Digital Signal Processing

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Lecture - 15

Simple Digital Filters

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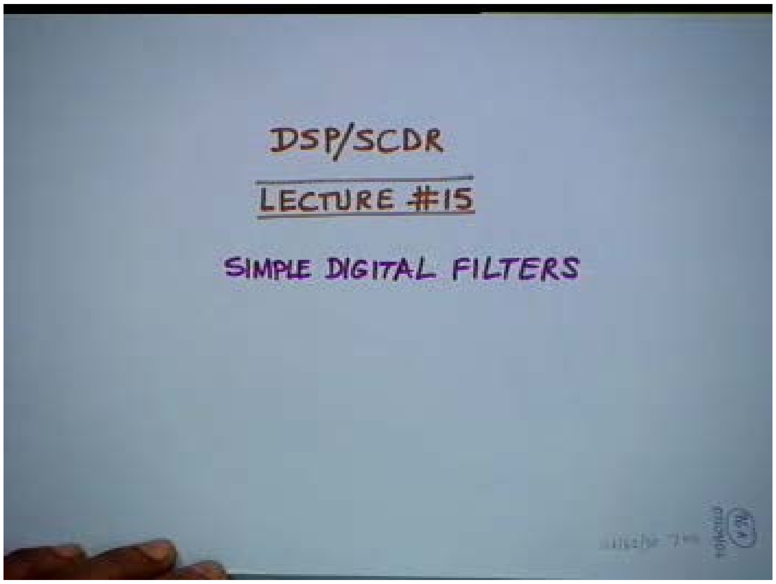

This is the 15th lecture on Digital Signal Processing and we continue our discussion on simple digital filters. In the last lecture, we introduced the concept of filtering and we explained and derived why an ideal characteristic cannot be realized. There are two reasons for this; one is that the impulse response is not zero for n less than 0, that is, an ideal filter is necessarily non-causal and it is also unstable because h(n) is not absolutely summable. Since an ideal characteristic cannot be realized, the actual characteristic must have a pass band, a stop band and a transition band. The character of the transition band determines the quality of the filter; the narrower it is, better is the filter, and sharper is the rejection of unwanted frequencies. We took an example of a high pass filter and we showed how to design the filter by taking a minimum possible order.

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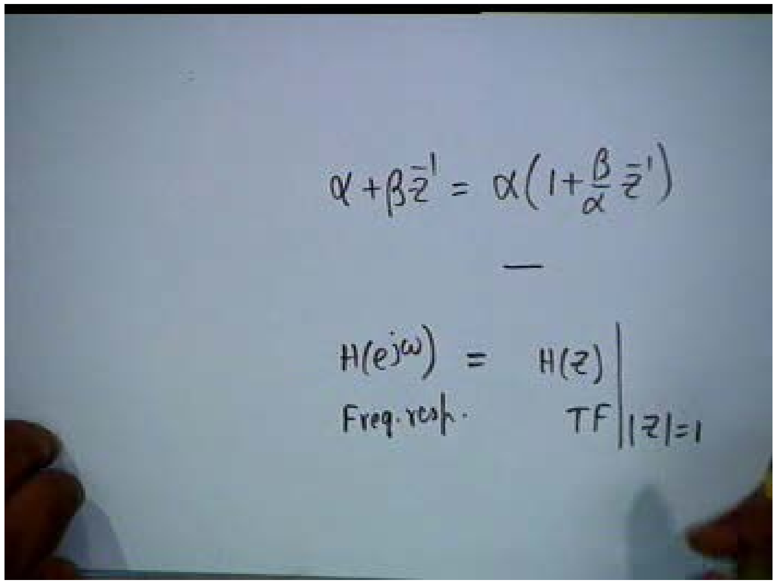

In the example of the band pass filter, could we have a first order filter to satisfy the specifications? Let us see. But if you have only two samples in the impulse response, the transfer function would have been H(z) = alpha + beta z^{-1}. Virtually, it is only one constant that can be determined. On the other hand, if we have alpha, beta and again alpha, (it is a special kind of filter, we shall look at it more carefully at a later date), we can satisfy specification at two frequencies, as the example demanded. At one frequency, the magnitude is 1 and at another frequency the magnitude was 0. If it was a band of frequencies over which the response is 1 or 0 obviously you cannot satisfy with any order. Why not? This is also an ideal characteristic. Absolutely flat characteristic over a certain band is not possible to realize. So, if it was a band of frequencies then all you can do is approximate by 1 or 0 within a certain tolerance. We shall have to specify the tolerance (in decibels) in the pass band, which will be very small, 3 dB or less. 3 dB tolerance is a rather low quality filter. Attenuation in the stop band also has to be specified. We also said that, in general, the frequency response H(e^{jomega}), which is the Fourier Transform of the impulse response, will be a rational function of $e^{-jomega}$. There is no reason why we cannot obtain this from the z transform of h(n), provided the ROC of H(z) includes the unit circle. If the variable is z, then we call it the transfer function and if the variable is e^{jomega} then we call the function as frequency response. There is no difference between the two, that is $H(e^{jomega})$ is H(z) on the unit circle where magnitude z = 1. Then we took some simple FIR filters.

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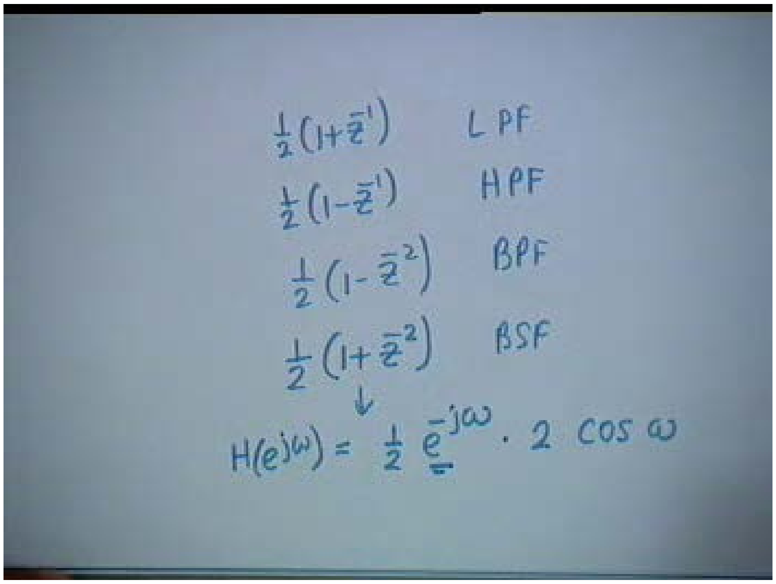

We showed that $(1/2)(1 + z^{-1})$ is a low pass filter. This is an elementary filter. Another elementary filter is $(1/2)(1 - z^{-1})$ which is HPF (High Pass Filter). We also told you that, if z in a low pass function is replaced by – z, it becomes a high pass function. I also gave you the reason: because –1 is e^{jpi}, therefore what is true at omega = 0 in LPF/HPF shall now be true at omega = pi in HPF/LPF and therefore low pass becomes a high pass, and vice versa. This is in general true. This means that in an FIR filter, alternate h(n) samples are changed in sign. Then we talked about a band pass filter; we said that if we cascade an elementary LPF and an elementary HPF and change the normalizing constant to ½, i.e. if we take the transfer function $H(z) = (1/2)(1 - z^{-2})$, then we get a band pass filter. We concluded the class with a discussion of the band stop filter which is $(1/2)(1 + z^{-2})$. This is band stop filter because if you take the frequency response, that is, if you put $z = e^{jomega}$ then you get $H(e^{jomega}) = (1/2) e^{-jomega} \times 2 \cos(omega) = e^{-jomega} \cos\omega$.

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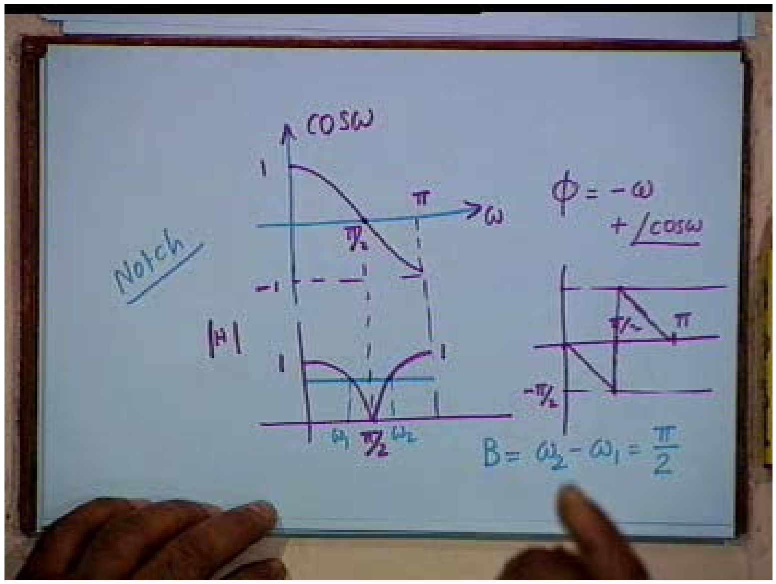

Now one has to be very careful about the phase. Cosine omega starts from 1 at $\omega = 0$ and goes to – 1 at $\omega = \pi$, changing sign at $\omega = \pi/2$. Once it transits from positive to negative value the phase abruptly changes by pi. Therefore the magnitude of H shall be the magnitude of cosine omega. At pi/2 it is 0 and then it flips because we are taking magnitude. Therefore, it is indeed a band stop filter. If you want the angle phi, it shall be = – omega + angle of cos(omega). Therefore if I plot phi, then it will go from 0 at $\omega = 0$ to – pi/2 at ω = pi/2. Then at pi/2, angle of cosine omega becomes pi so phase jumps from – pi/2 to + pi/2; then it again falls linearly to 0 at pi. The phase plot is piece-wise linear. One must be careful about the phase. As far as magnitude is concerned, it is indeed a band stop filter having a center frequency of pi/2. Center frequency is not the term for band stop. For band stop, we say it is the notch frequency. Band stop filter is also known as a notch filter. Notch is a common word, wherever there is a small hole we call it a notch. The notch frequency is pi/2. The bandwidth in this case is not determined from magnitude = 1/square root of 2. The two 3dB frequencies here are omega$_1$ and omega$_2$, and omega$_2$ – omega$_1$ is also equal to pi/2. Actually omega$_1$ would be pi/4 and omega$_2$ would be pi/4 + pi/2. The rejection band width for a band stop filter will be defined by the minimum tolerable attenuation. If this is α (typically .001 $\equiv$ 60 dB or .0001 $\equiv$ 80 dB) then the rejection bandwidth is ωs1 – ωs2 where ωs1 and ωs2 are the frequencies at which magnitude $= \alpha$. These are the elementary filters. For the band pass filter, for example, the center frequency is exactly at pi/2. For band stop filter the notch frequency is at pi/2. How do you design arbitrary band pass and band stop filters? We shall come back to a detailed discussion of this topic at a later date. But at this point of time, you have been introduced to four elementary filters, namely, LPF, HPF, BPF and BSF.

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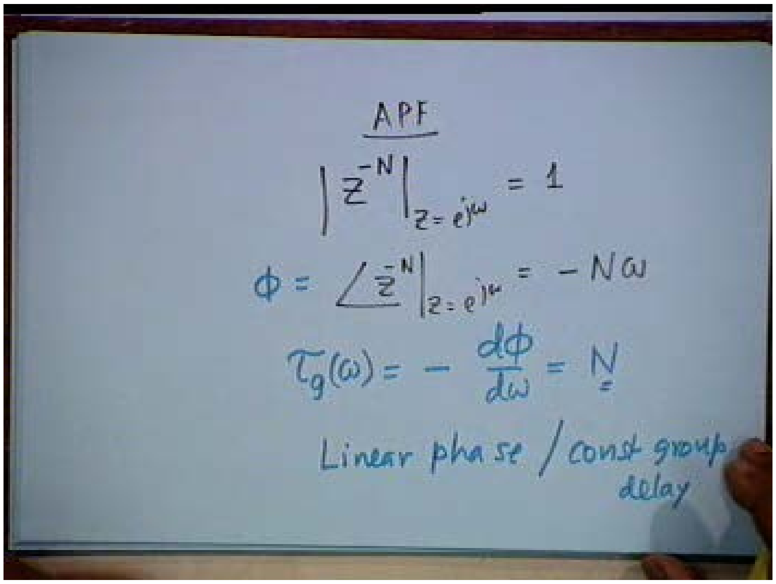

What about APF, i.e. All Pass Filter?

FIR All Pass Filters are trivial ones. They have the transfer function of the form z^{-N}. If you put z = e^{jomega} then the magnitude is 1. What is an All Pass Filter? It is a filter whose magnitude is 1 or a constant over all frequencies. Its angle will be different at different frequencies. For example, the magnitude of z^{-N} for z = e^{jomega} = 1 and the angle is = – N omega. So the angle is linear with respect to frequency. If you call the phase as phi, then the group delay tau_g(omega), which is defined as the negative gradient of phase, is simply equal to N. In other words,, if there is a group of frequencies transmitted over a channel whose transfer function is z^{-N} then all of these frequencies shall arrive at the output N samples later. That is what you mean by delay. The group delay is the delay suffered by a group of frequencies. If the group delay is not a constant, then we have a problem, because several frequencies start together and if they do not arrive together, then there is distortion in the received waveform or the received signal. The group delay will be constant only when the phase is linear. So, linear phase which is equivalent to constant group delay is an extremely important objective in designing any filter or any communication channel. Now let us look at some simple IIR filters.

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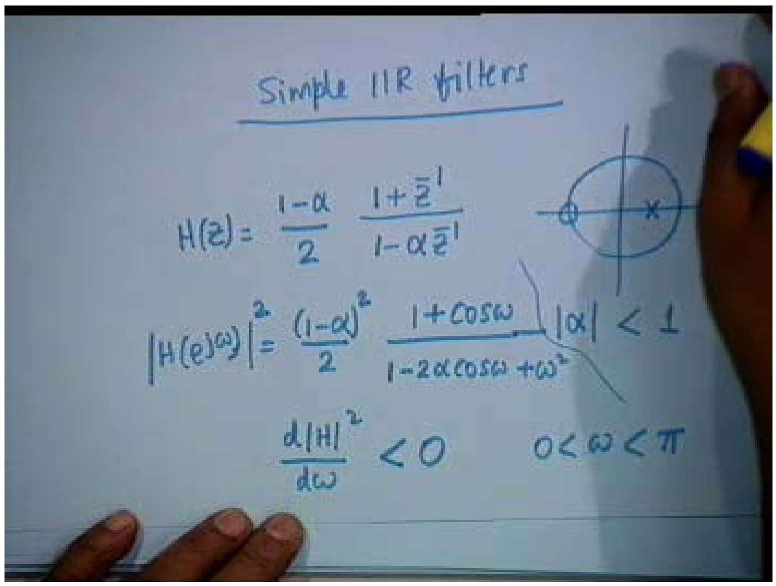

You will see how we derive such filters using common sense. We know that $(1 + z^{-1})$ is a low pass filter. This is FIR and therefore you must add a pole to make it IIR. Let the pole be at alpha; then this becomes an infinite impulse response filter. This is the simplest IIR LPF which has a zero at z = – 1 and a pole at z = alpha. We take alpha as a real positive quantity. But obviously, for stability, alpha, whether it is real or complex, positive or negative, must be less than 1 in magnitude; i.e. the pole must be inside the unit circle. To normalize the filter, we make the dc response as 1. The dc response of $(1 + z^{-1})/(1 - \alpha\ z^{1})$ would be 2/(1 – alpha). So we multiply by (1 – alpha)/2; this is the scaling constant. So my simple IIR low pass filter becomes [(1 – alpha)/2] × $(1 + z^{-1})/(1 - \text{alpha } z^{-1})$; this is the simplest possible first order low pass digital filter. We want the transmission at z = – 1, that is omega = pi to be 0; this is satisfied by the numerator. If you put $z = e^{jomega}$ and find out the magnitude response, you get $|H(e^{jomega})|^2 = [(1 - \text{alpha})^2/2]$ × (1 + cos(omega))/(1 – 2 alpha cos(omega) + alpha^2). This is the magnitude response. You can show by differentiating that the magnitude response is monotonic, that is $d|H|^2/d(\text{omega})$ is always less than 0 in the range 0 to pi. This is our range of vision. Therefore the magnitude response is monotonic, that is it falls monotonically from 1 to 0.

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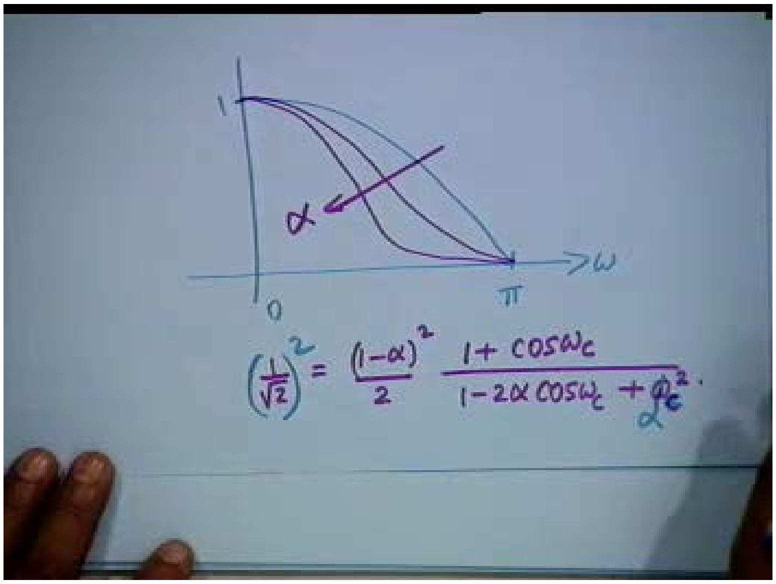

The only parameter for this LPF is alpha. Unlike the simplest FIR which is (1/2) (1 + z inverse), the cut-off frequency can now be controlled by alpha. And if you plot the characteristics, it would be like that shown in the figure. As alpha increases, the characteristic becomes better and better. How do you find the 3 dB cut-off frequency? The maximum is 1.

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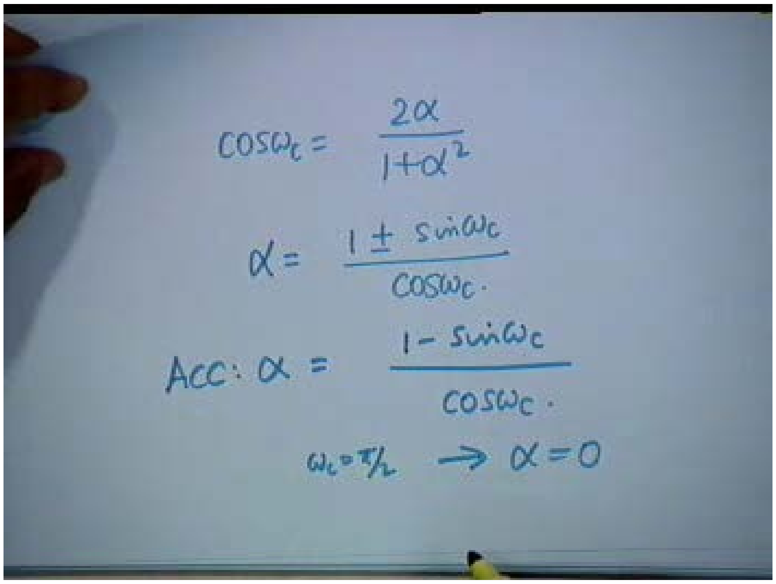

So you require (1/square root of 2)^2 = [(1 – alpha)^2/2] × (1 + cos(omega$_c$))/(1 – 2 alpha cos(omega$_c$) + alpha^2) where omega$_c$ is the cut-off frequency. If you solve this then you get the value of cos(omega$_c$). It is a linear equation in cos(omega$_c$) and cos(omega$_c$) comes as 2alpha/(1 + alpha^2). Now, omega$_c$ is controlled by alpha. There is only one parameter alpha and therefore you can only control omega$_c$. If omega$_c$ is given then to find out alpha we have to solve a quadratic equation. The solution, for a given omega$_c$ comes as (1 ± sin(omega$_c$))/cos(omega$_c$). It turns out that (1 + sin(omega$_c$))/cos(omega$_c$) is a quantity greater than 1 and (1 – sin(omega$_c$))/cos(omega$_c$) is less than 1. Therefore acceptable alpha = (1 – sin(omega$_c$))/cos(omega$_c$). This is the simplest IIR low pass digital filter. If omega$_c$ is given as pi/2 what is alpha? From α = (1 – sin ω_c)/cos ω_c , α becomes of the form 0/0. So go back to the defining equation for ω_c, put $\omega_c = \pi/2$ and get α = 0. This is nothing but the elementary FIR LPF.

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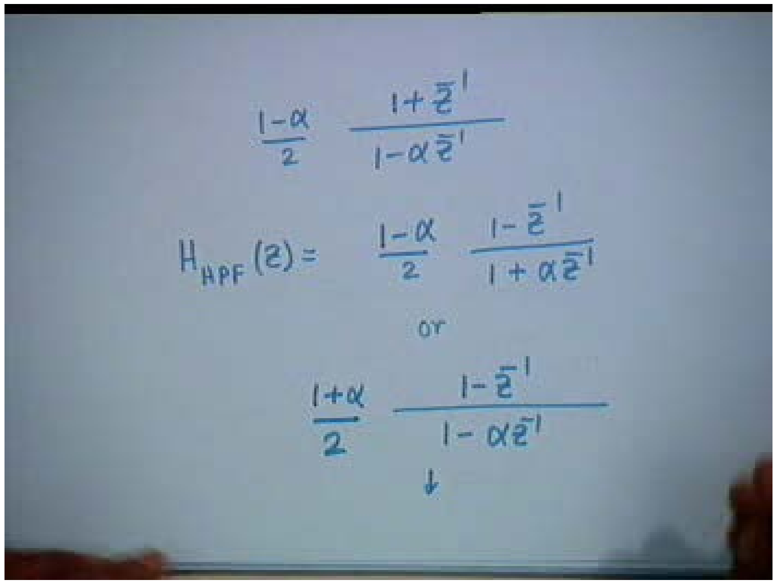

Now, if you want a high pass filter, $H_{HPF}(z)$, change z to – z. So, the transfer function would be $H_{HPF}(z)$ = [(1 – alpha)/2] × (1 – z^{-1})/(1 + alpha z^{-1}). Suppose I want to keep the pole at the same place as that of the LPF, that is at z = alpha. High pass filter is assured because the numerator is (1 – z^{-1}). All that I have to change is the multiplying constant. The transfer function of the HPF is now [(1 + alpha)/2] (1 – z^{-1})/ (1 – alpha z^{-1}). The magnitude characteristics would be identical irrespective of whether the pole is at $+\alpha$ or $-\alpha$. You can verify this.

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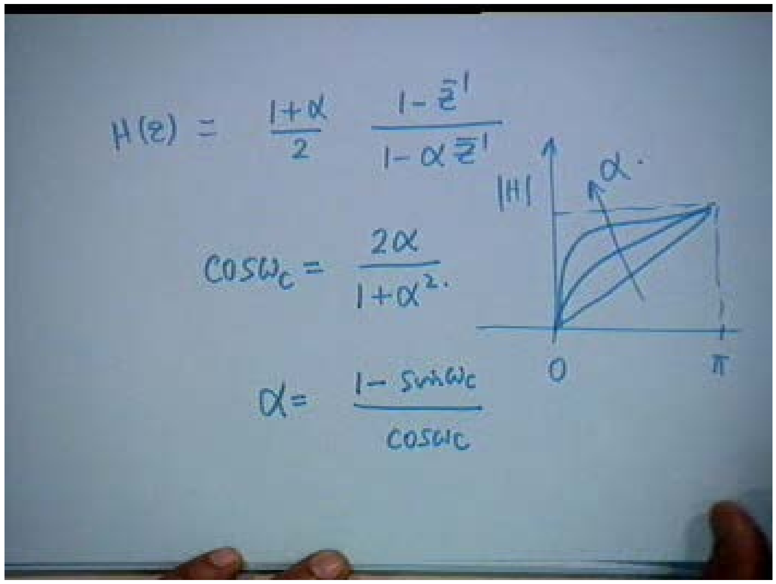

Let us consider H(z) = [(1 + alpha)/2] × (1 – z^{-1})/(1 + alpha z^{-1}), where we have dropped the subscript HPF. The magnitude characteristics versus omega in the range (0 to pi) would be just the complement of the low pass characteristic. Now can you tell me in which direction alpha increases? It is upward. As alpha increases, the characteristic approaches the ideal more and more. You can once again put z = e^{jomega}, then find out the magnitude response and the cut off frequency. It turns out that the cut off frequency is identical to that in the previous case, that is $\cos\omega_c$ = 2alpha/(1 + $alpha^2$). Once again if you have to design a filter for a specified $omega_c$, then you have to solve for alpha. The acceptable alpha would be (1 – sin($omega_c$))/cos($omega_c$). You can do this algebra yourself. Next we consider the question of a band pass filter.

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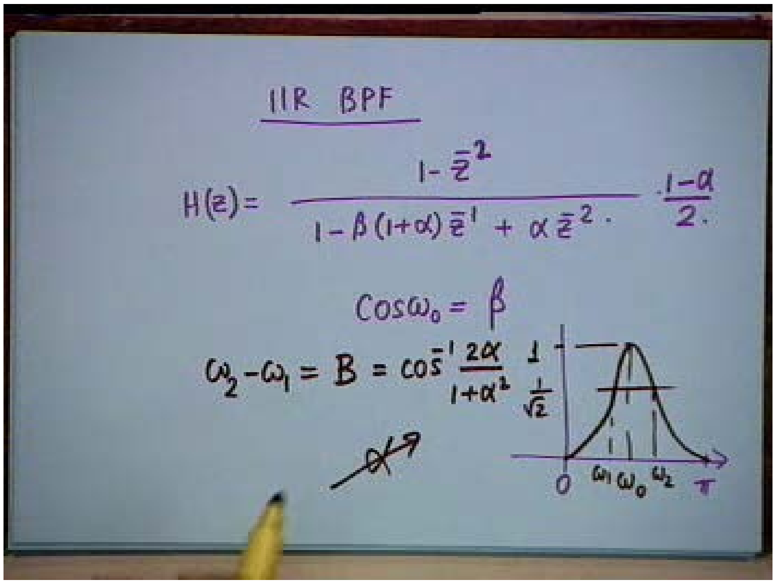

We are looking for the simplest band pass filter. As you saw in the FIR case, the minimum order needed was 2. In fact, you cannot make a first order band pass filter. The minimum order that you need is 2 and by trial and error the transfer function that has been worked out is the following: H(z) = P(z)/Q(z) where Q(z) = 1 – beta (1 + alpha) z^{-1} + alpha z^{-2}. The numerator P(z) is the same as that of the FIR case, that is (1 – z^{-2}) and to make sure that the maximum value is unity a multiplying constant of (1– alpha)/2 is needed. Now you can see that at z = $\pm$ 1, the magnitude will be 0. Therefore, the first condition for band pass is met.

Second, where does the maximum occur? It involves a bit of algebra. If you carry this out, you can show that the maximum occurs at cosine (omega_0) = beta. That is, the characteristic would be something like this. The range of ω is from 0 to pi. The maximum value is 1 at omega = omega_0 and then you can find out the cut off frequencies by equating $|H(e^{j\omega})|$ to $1/\sqrt{2}$. Let these frequencies be called omega_1 and omega_2. You can show that the magnitude is down to 3db at frequencies omega_1 and omega_2 such that omega_2 – omega_1, which is defined as the bandwidth, is again given by cosine^{-1} 2 alpha/(1+alpha^2). You saw that the same holds for low pass as well as high pass cosine (omega_c) where omega $_c$ is the cut off frequency and also the bandwidth for the low pass filter. For the high pass filter, what is the bandwidth? It is pi – omega_c. Here there are two cut off frequencies, because it is a band pass filter. The two cut off frequencies differ by the bandwidth which is given by the expression cosine^{-1} (2 alpha)/(1+alpha^2).

Once again, if you want to find out alpha for a given bandwidth then you have to solve a quadratic. You notice two interesting things, namely that the two constants alpha and beta independently determine one performance parameter of the band pass filter. Beta determines the center frequency and has no effect on the bandwidth, whereas alpha determines the bandwidth. So for the same center frequency, you can get various bandwidths by simply varying alpha and alpha is simply a multiplier. If you change the multiplier, you change the bandwidth and can get the desired value. This is an extremely important feature of any filter, that the parameters should be controllable independently of each other. What is omega_0/bandwidth? That is called the Q of the filter. So the Q of the filter cannot be changed independently of the center frequency. This transfer function has been worked out by trial and error; it is a very versatile transfer function. It has two parameters which can independently control the center frequency and the bandwidth. Where are the zeros of this transfer function? It is zero at z = + 1 (needed for making the dc response = 0), and at z = – 1 (needed to make the response at pi = 0).

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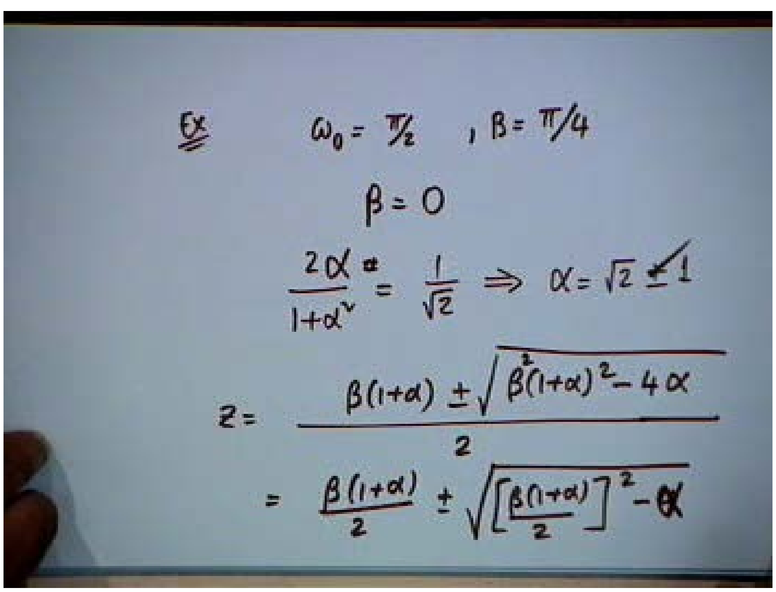

Where are the poles? You can work it out. They are complex. With real poles, you can achieve very little; the Q is very restricted. You have to make complex poles to get high Q. As an example, suppose omega$_0$ = pi/2 and bandwidth = pi/4. The value of Q is 2. Then here beta = cosine (pi/2) = 0. Then 2 alpha/(1+alpha^2) = cosine (pi/4) = 1/root 2. This gives two values of alpha; alpha is $\sqrt{2} \pm 1$. Obviously we should choose – sign, because then, you can show the poles are inside the unit circle. Can alpha be chosen to be (root 2) + 1? You have to look at the poles. If the poles go out of the unit circle then you are in trouble. My guess is that alpha (root 2) + 1 may take the poles outside the unit circle. Let us verify. Poles are at [beta (1+alpha) $\pm$ square root of (beta^2 (1+alpha)^2 – (4 alpha))]/2. So it is [beta (1+alpha)]/2 $\pm$ square root of ([(beta (1+alpha)/2]^2 – alpha).

Since $\beta = 0$, the poles are at $\pm j\sqrt{\alpha}$. Indeed, $\alpha = \sqrt{2+1}$ takes the pole outside the unit circle. Hence $\alpha = \sqrt{2-1}$ will be our choice. The transfer function now becomes H(z) = $\frac{1-\alpha}{2}\frac{1-z^{-z}}{1+\alpha z^{-2}} = (1-\frac{1}{\sqrt{2}})\frac{1-z^{-z}}{1+\left(\sqrt{2}-1\right)z^{-z}}$.

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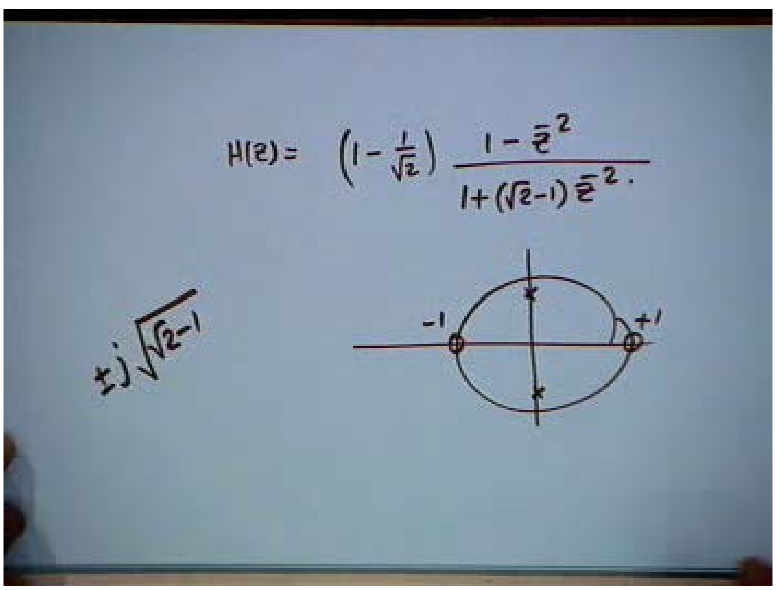

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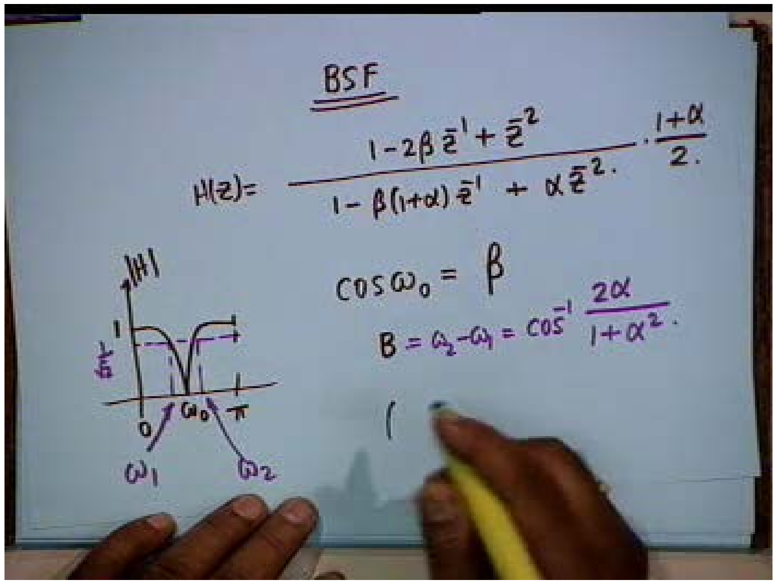

Next we consider a band stop filter. The band-pass and band-stop filters do not obey the correspondence that low pass and high pass obey. In a low pass filter if you replace z by – z, you get a high pass filter and vice versa i.e. if you take a high pass and replace z by – z, you will get a low pass filter. That is not true about band stop filter and band pass filter; they have to be derived ab initio. The simplest second order band stop filter that has been worked out has the same poles, that is the denominator is 1 – beta (1+alpha)z^{-1} + alpha z^{-2}. The numerator is not 1 + z^{-2} because that fixes the notch frequency at $\omega_0 = \pi/2$. So I introduce another term – 2 beta z^{-1} to be able to control the notch frequency. The multiplying constant is now (1 + alpha)/2. It is not simply changing z to $-z^{-1}$ in the band pass function. It has to be worked out independently.

Now, if you put z = $e^{j\ omega}$, find the magnitude and equate it to zero, then you get the notch frequency $omega_0$; you get a characteristic like that shown in the figure. The dc response as well as the response at pi are equal to 1; this has happened because the normalizing scaling constant was chosen as (1 + alpha)/2. We can show that $omega_0$ is given by the same relationship as in the band pass filter, that is, cosine ($omega_0$) = beta, $omega_2$ – $omega_1$, which are the frequencies at which the response is 1/root 2, once again, is given by the same expression, that is $cosine^{-1}$ [2 alpha/(1 + $alpha^2$)]. If alpha = 1 we get an all pass filter. But this all pass filter is useless, it is a trivial all pass filter. A straight connection is also an all pass filter. Here also alpha has to be less than 1 in order that the poles are inside the unit circle. Note that omega_2 – omega_1 is not the rejection bandwidth. That will be fixed by the tolerance permitted in the stop or rejection band. If $|H| \leq \gamma$ in the stopband, then the stop bandwidth will be $\omega_{s2} - \omega_{S1}$, when ω_{S2} and ω_{S1} are the frequencies at which $|H| = \gamma$. Normally, γ is much less than unity.

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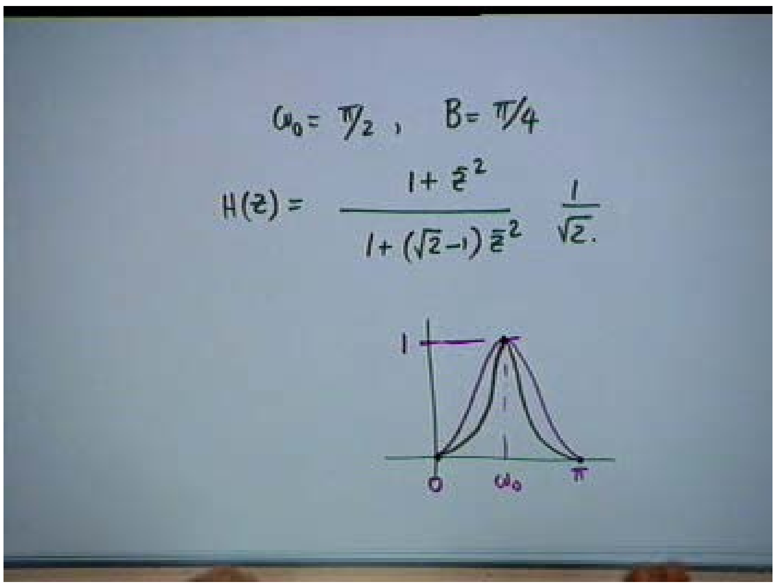

Let us take an example. If we want omega_0 = pi/2, then $\omega_2 - \omega_1$ = pi/4 as in the band pass filter. You can show that the required transfer function is $H(z) = (1 - \frac{1}{\sqrt{2}})(1 + z^{-2})/[1 + (\text{root } 2 - 1)z^{-2}]$.

The poles are at the same frequencies as in the previous example of band pass filter. This is the required IIR band stop filter. Suppose we wish to narrow down the bandwidth in the example band pass filter; a simple thing you can do is to cascade another such filter. Then all magnitude points, except 1 and 0, will come down, as shown in the figure for band-pass case. The bandwidth will shrink and the Q will increase.

If you cascade two band-stop filters, what would you get? Would you get a better characteristic or a worse characteristic? You will get a worse characteristic, because the stop bandwidth will increase. For a low pass filter, you can cascade more low pass filters to get a better characteristic. In fields like bio medical signal processing, such intuitions are used. The basic blocks used are, ½ $(1 + z^{-1})$, ½ $(1 - z^{-1})$, FIR band pass, FIR band stop, IIR bandpass and IIR bandstop. If one is not sufficient, cascading will be tried. But do not make this mistake in the case of band stop filter.

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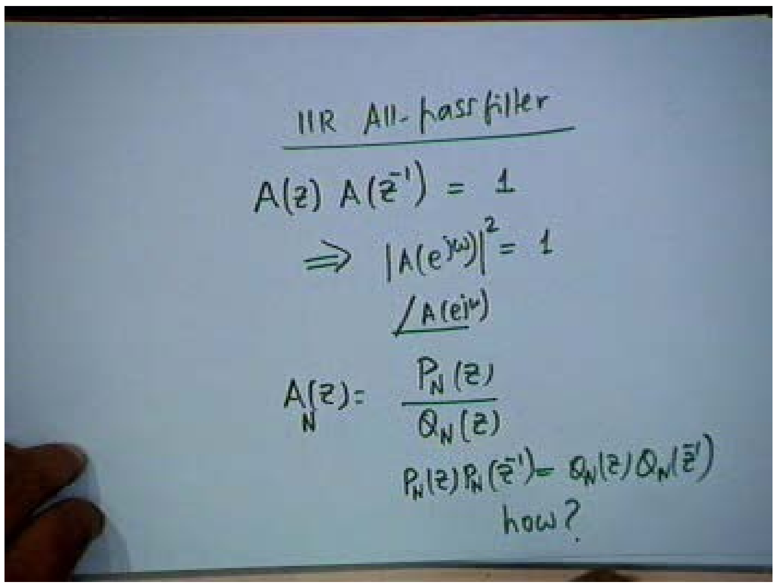

Now, we will consider IIR All Pass Filters. And to distinguish between all pass and all other kinds of filters, we shall use this symbol A for the transfer function for All Pass Filter. A normalized All Pass Filter is one in which $A(z)\ A(z^{-1}) = 1$ which means that the magnitude of $A(e^{j\ omega}) = 1$ and the angle of $A(e^{j\ omega})$ can be any desired value. If $A(z) = P_N(z)/Q_N(z)$, where N is order of the transfer function, we require that $P_N(z)\ P_N(z^{-1}) = Q_N(z)\ Q_N(z^{-1})$. We want the product $A(z)\ A(z^{-1}) = 1$. How to do this? To get an insight into the problem, let us first consider a first order all pass. First order will have one pole and one zero.

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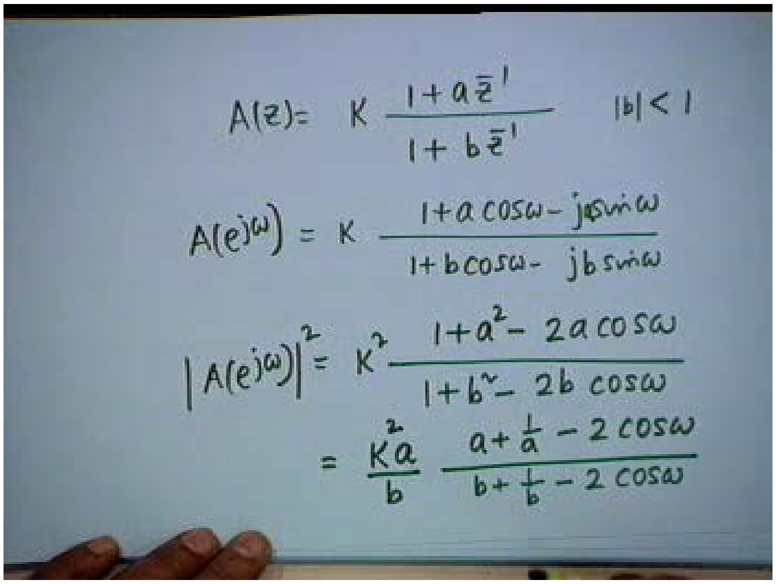

So let us consider $A(z) = k\,(1+az^{-1})/(1+bz^{-1})$. Of course, in order that the filter is stable, b should be bounded by unity because pole is at z = – b so mod b should be less than 1. Now we shall choose k such that $A(z)\,A(z^{-1}) = 1$. If I take the frequency response, i.e. we put $z = e^{j\ \mathrm{omega}}$, then I get $A(e^{j\omega})$ k[1 + acosine (omega) – j sine (omega)]/[1 + bcosine (omega) – jbsine (omega)] and if I take the magnitude square, I shall get k^2 [1 + a^2 – 2acosine (omega)]/(1 + b^2 – 2bcosine (omega))] which I can write as k^2a/b [a + (1/a) – 2cosine (omega)]/[b + (1/b) – 2cosine (omega)]. One way of satisfying that magnitude $A(e^{j\ \mathrm{omega}}) = 1$ is to make a = b and k = 1. But if I make a = b it is a trivial filter. It becomes A(z) = 1, it does not filter at all. It is a straight connection. The other way is to make a = 1/b, isn't that right? a + (1/a) and b + (1/b) would be identical if a = 1/b. That is what gives you a useful All Pass Filter.

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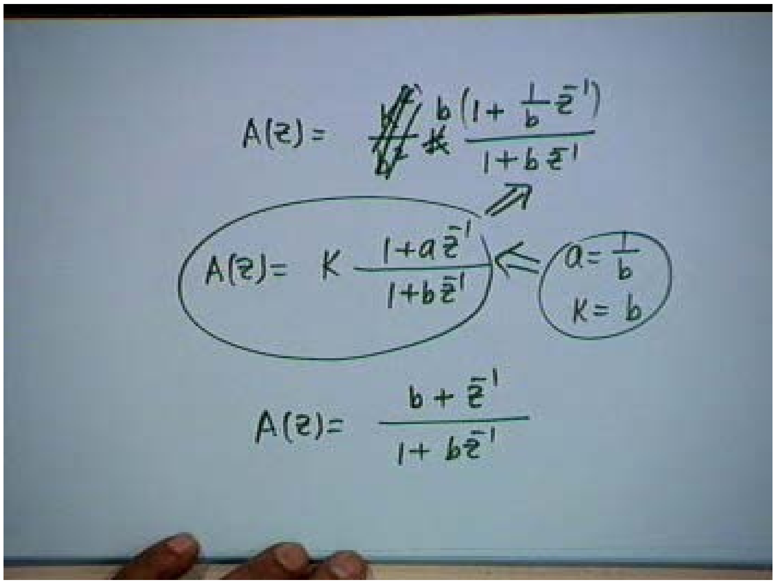

In other words, we get A(z) = $(k^2/b^2)[(1 + (1/b)z^{-1})/(1 + bz^{-1})]$. For A(z) $A(z^{-1}) = 1$, we need k = b. I substitute these here and get A(z) = $(b + z^{-1})/(1 + bz^{-1})$. This makes sure that the magnitude is 1, and also points to one very interesting thing.

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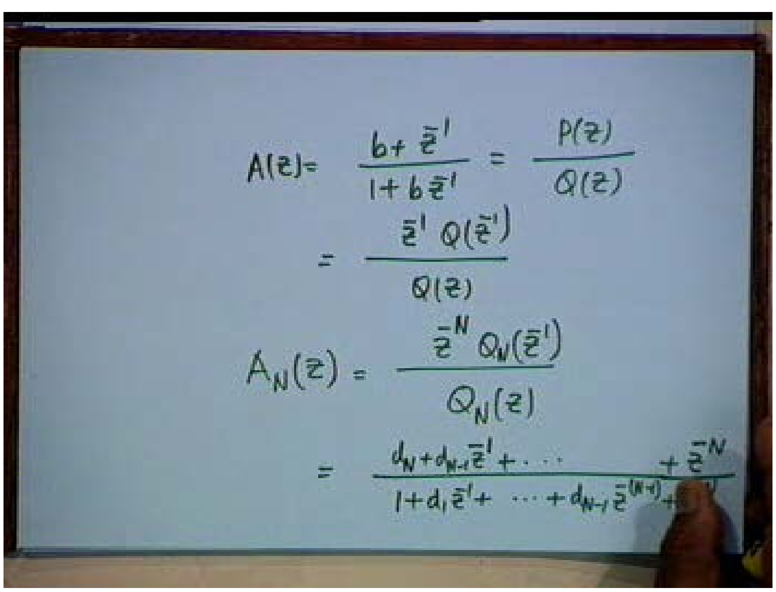

You look at this: $A(z) = (b + z^{-1})/(1 + b z^{-1})$; if I write this as $P(z)/Q(z)$ then you notice that $P(z)$ can be derived from $Q(z)$. Let us take $Q(z^{-1})$, what is $Q(z^{-1})$? $Q(z)$ is $(1 + bz^{-1})$. So $Q(z^{-1})$ must be $(1 + bz)$; then I multiply this by z^{-1}, what do I get? I get precisely $b + z^{-1}$; this is true in general. That is any IIR all pass function of order N whose magnitude is unity at all frequencies can always be written as $z^{-N} Q_N(z^{-1})/ Q_N(z)$. This can always be done. In other words, if my transfer function denominator polynomial is $1 + d_1 z^{-1} + \text{etc} + d_{N-1} z^{-(N-1)} + d_N z^{-N}$, then I can construct the numerator very easily. What is the numerator? Instead of z^{-1} you write z and multiply by z^{-N}. What will happen in the numerator? The coefficients will simply be reversed. So I shall get $d_N + d_{N-1} z^{-1} + \ldots. + z^{-N}$. Don't you see that this has reflected in the first order all-pass function?

All pass functions are very interesting functions, to the extent of being called romantic functions. You can do almost any thing with all pass. From all pass, you can derive a low pass, high pass, band pass, band stop and so on. Professor Mitra who is the author of our book is fascinated by all pass functions; not simply Professor Mitra, but the whole community. You can do anything with an all pass function. You see all pass functions also have a very interesting property of poles and zeros. If you know the poles, obviously you know the zeros also because they are determined by $z^{-N} Q_N(z^{-1}) = 0$; the poles and zeros are related to each other. And we shall show next time that they are in reciprocal pairs. That is if p_0 is a pole, $1/p_0$ must be a zero. So all pass functions are very easy to conceptualize, they are very easy to design and they are very easy to implement. We shall continue this discussion next time.