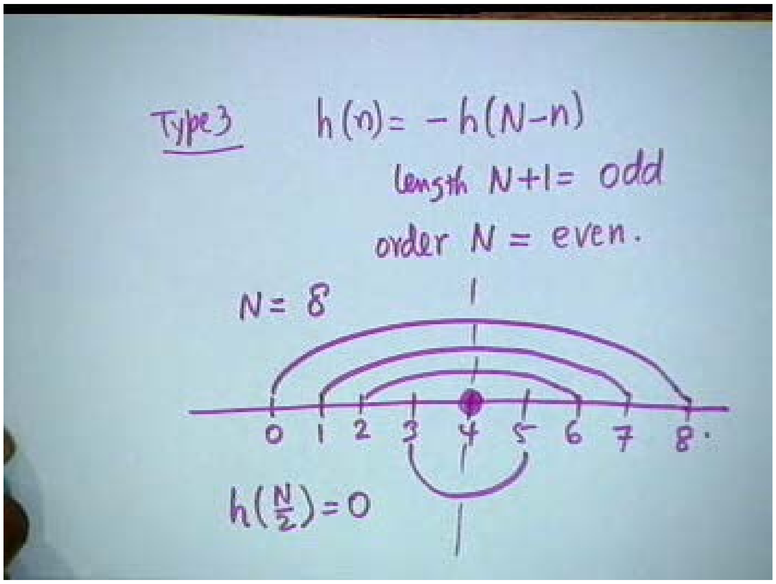

Type 3 has anti symmetry, that is, h(n) = – h (N – n) and a length N + 1 which is odd. Therefore N, the order of the filter is even. To concretize our idea we take an example. Let N = 8; then we have samples at 0, 1, 2, 3, 4, 5, 6, 7 and 8. The pairing of h_0 is with h_8 but they are not equal now, they are equal and opposite because of anti symmetry. So $h_8 = -h_0$, similarly $h_1 = -h_7$, $h_2 = -h_6$, $h_3 = -h_5$, and $h_4 = -h_4$; the last relation forces h_4 to be identically = 0, and h_4 is the axis of symmetry. Therefore, in general, h(N/2) = 0 and this introduces further restrictions, as we shall see. But let us first see the form of the frequency response.

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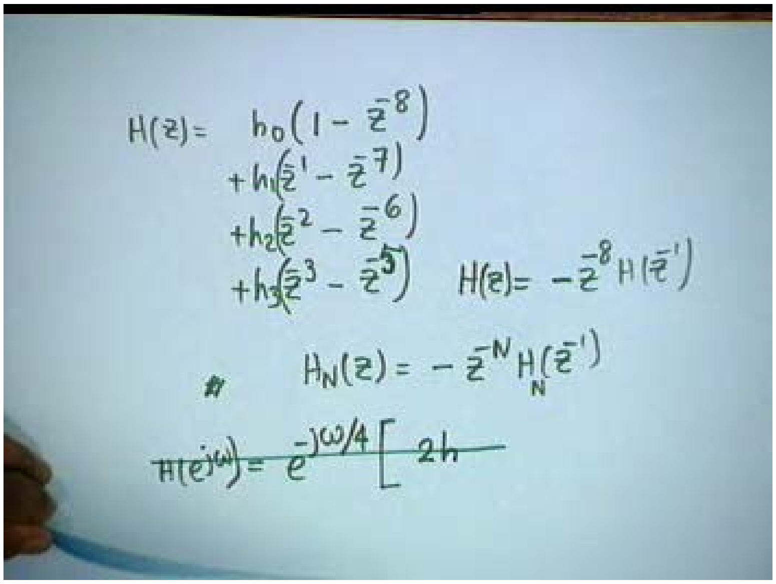

For this particular example, $H(z) = h_0(1-z^{-8}) + h_1(z^{-1}- z^{-7}) + h_2 (z^{-2}-z^{-6}) + h_3 (z^{-3}-z^{-5})$. This is the transfer function. Suppose we change z inverse to z; then how can we regain H(z) from this? We multiply by z^{-8} but the signs are interchanged. If I multiply $1 - z^8$ by z^{-8}, I get $z^{-8} - 1$ and therefore I shall require a negative sign. So in general $H_N(z) = -z^{-N}H_N(z^{-1})$; this is the difference from types 1 and 2. For types 1 and 2, symmetrical impulse response, $H_N(z)= + z^{-N} H_N(z^{-1})$.

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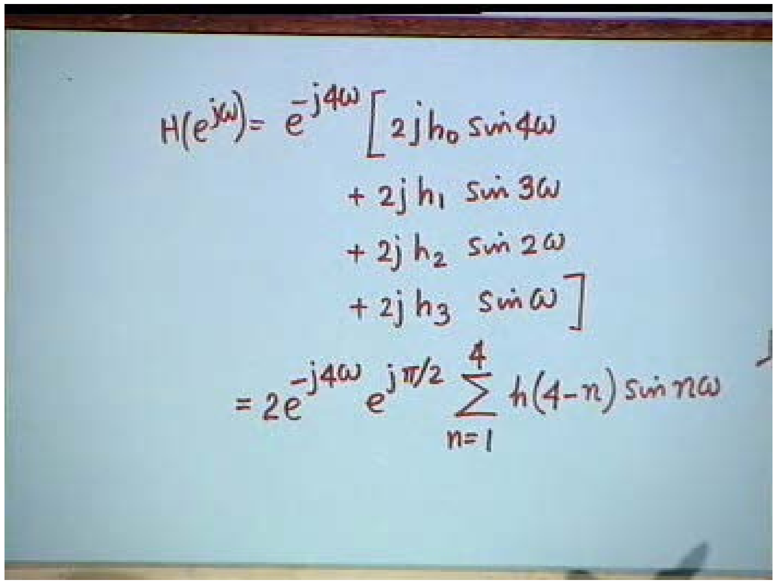

For both of types 3 and 4, $H_N(z)= -z^{-N}H_N(z^{-1})$. This is where further restrictions come. In this particular case of type 3 example, the frequency response is $H(e^{jomega}) = e^{-j4omega}$ [2j h_0 sine 4omega + 2j h_1 sine 3omega+ 2j h_2 sine 2omega + 2j h_3 sine of omega] The factor j did not come in types 1 and 2; h(n) was symmetric and $e^{jtheta} + e^{-jtheta}$ was a real quantity, twice cosine theta. Here $e^{jtheta} - e^{-jtheta}$ is twice j sine theta. You see that in symmetry, cosine functions come. In anti symmetry sine functions come; in addition a factor j comes which can be taken care of by a phase shift of π/2 and therefore I can write this as $H(e^{j\omega})=2\ e^{-j4omega}\ e^{jpi/2}$ summation h (4 – n) sine n omega where n goes from 1 to 4.

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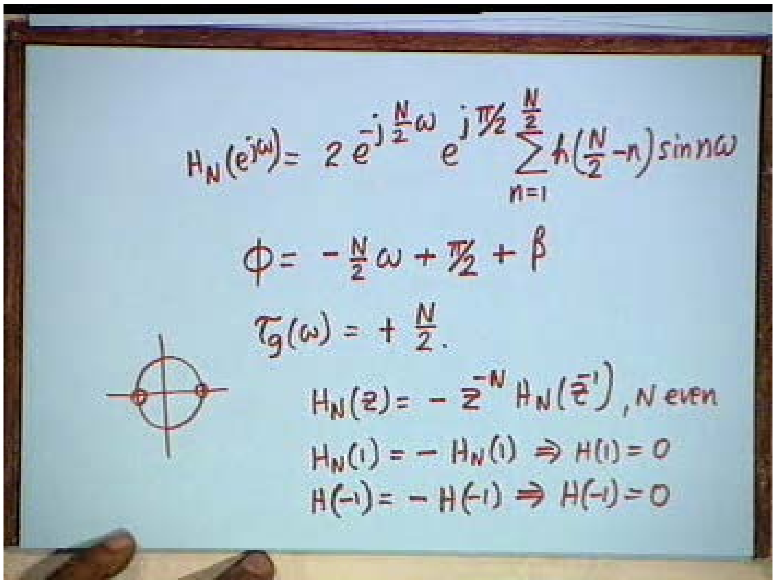

In general, for Nth order H_N (e^{jomega}) = $2e^{-j(N/2)omega}$ $e^{jpi/2}$ summation (n = 1 to N/2) h(N/2 – n) sine of n omega; this is the general expression. You see the phase shift now is phi = – (N/2) omega + pi/2 + possible addition of integral multiples of pi. The summation is the pseudo magnitude can have a change of sign; therefore you must admit a beta= integral multiple of π. But tau_g(omega) which is the negative gradient of phase = + N/2. The other thing that one must notice is that $H_N(z) = -z^{-N} H_N(z^{-1})$. So as far as zeros are concerned, they occur in reciprocal pairs. But you notice now, for type 3, N, the order is even, the length is odd and you notice that $H_N(1) = -H_N(1)$ which means that H(1) =0. Similarly, H(– 1) = – H (– 1) which means that H(– 1) = 0. In terms of zeros of H(z), obviously we have zeros at + 1 as well as – 1.

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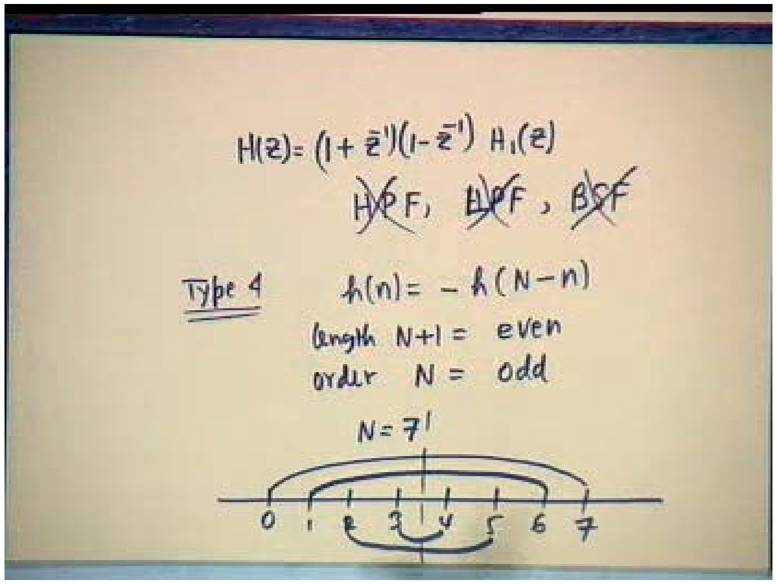

Therefore for type 3, the transfer function H (z) shall have a factor of $1 + z^{-1}$ and also $(1 - z^{-1})$. Call the remaining transfer function as $H_1(z)$. Is $H_1(z)$ a linear phase? It is, because $(1 + z^{-1})$ is linear phase and $(1 - z^{-1})$ is also linear phase, so from linear phase you are taking out two linear phase factors. The rest must also be a linear phase, otherwise the product cannot be linear phase. So H_1 (z) is also a linear phase; it contains the other zeros of H(z) which occur in reciprocal pairs. What about this $(1 \pm z^{-1})$; why the corresponding zeros not occur in reciprocal pair? It is because the reciprocal of + 1 is + 1 itself, so it can occur as a single zero. The reciprocal of – 1 is also –1 and therefore that can also occur as single zero. $H_1(z)$, the rest of the function, may not contain another 0 at + 1 or – 1. I said may not, it may also contain. For example, the zero at z = 1 can occur with multiplicity. So, at least there is one zero at + 1 as well as at – 1. And therefore HPF, LPF and BSF are not possible. The only thing that you can do with type 3 is a band pass filter.

Now let us talk of type 4. In type 4, h(n) = – h(N – n), and length N + 1 = even. So order N = odd. If we take N = 7 as an example, then my pairing would be like this: $h_0 = - h_7$, $h_1 = - h_6$, $h_2 = - h_5$ and $h_3 = - h_4$. There is no loner here. The symmetry is around 3.5 where there exists no sample.

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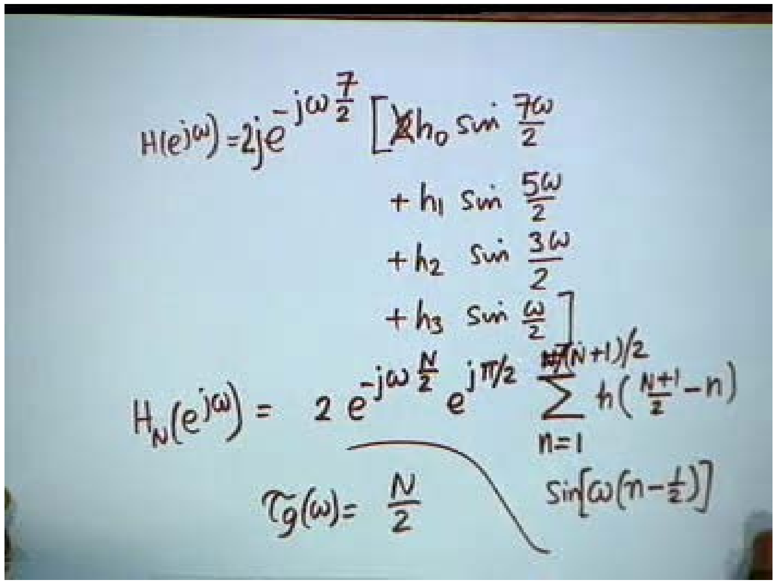

So we can write the frequency response directly as $H(e^{jomega})$ =2j $e^{-jomega7/2}$ [(h $_0$ sine of 7 omega/2 + h_1 sine of 5omega/2 +$h_2$sin of 3omega/2 +h_3 sine of omega/2]. In general $H_N(e^{jomega})$ = $2e^{-j\ omega\ N/2}$. $e^{jpi/2}$ summation (n = 1 to (N + 1)/2) h([(N + 1)/2] – n) sine of omega(n – ½). This is the correct expression and here also the phase is – omega N/2 + pi/2 + beta, so tau_g(omega) = N/2; the extra half sample delay creates problems in actual hardware realization but the problem can be surmounted.

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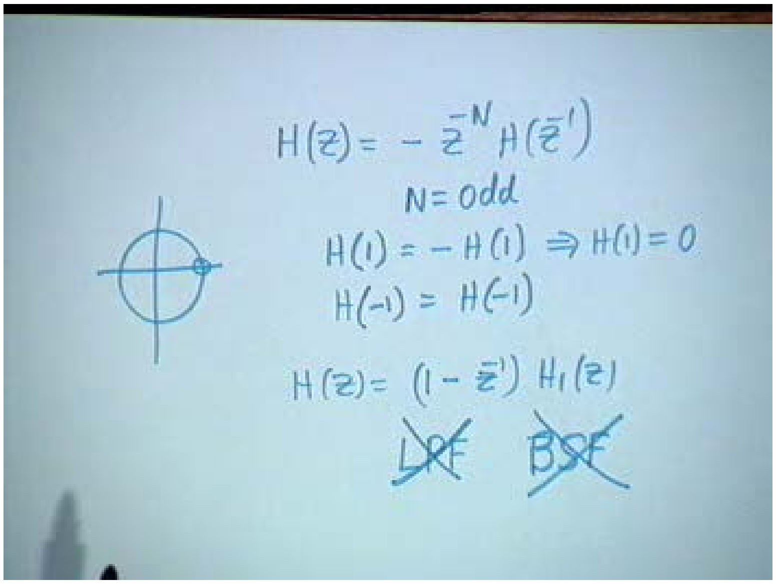

Now $H(z) = -z^{-N} H(z^{-1})$ and N = odd. So H(1) = – H(1), but H(– 1) = H(– 1). In other words, you have a zero at z=1. H(z) shall have a factor $(1 - z^{-1})$. Therefore a Low pass Filter is not possible because z = 1 corresponds to omega = 0. If LPF is not possible, Band Stop Filter is also not possible. Thus types 2, 3 and 4 are restricted. Type 3 is the most restricted one, we can only design Band Pass Filters. Type 1 is the most versatile, you can design any kind of filter. Type 1 also does not give a non integer delay, it gives delay of an integral number of samples. Nevertheless in situations where you require zero at z = 1 or z = – 1 or both you use the other appropriate type.

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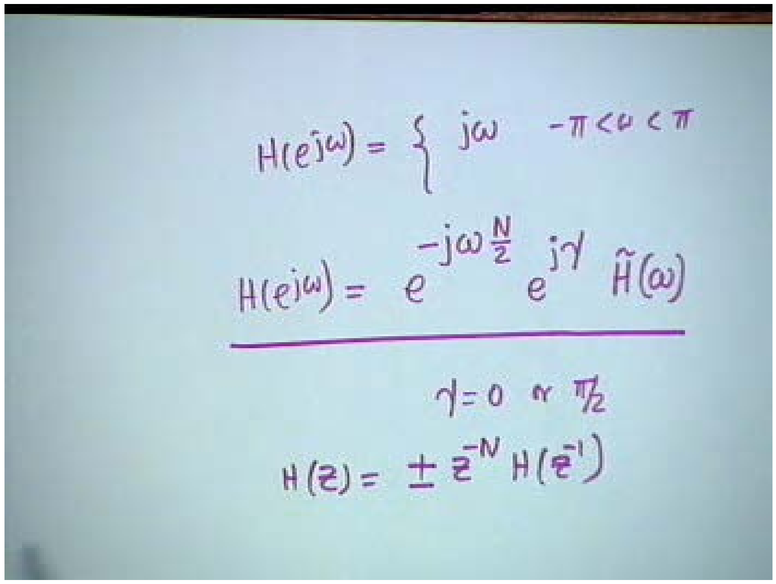

For example, you may want to design a differentiator. What is the magnitude response of a differentiator? It is a straight line. For an ideal differentiator, $H(e^{jomega})$ would be of the form j omega, – pi less than omega less than pi. Therefore its magnitude is 0 at omega = 0. What type would be suitable? At omega = 0 that is z = 1 it is 0; so type 4 filter would be the most suitable one for this application. So there are situations where one requires one of these types. But if it is a general filter design problem, you have to design Low Pass, High Pass, Band Pass and Band Stop and so on, then type 1 is the most favored one. In general, we have also seen that for a linear phase FIR, the expression is of the form $e^{-jomega\ N/2}$ $H_1(\omega)e^{j\gamma}$. N even or odd. All of them have N/2 number of delays and then you have an additional factor of e^{jgamma} where gamma can be 0 or $\pi/2$; gamma is 0 for types 1 and 2, gamma is $\pi/2$ for types 3 and 4. $H_1(\omega)$ is the pseudo magnitude function. So the phase is – omega (N/2) + gamma + beta which can be 0 or an integral multiple of π. The group delay is still N/2. Now let us look at the zeros. The transfer function, in general, obeys the relationship $H(z) = \pm z^{-N} H(z^{-1})$, so what we said about zeros of types 1 and 2 are valid here also. I mean, even if there is negative sign it does not change the reciprocal pair character of the zeros.

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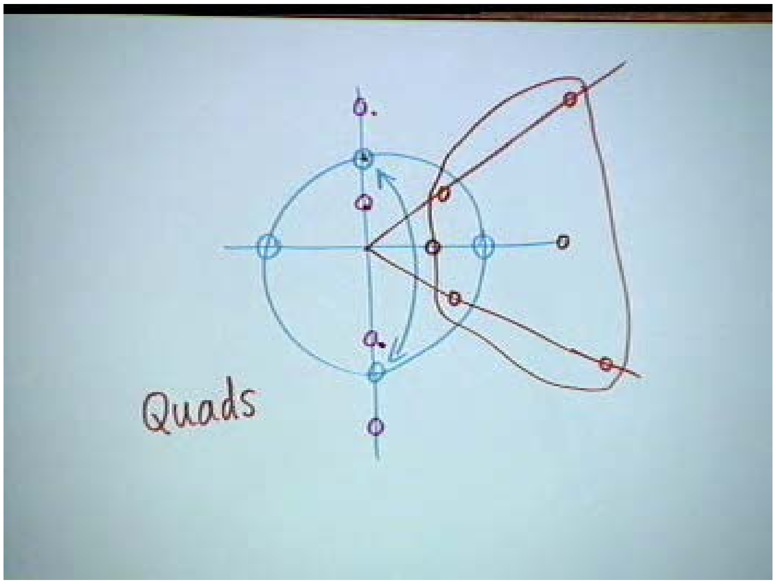

I am going to give you a location of a zero and I want you to tell me where its reciprocal would be. If we have a zero at + 1 then we do not require its reciprocal. It can occur as singly or multiple times. Similarly a zero at z = – 1 does not require a reciprocal. But if I have a zero at z = + j, what is its reciprocal? That is – j, but do I have to force the function to have a zero at – j or does it automatically come? It automatically comes with real impulse response. If I have a zero at + j, I must have a zero at – j, not only because 1/j is – j, but also because complex zeros must occur in conjugate pairs for real h(n). If I have a zero at j r where r is less than 1, and then I must have a zero at – j/r. But since zeros must occur in complex conjugate pairs, we conclude that an imaginary zero must occur in a quad. If I have a zero at + r, then I must have 0 at 1/r; if it is at – r, I must have one at –1/r also. But if I have zero at $re^{j\theta}$ then I must have zero at $re^{-j\theta}$ also for real coefficient, and I must have zeros at $1/r\ e^{-j\theta}$ and $1/r\ e^{j\theta}$, these being the reciprocals of $re^{j\theta}$ and $re^{-j\theta}$ respectively. Purely imaginary zeros, as we have seen, also occur in groups of 4. Why group of 4? It is because real coefficient demands that complex zeros must occur in conjugate pairs. If a zero exists outside the unit circle on the imaginary axis then it must occur in sets of 4. The term "Quad" is a short term for quadruplet.

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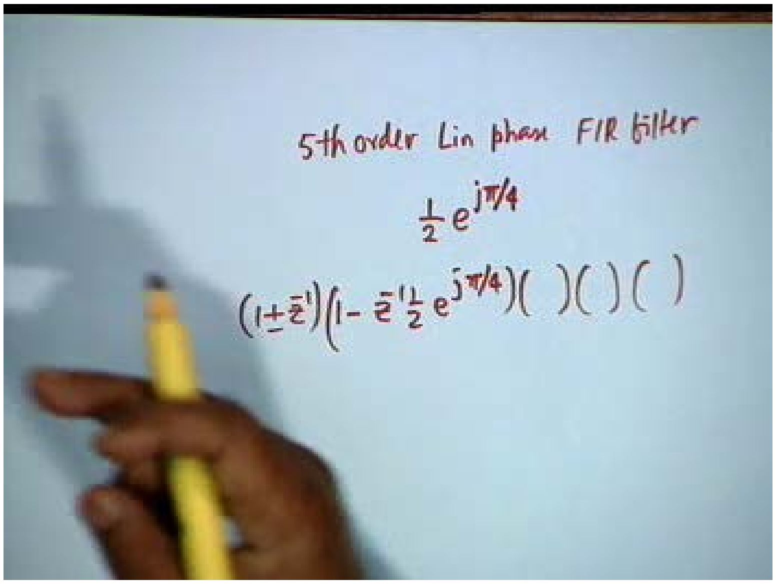

Suppose I have a 5th order Linear phase FIR filter, having one of the zeros at (1/2) $e^{jpi/4}$; can you construct the total filter? I have given you only this piece of information. How many choices do you have? Four of these zeros are determined by the "quad" property. You shall have (1-z^{-1} (1/2) $e^{j\ pi/4}$) as one of the factors. Then you shall have (1 – z^{-1} (1/2)$e^{-\ j\ pi/4}$), (1 – z^{-1} 2$e^{jpi/4}$) and (1 – z^{-1} 2$e^{-j\ pi/4}$) also as factors. Now what would be your choice for the fifth factor? I have said the order is five; we have been able to construct four factors, what will be the fifth one? You require a zero which occurs singly, either at + 1 or – 1, and therefore you have two choices, (1 ± z^{-1}); you also have the choice of simply z^{-1}, which has linear phase. If you choose a zero at +1, then you have chosen which type? With a zero at – 1, is it type 1? No, type 1 cannot have a zero at – 1. Type 2? No, it cannot be. Type 3? Type 3 requires a zero at – 1 and also at + 1. You must be careful, it is type 2. On the other hand if you choose a zero at + 1 then you have a type 4. So this is the kind of judgment that one has to exercise if partial information is provided. You can construct the whole, given only partial information because of the property of Linear Phase Functions. To summarize, type 1 is versatile having no zero at + 1 or – 1; all kinds of filters can be designed. Type 2 is not versatile; HPF is not possible in type 2 and if HPF is not possible, Band Stop is also not possible. Type 3 is severely restricted: no low pass, no high pass, no Band Stop it is only suitable for Band Pass Filter. With Type 4, no low pass, no Band Stop. Only high pass and band pass are provided. That summarizes our discussion of Linear Phase.

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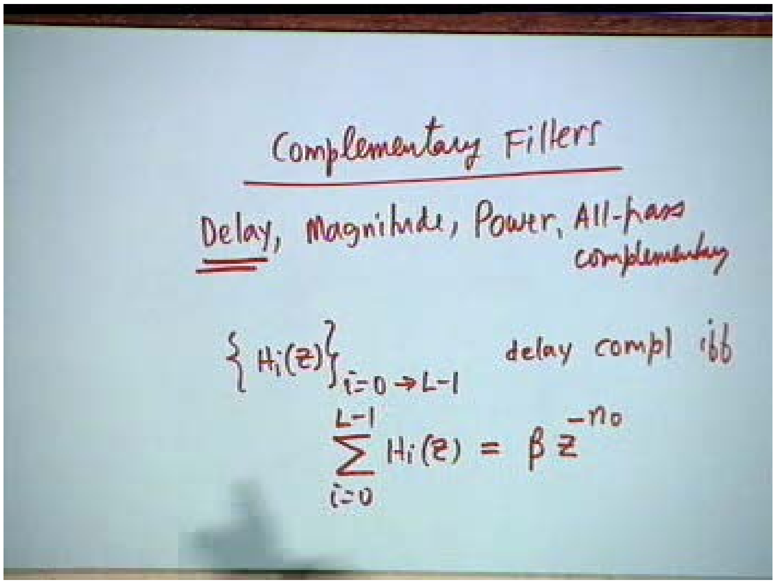

Now we go to Complementary Filters. The Complementary Filters can be of various types: it can be Delay Complementary, Magnitude Complementary, Power Complementary or All Pass Complementary. First we consider Delay Complementary The definition is that a set of filters $H_i(z)$ where i = 0 to L – 1, i.e. a set of L number of filters $H_i(z)$ are said to be delay complementary if and only if the summation of $H_I(z)$, i = 0 to L – 1, is some constant beta multiplied by a pure delay z^{-n0} where n_0 is a positive integer. The sum of the filters makes a pure delay, the magnitude may not be unity, magnitude may be a constant beta. For example, if you have L = 2 then $\{H_0(z), H_1(z)\}$constitute a delay complementary set if $H_0(z) + H_1(z) = \text{beta } z^{-n0}$.

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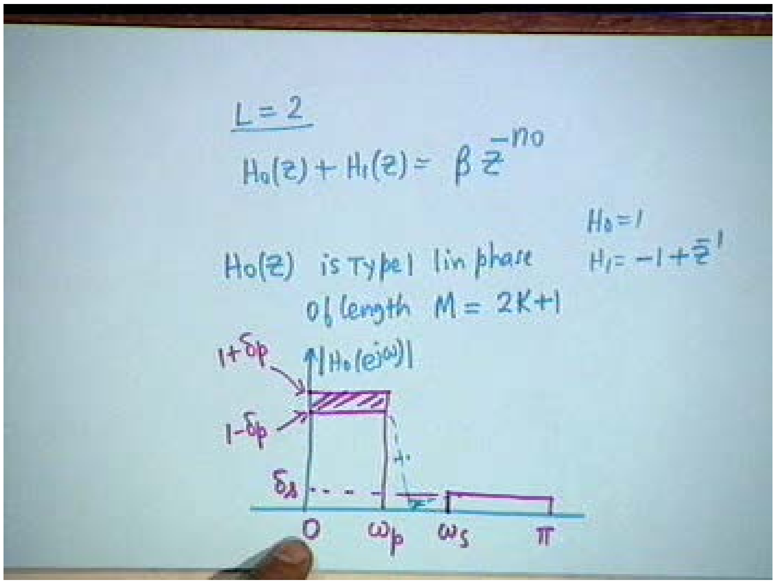

A very simple example is $H_0 = 1$ and $H_1 = -1 + z^{-1}$; these are delay complementary. This is a trivial example. Suppose H_0 is type 1 Linear Phase of length M = odd. And suppose the magnitude characteristics of H_0 (e^{jomega}) is Low Pass. A Low Pass Filter, as you know, has to have a tolerance in the Pass Band and a tolerance in the Stop Band. Suppose the tolerance scheme is this: in the range 0 to ω_p, the magnitude must lie between two levels 1 + delta p and 1 – delta p. It can be monotonic, or with ripple. Then in the Stop Band, extending from $omega_s$ to pi, the magnitude must be less than or equal to $delta_s$. The magnitude can be zero at some frequency or frequencies between $omega_s$ and pi. At such a zero, there will be an abrupt jump of π in phase. So in the pass and transition bands, the pseudo magnitude and magnitude are identical. Since the Stop Band has started at omega s, if at all there is a change of sign in the pseudo magnitude, it shall occur between omega s and pi.

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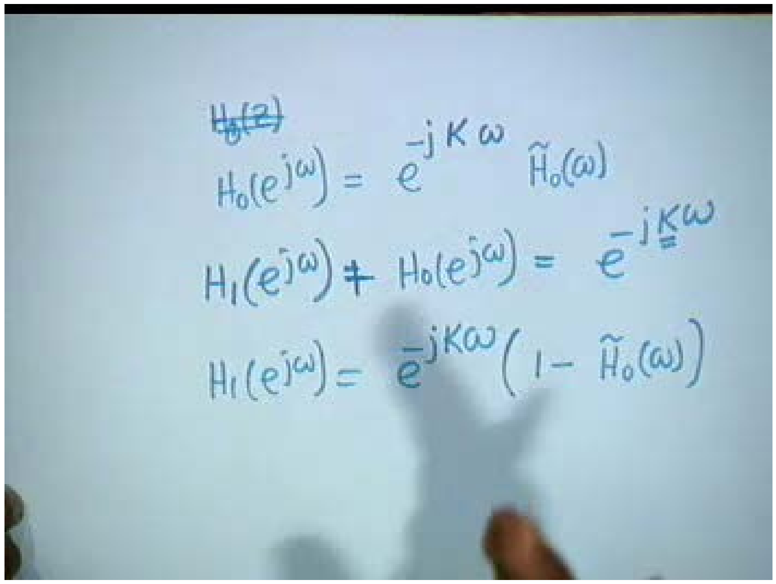

Now, suppose we want to construct, from this $H_{0,}$ a Delay Complementary $H_1(z)$, but first let us find out the form of $H_0(e^{jomega})$. It is of the form $H_0(e^{j\omega})= e^{-j\omega(M-1)/2}\ \tilde{H}_0(\omega)$, where $\tilde{H}_0(\omega)$ is the pseudo magnitude function. We want a Delay Complementary function $H_1(e^{jomega})$. What can be the form of $H_1(e^{jomega})$?

Clearly, $H_1(e^{j\omega})=1- H_0(e^{j\omega})$ should be of the form $e^{-j\omega N/2}\ \tilde{H}_1(\omega)$ in order that $\tilde{H}_0(\omega)+ \tilde{H}_1(\omega)=1$. (We normalize β to unity).

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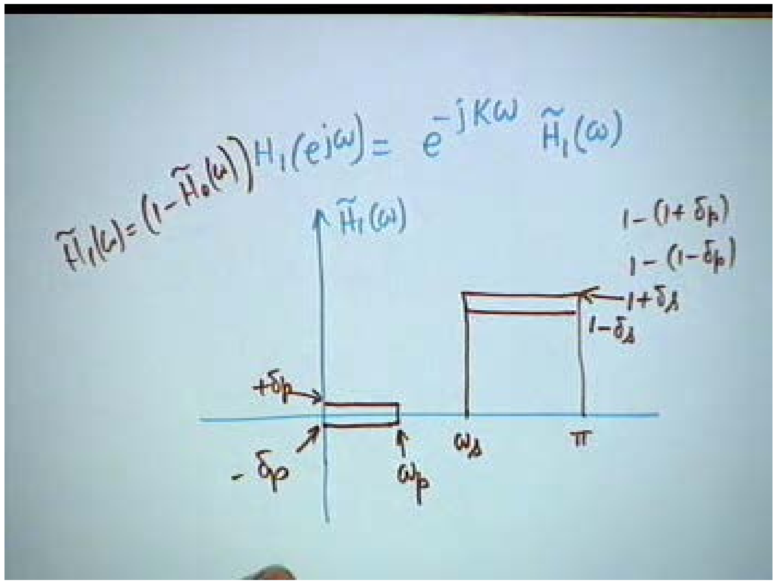

Consider pseudo magnitude $\tilde{H}_1(\omega) = 1 - \tilde{H}_0(\omega)$. Now, in the Pass Band of H_0, the magnitude (which is the same as $\tilde{H}_0(\omega)$) lies between 1 + delta p and 1 – delta p; so, $\tilde{H}_1(\omega)$ will lie between + delta p and – delta p in the frequency range 0 to omega p. In the stop band, H_0 magnitude (which may be different from $\tilde{H}_0$), lies between zero and delta s, within the frequency range omega s to pi. $\tilde{H}_0$ therefore could lie between – δ_s and +δ_s. Corresponding, $\tilde{H}_1(\omega)$ will lie between 1 + δ_s and 1 – δ_s. Clearly, $H_1(e^{j\omega})$ is a high pass filter with stop band between 0 and ω_p, with tolerance δ_p and pass band between ω_s and π with tolerance $2\delta_s$. On the other hand, in the Low Pass Filter, the tolerance was 2delta p and delta s only. So, in general it stands to reason that the Delay Complementary Filter of a Low Pass Filter shall be a High Pass Filter and vice versa but the tolerance schemes are different; it has to be worked out. For a Band Pass Filter the Delay Complementary Filter will be a Band Stop Filter. For an All Pass Filter, the Delay Complementary Filter shall also be All Pass. [Although the example we took was that of FIR, complementary filters need not necessarily be FIR; they can be FIR as well as IIR.]

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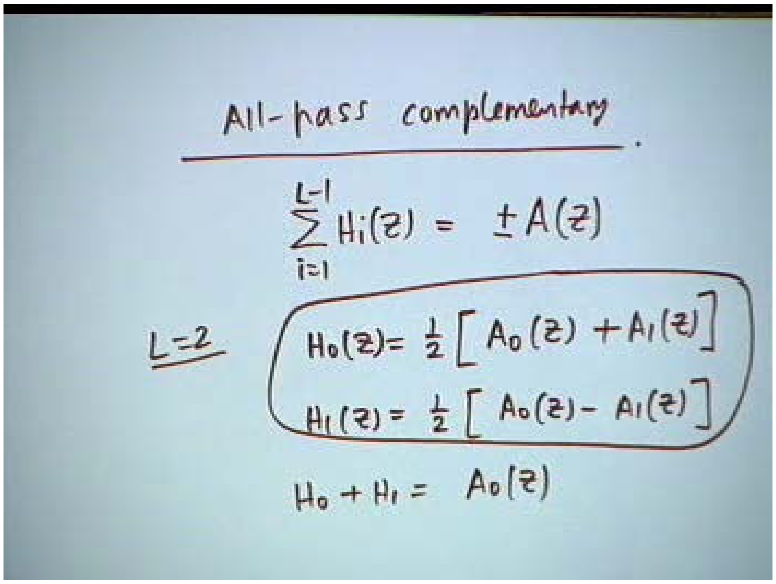

Let us now talk about All Pass Complementary Filters. A set of transfer functions H_i, where i = 0 to L – 1, is All Pass Complementary if their sum is an all pass filter. Delay Complementary Filter is also All Pass Complementary. Two delay Complementary Filters, H_0 and H_1 have H_0+H_1 = beta times z^{-n0}. What is this filter? It is all pass. If the components of this set are FIR then Delay Complementary is a special case of All Pass Complementary. Delay complementary filters can be realized with all pass filters. As a simple example, suppose L = 2 and suppose I take $H_0(z)$ = (1/2) ($A_0(z) + A_1(z)$) where $A_0(z)$ and $A_1(z)$ are All Pass, then $H_1(z)$ shall be equal to (½)(($A_0(z)$ – $A_1(z)$)). Now let me also point out that this sum can be either positive or negative, as far as magnitude is concerned, it does not matter. So when I add these two, I get $H_0 + H_1 = A_0(z)$. Now you can see how All Pass is gradually coming into the picture. I have already stated that if you can design an All Pass Filter, you can design any other kind of filter.

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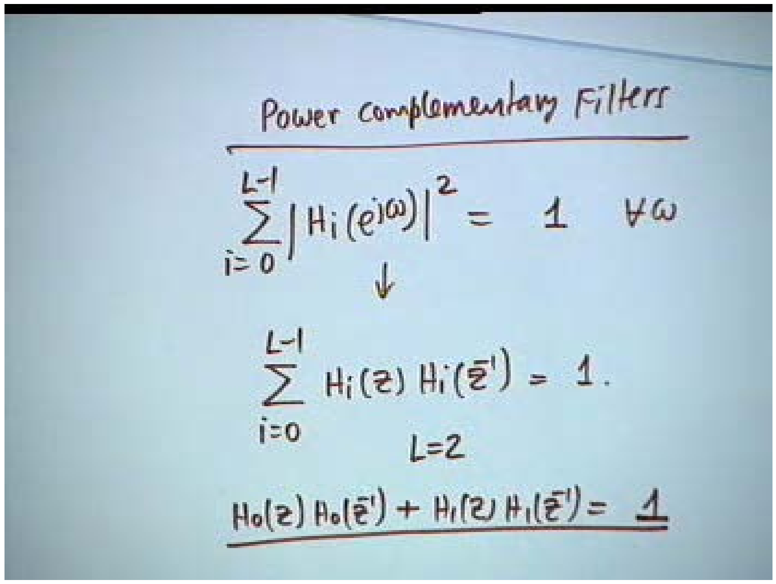

We will conclude this class with the definition of Power Complementary Filters. The set H_i is power complementary if their powers (which are characterized by magnitude squared of the transfer function) add up to a (normalized) value of 1; We did not say energy, because magnitude squared is not energy, but why is it power? It is power because for energy you require (1/(2 pi)) integral of magnitude squared with respect to omega. Since the sum is unity for all omega, then by analytic continuation, you can also say that summation $H_i(z)$ $H_i(z^{-1})$ i = 0 to L – 1, is unity.

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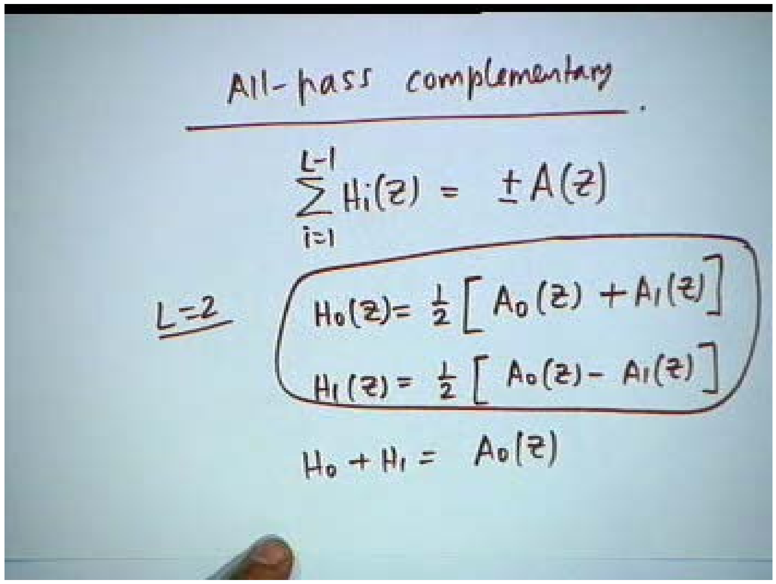

Suppose L = 2. Next time we will start with a very interesting observation that two filters which are All Pass Complementary are also Power Complementary. That is, if $H_0(z)\ H_0(z^{-1}) + H_1(z)\ H_1(z^{-1}) = 1$, then H_0+H_1=all-pass and vice-versa. You can prove it by taking H_0 as (½) $[A_0 + A_1]$ and H_1 as (1/2) $[A_0 - A_{1,}]$.

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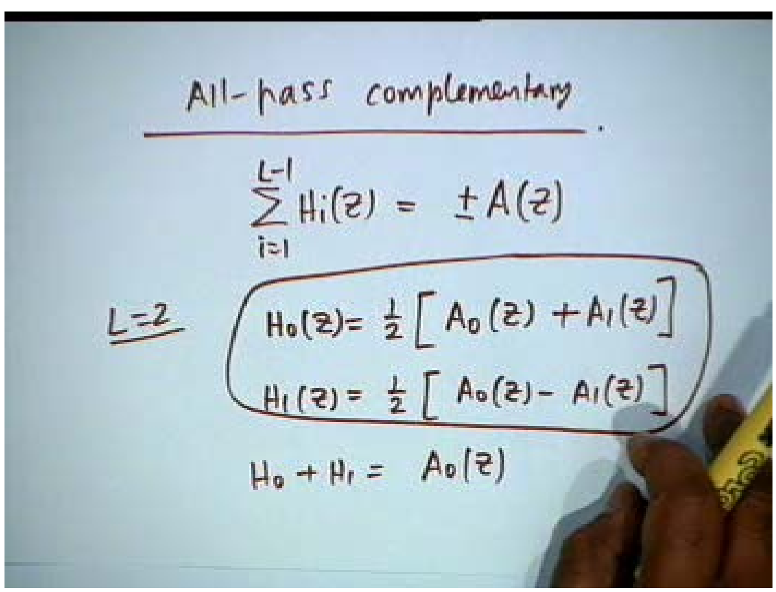

Therefore this set of filters is All Pass Complementary as well as Power Complementary and they are known as Doubly Complementary Filters. There are very important applications, and we shall look at some of them at a later date.