Digital Signal Processing

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Lecture - 13

Z transforms (contd...)

This is the 13th lecture on DSP and we continue our discussion of Z transforms today. In the last lecture, we talked about the Inverse Z transform in terms of an integral. That integral is also a special integral, because it is a contour integral, the variable being a complex one, $z = re^{j\ omega}$, so we have to choose the contour and then carry out the integration. If you have to evaluate the integral, then Cauchy's residue theorem can be used. Fortunately for us, z-transforms for LTI systems of importance are rational functions and therefore we can evaluate the inverse Z transform by either partial fraction expansion and then inverting term by term, or by long division. The disadvantage of long division is that you may not be able to guess a general formula for the inverse Z transform.

On the other hand, partial fraction expansion always gives you a closed form formula. I also pointed out that if the rational function is P(z)/Q(z) with degree of P(z) greater than the degree of Q(z), then this will be written as some polynomial in Z inverse + a proper rational function that is $A(z) + P_1(z)/Q(z)$ where the degree of P_1 is less than the degree of Q, that is, P_1/Q is a proper rational function. Then you invert A(z) term by term which is very easy. If you have term like $5z^{-5}$, then its inversion gives 5 delta (n – 5). For P_1/Q you have to carry out partial fraction expansion. I also pointed out that if there are multiple poles, then in general, you have to carry out differentiation. But the differentiation can be avoided by using a simple trick that I have asked you to follow. That is, write a set of linear equations by putting specific values of z and then evaluate the required constants. We pointed out the importance of region of convergence (ROC) and we showed that u(n) and – u(– n – 1) both have a z-transform $1/(1 - z^{-1})$. The ROC for u(n) is mod(z) greater than 1 and ROC for u(–n – 1) is mod(z) less than 1.

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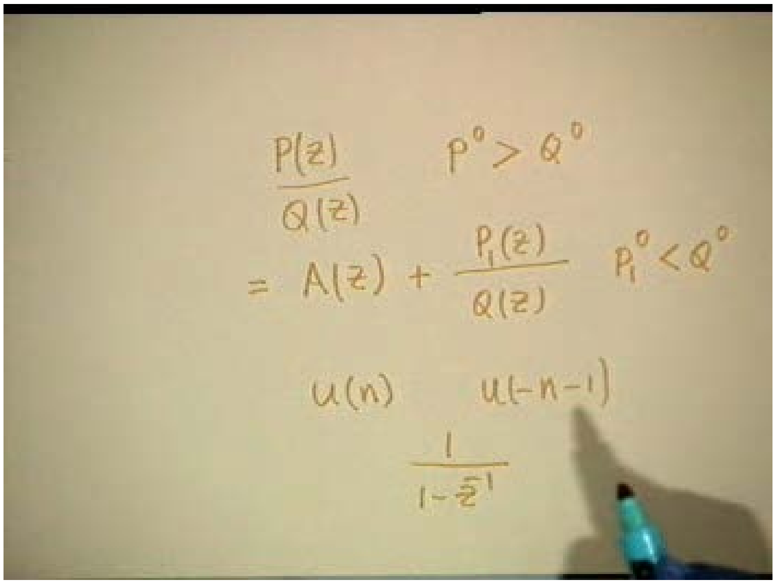

We also took an example in which there were three possible ROC's and three different sequences. The Z transform is a one to one transformation only if ROC is specified. Otherwise the answer can be non unique. We started with the properties of Z transform. Before that I had introduced some notations that is, for G(z), the ROC is denoted by R_g and for H(z) the ROC is denoted by $R_{h,}R_g$ is the symbol for inequality: R_{g1} is less than mod z less than $R_{g2.}$ Then we also used the notation $1/R_g$ which means $1/R_{g2}$ less than mod z less than $1/R_{g1.}$ Also the meaning of R_gR_h is: $R_{g1}R_{h1}$ less than mod z less than $R_{g2}R_{h2.}$ And then we talked about several properties of z-transforms.

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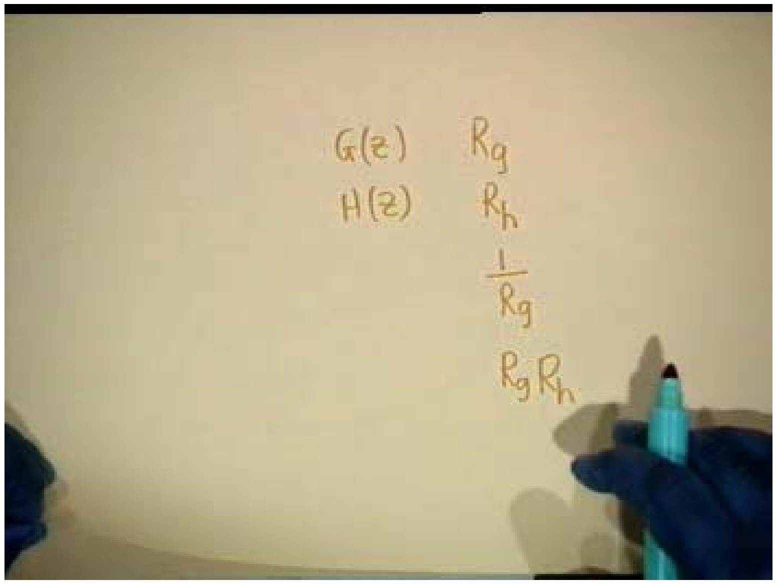

Firstly, g*(n) has the transform G*(Z*). The region of convergence is the same as $R_{g,}$ because mod z* is same as mod z. Then g(–n) transforms to G(1/z) and the region of convergence is $1/R_{g.}$ Alpha g(n) + beta h(n) transforms to alpha G(z) + beta H(z) and ROC includes the intersection of R_g and $R_{h.}$, but it can be wider. I explained why it can be wider; the possibility arises because there can be cancellation of zeros and poles.

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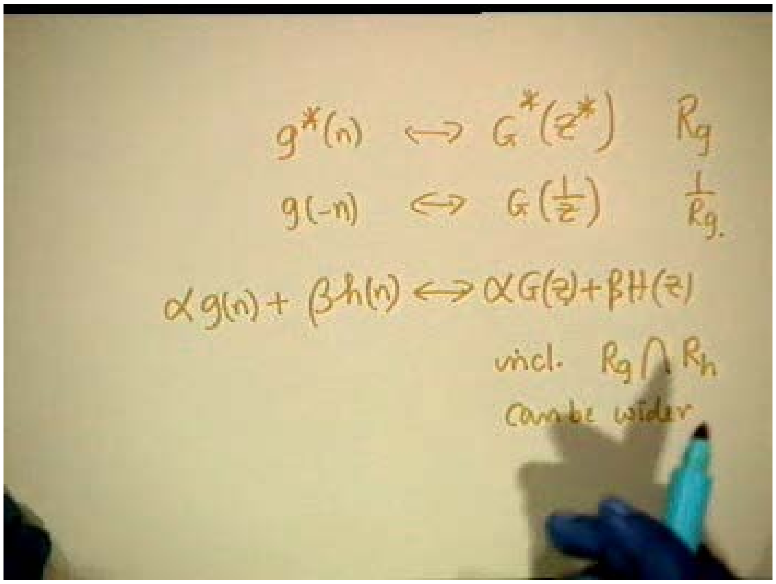

And then, if I shift a sequence by n_0 that is, if I take $g(n - n_0)$, the corresponding z-transform is z^{-n0} G(z) and region of convergence is R_g, except for either z = 0 or z = infinity depending on n_0 and cancellations. Suppose your n_0 is 1 and there is a term in the denominator which is z^{-1}; they may cancel. Therefore it is R_g with possible exception of z = 0 or z = infinity. If I multiply the sequence by $alpha^n$, then we get G(z/alpha) and the region of convergence is mod alpha times R_g. The next property is that of differentiation. That is, if you multiply a sequence by n then the corresponding z-transform is – z[dG(z)/dz]. You have to carry out the differentiation and this will give you a clue for inversion of $1/(1 – alpha\ z^{-1})^r$; that is the case of multiple poles. We shall carry out an example a little later.

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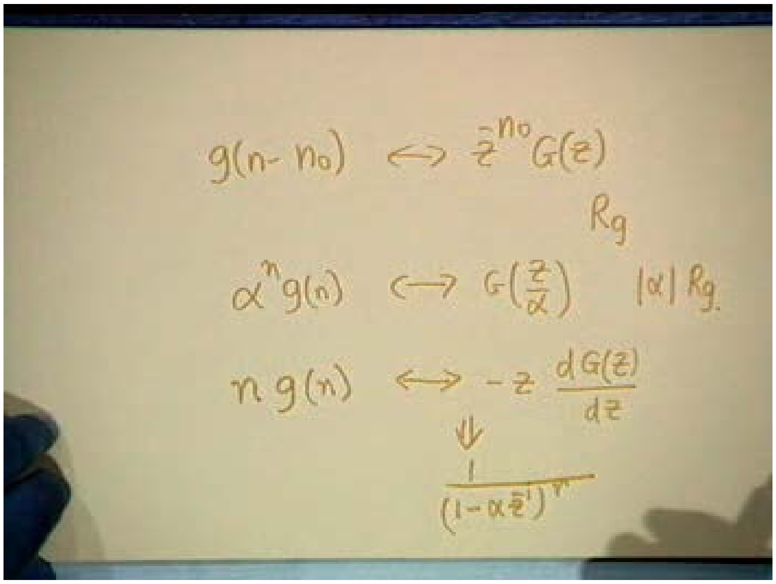

If you have a linear convolution of two sequences g(n) and h(n), the corresponding Z transform is the product G(z)H(z). What is the region of convergence here? I did not mention this. It would be the overlap of R_g and R_n, but it can be wider. There can be cancellations.

Finally, the modulation property: that is, if you multiply g(n) by h(n) the corresponding Z transform is a contour integral. This is a convolution in the Z domain. The corresponding Z transform is [1/(2pi j)] contour integral over some contour [G(v)H(z/v)v^{-1} dv; this is the formula for modulation. And I strongly urge you to prove each of these properties. For the modulation property, the clue is that you replace G(v) by its equivalent expression, that is, summation [g(n)v^{-n}]. Then interchange the summation and the integration. You should be able to prove it. Obviously, the ROC has the product $R_gR_{h,}$ in the sense that we have defined it; but again it includes R_g R_h, it can be wider because there can be cancellations.

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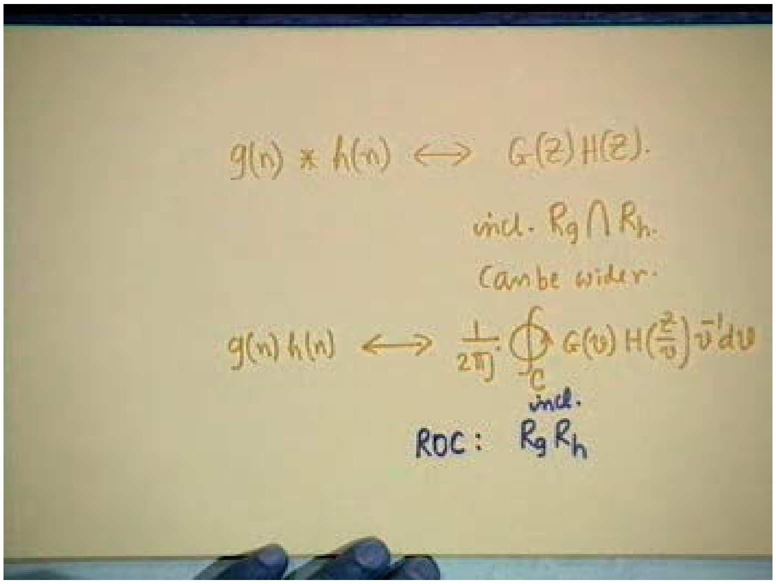

As a simple example of application of the table of z-transforms, let us consider, x(n) = r^nsin(n omega$_0$)u(n). We write this as [(1/(2j)] [r^n ($e^{jnomega0}$ – $e^{-jnomega0}$)u(n)]. The z-transform would be [1/(2j)][summation(r^n ($e^{jnomega0}$ z^{-n})– summation(r^n $e^{-jnomega0}$ z^{-n})], where n = 0 to infinity. We can calculate this; we can also view these two sequences r^n ($e^{jnomega0}$), and then r^n ($e^{-jnomega0}$) as complex conjugates of each other. In other words we could write this as [1/(2j)] [(v(n)) – v*(n)], where v(n) is (r^n $e^{jnomega0}$)u(n).

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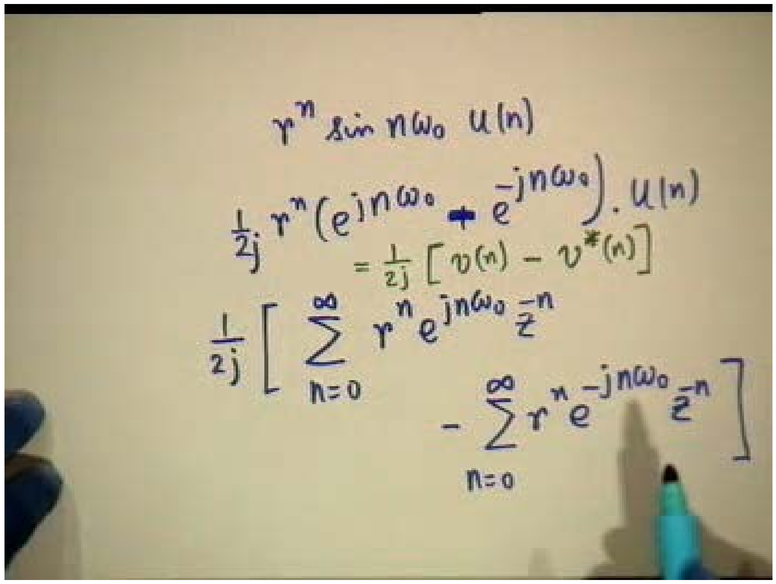

If we apply the formula then my desired transform is X(z) = [1/(2j)] [(V(z)) – V*(z*)]. Let us see what this is: [1/(2j)] [(1/(1 – $re^{j\ \text{omega}\ 0}$ z^{-1})) – (1/(1 – $re^{j\ \text{omega}\ 0}$ z^{*-1})) *]. Now what is this latter quantity? Obviously this is 1/(1 – $re^{-j\ \text{omega}\ 0}$ z^{-1}). Now you can add the two quantities and get the desired result. What is the result?

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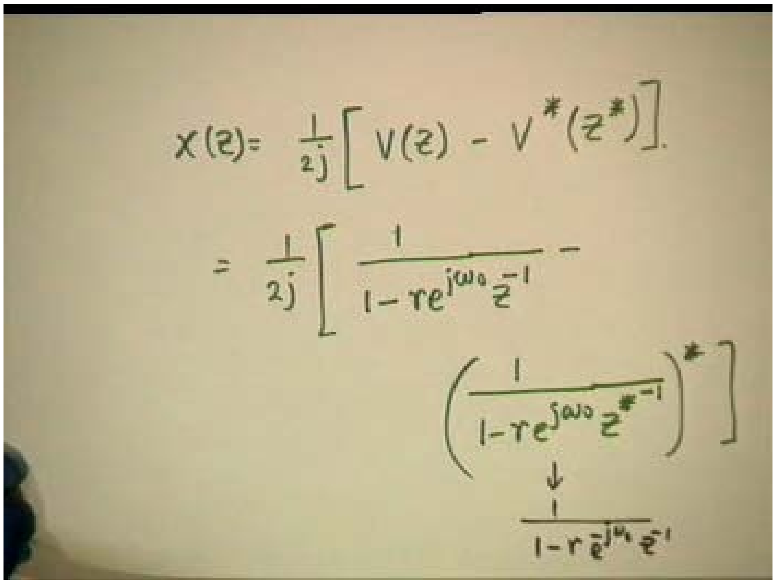

The result is (r cosine(omega$_0$) z^{-1})/[1 – 2r cosine(omega$_0$) z^{-1} + r^2 z^{-2}]. The transforms of sine, cosine, delta n, u(n), $e^{jnmega0}$ you should be able to write without consulting the table. You do not have to turn the pages of the book to find these out. If it is cosine, then what is the modification needed? The answer is (1 – r sine (omega$_0$) z^{-1})/(the same denominator).

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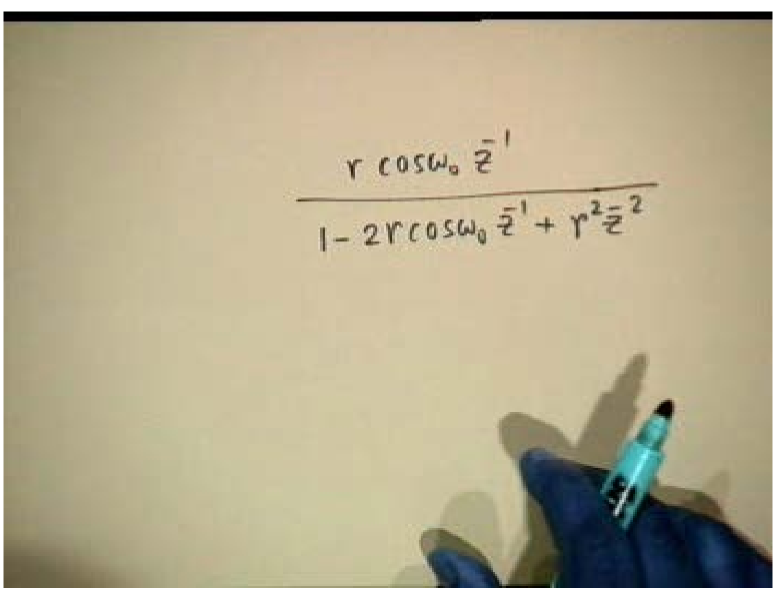

Now let us take an example of multiple poles and illustrate how we carry out the actual calculations. $X(z) = z^4/[(z - 0.5)^2(z - 0.2)(z + 0.6)]$; the ROC must be specified and in this case it is specified as mod z greater than 0.6 which means that it will be a right sided sequence and a causal sequence. So, the first thing we do is to write X(z) as a rational function z^{-1} that is we divide by z^4 then we get $X(z) = 1/[(1 - 0.5\, z^{-1})^2(1 - 0.2\, z^{-1})(1 + 0.6\, z^{-1})]$, Why do we do this? We do this because we know the inverse transform of $1/[1 - (\text{alpha})z^{-1}]$. We write $X(z) = A/(1 + 0.6\, z^{-1}) + B/(1 - 0.2\, z^{-1}) + C/(1 - 0.5\, z^{-1}) + D/(1 - 0.5\, z^{-1})^2$ and this creates the problem. A is $(1 + 0.6\, z^{-1})$ multiplied by X(z) with $0.6\, z^{-1} = -1$. Similarly, you can find B. You calculate D by multiplying both sides by $(1 - 0.5z^{-1})^2$ and by putting $0.5\, z^{-1} = 1$.

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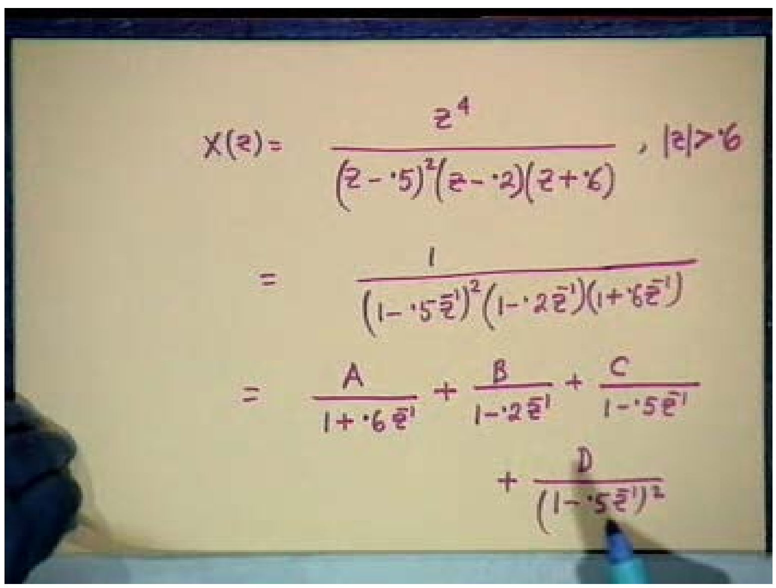

I have done this and the results are as follows: A = 27/121, B = 1/9, D = 25/33. There is a reason why I have kept them as fractions, I have told you the reason earlier also. The reason is that we do not truncate unless it is absolutely essential because in implementation we have to truncate anyway and because of finite word length constraint, we cannot represent every number exactly. So do not truncate till it is absolutely essential. Now the question of finding C. If you examine the partial fraction expression for X(z) what would be a convenient value of z to put? I have to write only one equation, because there is only one unknown. Convenient value of z obviously would be $z^{-1} = 0$ that is z = infinity. What would be X(z) at infinity? From the given expression it is 1; therefore 1 = A + B + C + D, you see it is as simple as that, you do not have to differentiate. You know A, B and D, so you can find C = 1 – (A + B + D); unfortunately, this number comes big but there is nothing to worry. You keep it as it is. The value is C = 116/1089. The problem now is that of inversion.

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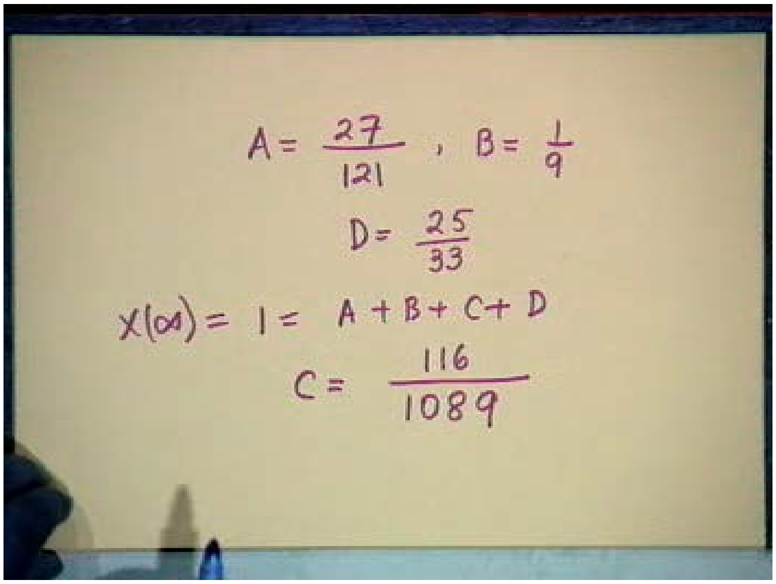

Inverse of the A, B, C term would be $A(-0.6)^n u(n) + B(0.2)^n u(n) + C(0.5)^n u(n)$. Now only the D term remains. Let us look at the general formula, then I will leave the rest for your calculation. General formula is that alpha^n u(n) has the transform $1/(1 - \text{alpha } z^{-1})$. If I multiply by n, then for $n(\text{alpha}^n)u(n)$ the z-transform would be $-z(d/dz)(1/(1 - \text{alpha } z^{-1}))$. If I differentiate with respect to z and simplify, the result is $\text{alpha } z^{-1}/1 - \text{alpha } z^{-1})^2$; what we want is that the numerator should be 1. Therefore first thing I do is to divide by alpha and advance the sequence by one sample. Hence the inverse of $(1 - \text{alpha } z^{-1})^{-2}$ would be $(n + 1)\text{ alpha}^n u(n + 1)$.

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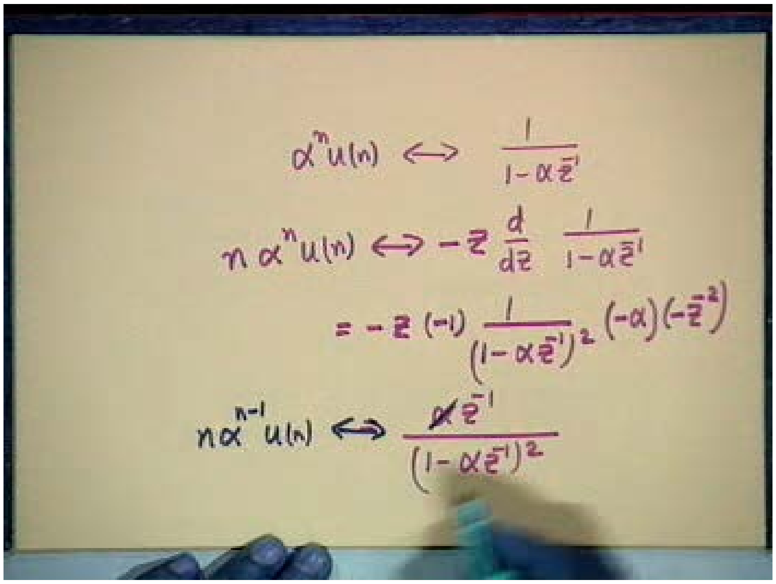

Here is something interesting that I want you to notice: 1/(1 – alpha $z^{-1})^2$, has the inverse transform (n + 1)alpha^n u(n + 1). Now this sequence at n = – 1 is 0. Therefore you can write this as (n + 1)alpha^n u(n). The whole thing is to be written in terms of u(n). You can now collate the final result and the expression that I get is x(n) = [0.2231(– 0.6)^n + 0.1111(0.2)^n – 0.0918(0.5)^n + 0.7576(n + 1) (0.5)^n]u(n). I have truncated the numbers here and n + 1 is the multiple pole term.

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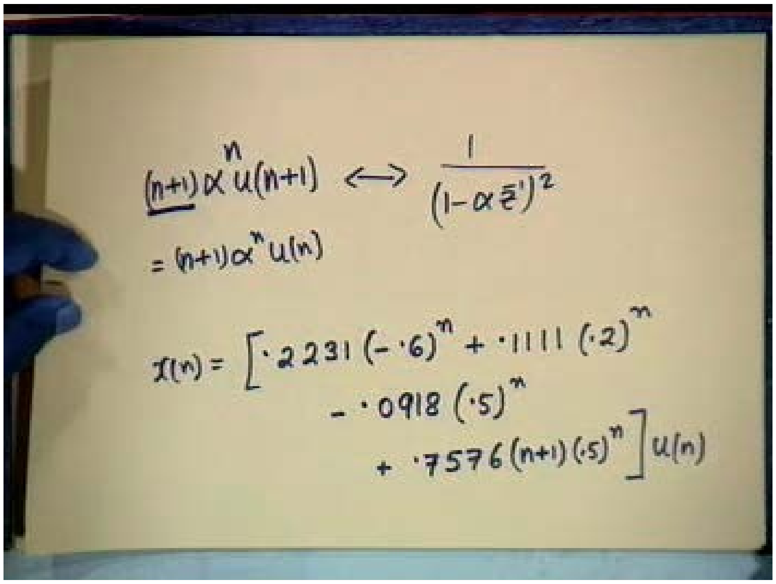

We go back to the modulation theorem that is if x(n) = g(n)h(n), then the z-transform is [1/(2 pi j)] contour integral[G(v)H(z/v)v^{-1}dv]. I can write summation[g(n)h(n)z^{-n}] = [1/(2 pi j)] contour integral[G(v)H(Z/v)v^{-1}dv] where n goes from – infinity to + infinity. In this expression, if I make g(n) = h(n) and put z = 1 then the left hand side simply becomes the energy of the sequence and the right hand side becomes [1/(2 pi j)] contour integral [G(v) G(v^{-1}) v^{-1} dv]. Since v is a dummy variable we can as well change it to z. So I can write the right hand side as: [1/(2 pi j)] contour integral [G(z)G(z^{-1}) z^{-1}dz]. Now if the region of convergence includes the unit circle then we know the Fourier transform exists. In other words the contour can be the unit circle. If FT of g(n) exists, then we could put z = e^{jomega}. A sufficient condition for the existence of Fourier transform is that g(n) is absolutely summable.

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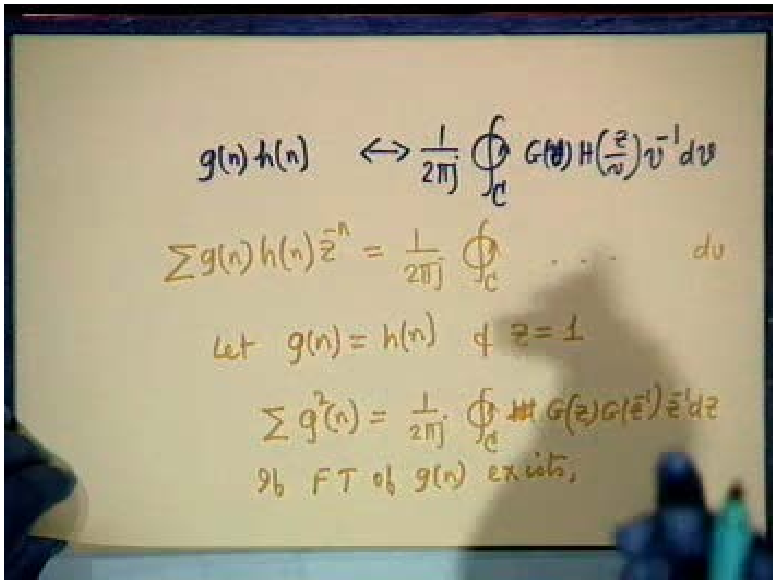

If g(n) is absolutely summable or in general if Fourier transform exists, then I get summation[g^2(n)] = [1/(2pi)] integral (– pi to + pi) |G(e^{jomega})$|^2$ d(omega). So we have proved Parseval‘s relation. Parseval’s relation is a consequence of the modulation theorem of Z transform. However I must caution you that there are sequences for which Fourier transform exists but z transform does not, and vice versa. If the Fourier transform does not exist, then we cannot apply this theorem.

However, we can apply the general modulation theorem in Z transform. Here g (n) was taken as a real sequence and we took g(n) = h(n). If g(n) was complex, then we would have taken h(n) = g*(n) and same result would have followed.

We have so far talked about signals. How do they interact with systems? That is, we take a discrete time system and see how it can be characterized in the transform domain.

We have learnt about Fourier transform, DFT and z transform of signals. How do they help in characterizing discrete time systems? One of the things you should appreciate is that any signal can be decomposed into sinusoids. When you take a periodic signal in the continuous time domain, we take Fourier serious which is a summation of sinusoids of frequencies which are harmonically related. There is a fundamental frequency that is 1/period, then we have second harmonic, third harmonic, fourth harmonic and so on. When the signal is not periodic, then what do we do? We take Fourier transform which means that the spectrum consists of all frequencies starting from 0 up to infinity. But it can also be looked upon as a continuum of sinusoids. Therefore any signal, be it continuous time or discrete time, can be thought of as a superposition of sinusoids. The separation between two adjacent sinusoids is infinitesimally small, d(omega), and that is why we use integral. You also know that any sinusoid can be written as a sum of two exponential signals by Euler's theorem. Therefore an exponential signal is the most elementary signal as far as characterization of a system is concerned. If we know the response to a sinusoid or an exponential signal, then we know the response to all other signals. So we try to find out the response to an exponential signal. We consider a Linear Time Invariant (LTI) system for which you know that the output is given by the convolution of the input and the impulse response. For a digital LTI system, the output is given as: y(n) = summation[x(n – k)h(k)], where k goes from – infinity to + infinity.

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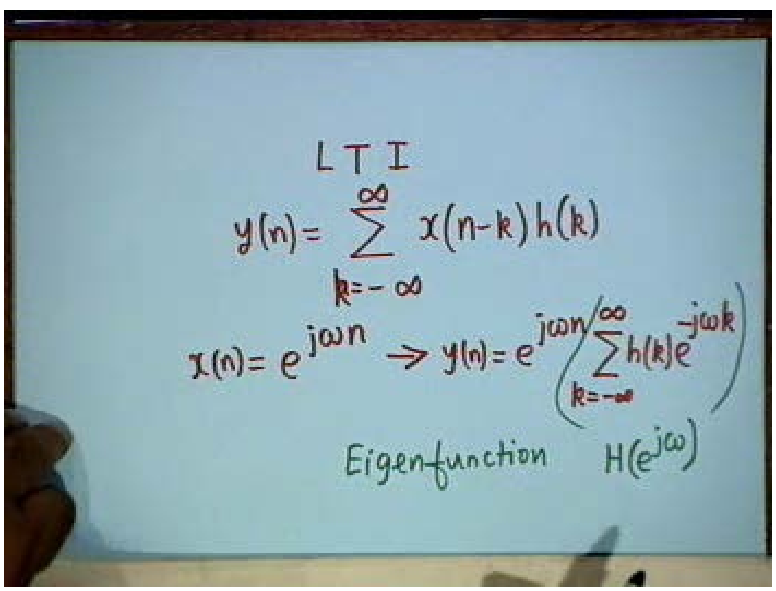

Now, to this system we apply an input x(n) = $e^{jomegan}$, so x(n – k) = $e^{jomega (n – k)}$; the $e^{jomega\ n}$ term can be taken outside this summation because the variable of summation is k so I get the output as ($e^{jomegan}$) summation [h(k)$e^{–jomegak}$] where k goes from – infinity to + infinity. You notice that since this summation is independent of n, it can be simply replaced by a constant H(e^{jomega}). So y(n) is H(e^{jomega})($e^{jomegan}$); such functions which when supplied as an input to the system produces an output which is a constant multiplied by the same function, are called Eigenfunctions. So $e^{jomegan}$ is an Eigenfunction of a Linear Discrete Time Invariant system. You notice that H(e^{jomega}) = summation[h(k)$e^{–jomegak}$] where k goes from – infinity to + infinity. This is simply the Fourier transform of h(n).

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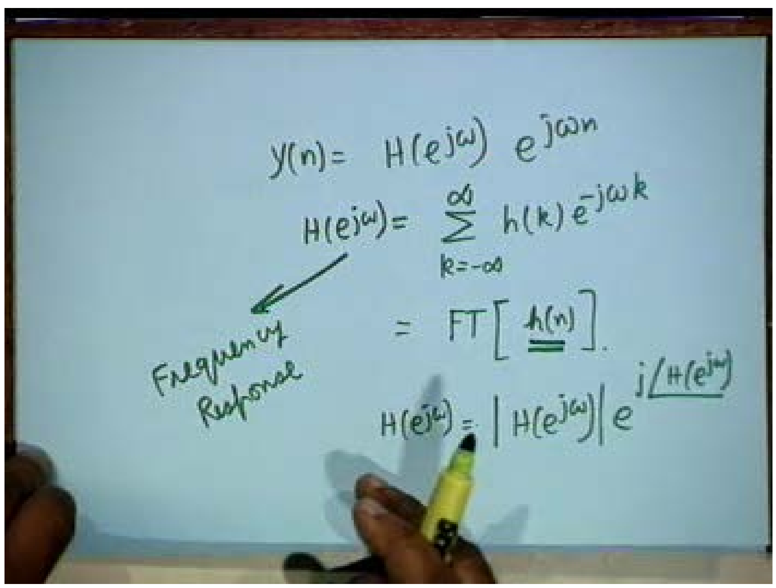

The quantity H(e^{jomega}) is known as the frequency response of the system. It is a characteristic of the system just as h(n) is a characteristic of the system. It is the same characterization as h(n), transformed to the frequency domain. It is called frequency response because it involves omega. It shows the complete frequency behavior of the system. In general, you see that H is a complex quantity. It can be written as H($e^{j\ omega}$) = magnitude[H(e^{jomega})]$e^{j\ angle\ of\ H(ejomega)}$. Magnitude must be real and even. Similarly, the phase is odd but also a real quantity. Angle and phase are equivalent terms. Now it is interesting to see that $e^{jnomega}$ leads to H($e^{j\ omega}$) $e^{jomegan}$ because this system is linear. If we take the real parts in both sides they should correspond to each other.

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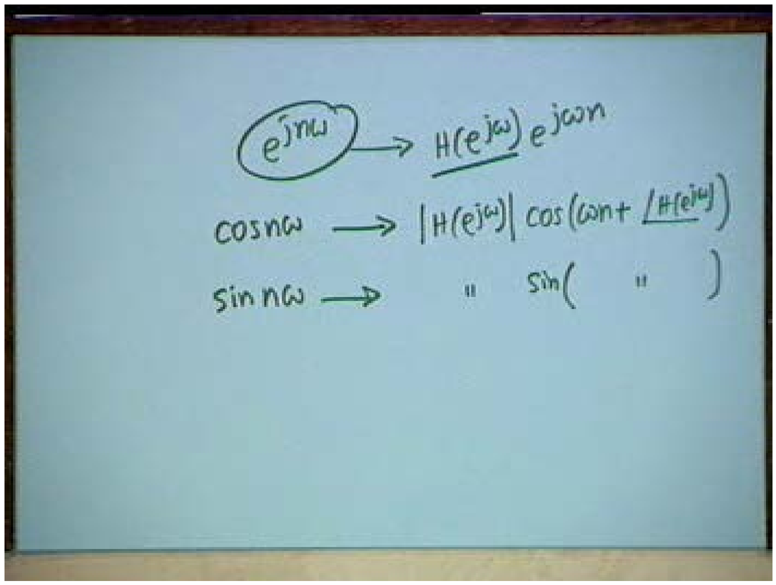

In other words, cosine(n omega), will give the response |H($e^{j\ omega}$)| cosine(omega n + angle of H(^{jomega})). In a similar manner, if I have an input of sine (n omega) then all that differs is that instead of cosine, I get a sine. This gives the steady state response of the system to either an exponential signal or a cosine signal or a sine signal. So this signal, $e^{jn\ omega}$ is the most important one in practice. In the time domain, the most important signal is delta n because if you know h(n) then you know the total characteristic of the system. Correspondingly in the frequency domain, it is $e^{jn\ omega}$; if you know the response to $e^{jn\ omega}$ or you know H(e^{jomega}) (they are equivalent characterizations), you can find the response to any other signal. Let us take an example.

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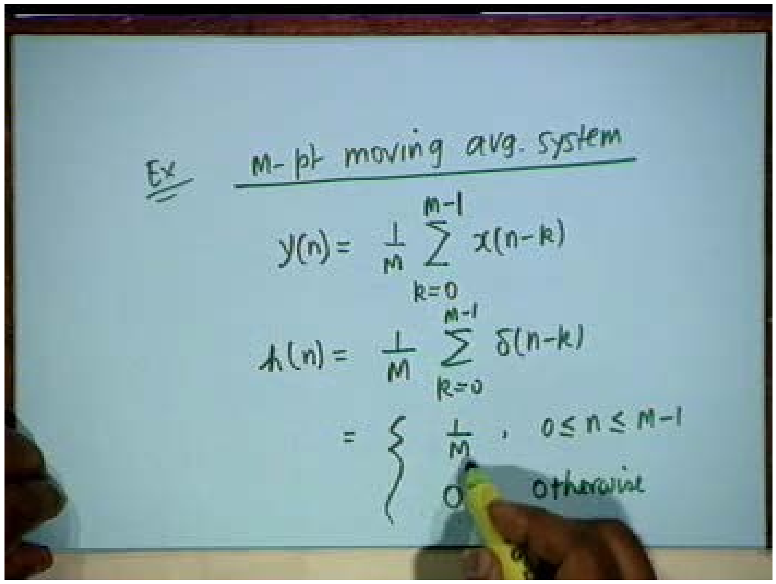

The example that we take is the M point moving average system. We have taken this example earlier also, in which the output at an instant n is the average of all inputs starting from the present instance to the (M – 1)th instant; that is, it is the average of all x(n – k), k = 0 to M – 1.What kind of system is this, is it FIR or IIR response? This is FIR because h(n) = (1/M)summation[delta (n – k)] where k = 0 to M – 1. So it is indeed an FIR system and for frequency response all we have do is to take the Fourier transform of h(n). So we shall get a summation, 1 + $e^{- j \text{ omega}}$ + …. up to the Mth term, $e^{(j \text{ omega})(M - 1)}$ and the whole thing divided by M.

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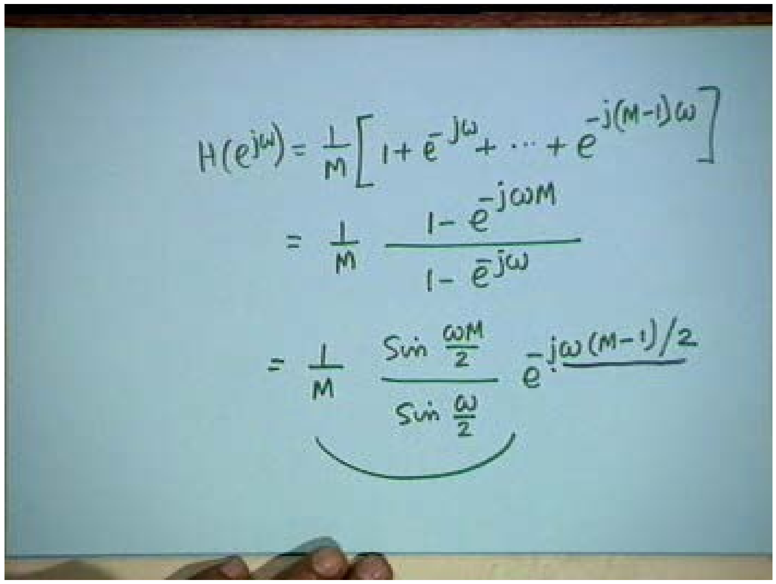

So the frequency response is $H(e^{jomega}) = (1/M)\ [1 + e^{-jomega} + e^{-j(M-1)\omega}]$. Since this is a geometric series we can sum it up as $(1/M)\ (1 - e^{-\ jomega\ M})/(1 - e^{-\ jomega})$. You know that this can be written in terms of series and (1/M) [sine((omega M/2)/sine(omega/2)][$e^{-jomega(M\ -\ 1)/2)}$]. It is very tempting to identify (1/M) × ratio of sines as the magnitude and the (power of e)/j as the phase, but this is not correct because [sine(omega M/2)/sine(omega/2)] changes signs. Whenever it changes sign, the phase changes by 180°.

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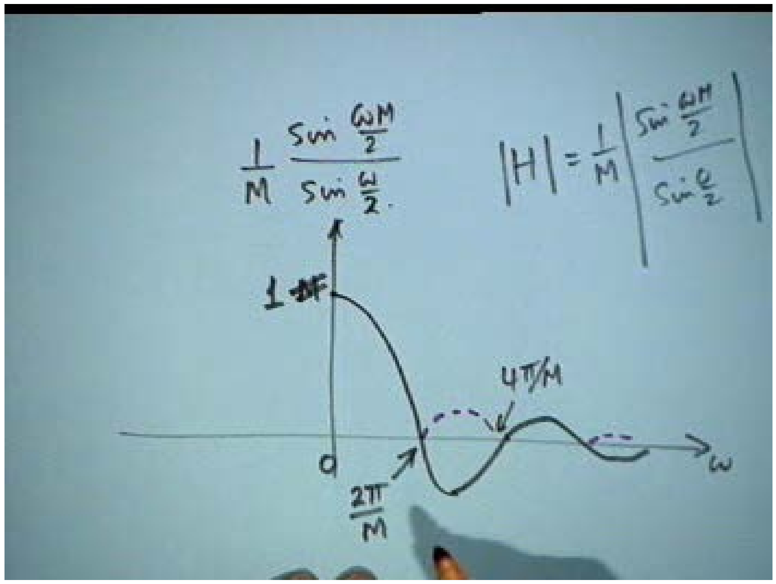

For example, if I plot (1/M) [sine (omega M/2)/sine(omega/2)] versus omega, then the omega = 0 value is 1. Then it decreases, and then it executes damped oscillations. Now where does the first 0 occur? It occurs when the numerator angle becomes equal to pi. That is, omega = 2pi/M. The denominator remains non-zero at ω = 2pi/M. The next zero crossing would be at ω = 4pi/M and so on. At every such transition, the phase increases by pi. So the magnitude is not [1/M] [sine (omega M/2)/sine(omega/2)], but its magnitude. The negative halves of the ratio of sines will flip over.

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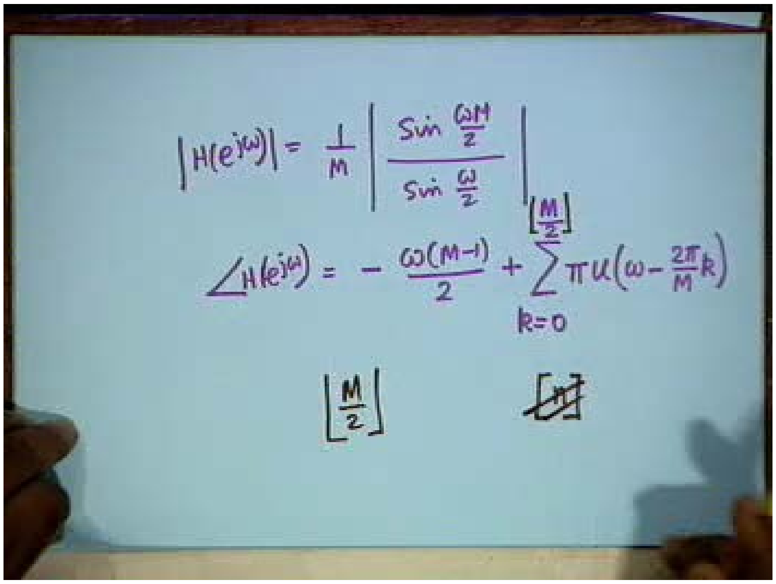

Therefore the magnitude and phase of the frequency response can be written like this: $|H(e^{jomega})|$ = |[sine (omega M/2)/sine(omega/2)]|. And the angle of H would be – omega (M – 1)/2 added to the phase transitions by pi. Wherever there is a 0 crossing there occurs a phase transition. So an angle pi is added there. Therefore the general expression becomes – (omega(M – 1)/2) + summation[pi u(omega – (2pi/M)k)]. There are many transitions. At each transition a step function of magnitude pi is added to the previous phase. Step functions of magnitude pi occur at omega = (2pi/M)k where k goes from 0 to M/2. Our range of vision is omega = 0 to pi and therefore when k becomes M/2, then the transition stops. But M/2 may or may not be an integer. So the last transition occurs at the integer closest to M/2 but less than M/2. For example, if M was 5 then the last transition shall occur at k = 2. We indicate this by $\lfloor M/2 \rfloor$ with a special sign. This sign is not the square bracket, the upper end is truncated. This $\lfloor M/2 \rfloor$ indicates that integer closest to M/2 but less than M/2. Similarly, if I write a quantity M like $\lceil M \rceil$, it indicates an integer closest to M but greater than M.

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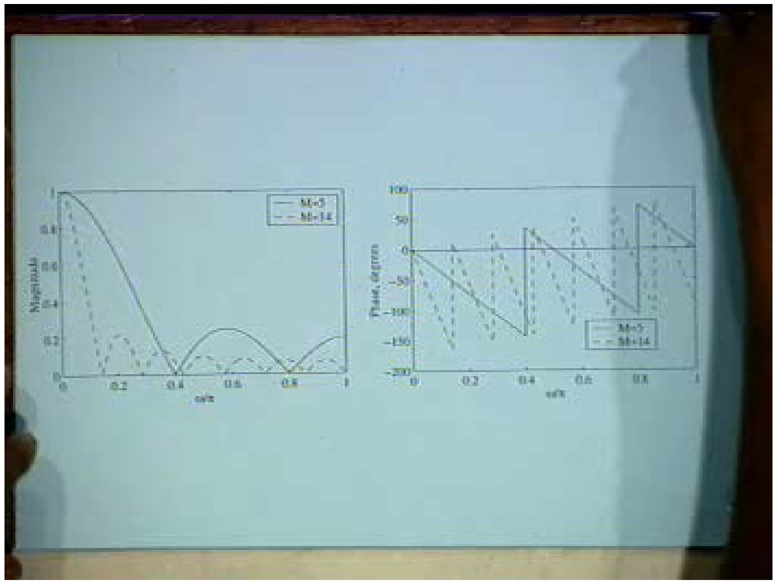

Figure shows the magnitude as well as phase for two values of M: one is M = 5 and the other is 14. If you have M = 5, then the first transition occurs at ω/π = 0.4. The next one shall occur at 0.8. There are only two transitions here because 5/2 is 2.5 and the integer closest to it is 2. Look at the phase, the phase starts from 0 and goes down to the value – 0.8 π. At 0.4, it jumps up exactly by 180°. Then the quantity [sine ((omega M/2)/sine(omega/2)] remains negative. Then it comes down again in a linear manner. Then at the next transition ω/π = 0.8, another pi is added. So the phase becomes a triangular waveform with a rising tendency. The phase is not – omega (M – 1)/2; you have to state the phase within the range of vision. If M is 14, then you see the transition occurs at less than 0.2. How many transitions now? Exactly 7; the 7th one is the end of the vision at omega = pi. So the phase behaves in a very peculiar manner; while in specifying magnitude we have no ambiguity. In specifying phase, the transition must be taken care of; it is not phase simply going linearly negative.