Digital Signal Processing

Prof. S. C. Dutta Roy

Department of Electrical Engineering

Indian Institute of Technology, Delhi

Lecture - 12

Z-transform

This is lecture 12^{th} on DSP and we continue our discussion on Z-transforms. In lecture 11, we had introduced the Circulant Matrix for computing circular convolution and then we showed how to compute two N point DFTs by computing a single N point DFT by combining the two sequences in an analytic manner. That is you take x(n) = g(n) + jh(n) and then compute the N point DFT of x(n). We gave an example. Then we introduced the Z-transform as the summation $g(n)z^{-n}$, summation going from n = – infinity to + infinity. We showed its relationship to Fourier Transforms. We also said that the ROC (Region of Convergence) is extremely important.

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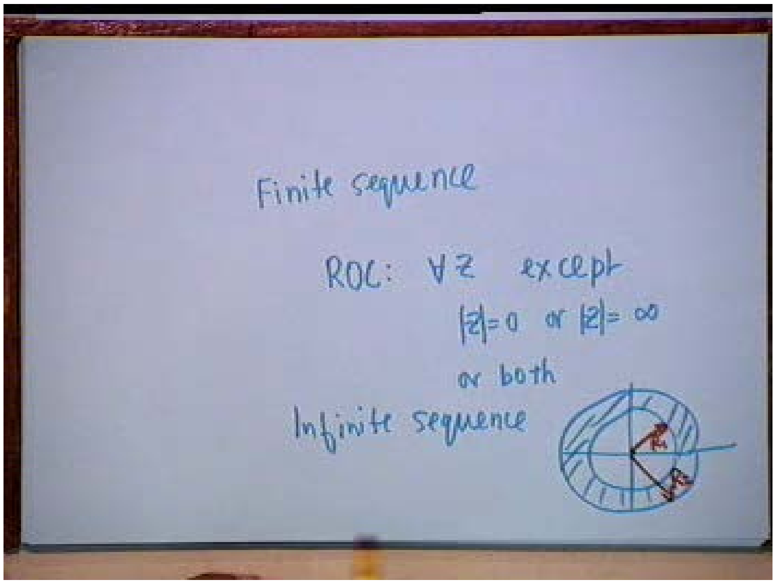

Then we made a special point that if you have a Finite Sequence then the ROC, irrespective of whether it is causal, anti causal or combination of causal and anti causal, is the whole z plane except possibly z = 0 and z = infinity. It can be both or it can be one of them. The ROC is the total z plane. But if it is an Infinite Sequence then the ROC in general is an annular region between two circles. Let the subscripts be 1 and 2 and let the smaller circle be of radius R1 and the larger circle be of radius R2, greater than R1.The ROC in general is this annular region, where R1 can be as small as 0, and R2 can be as large as infinity.

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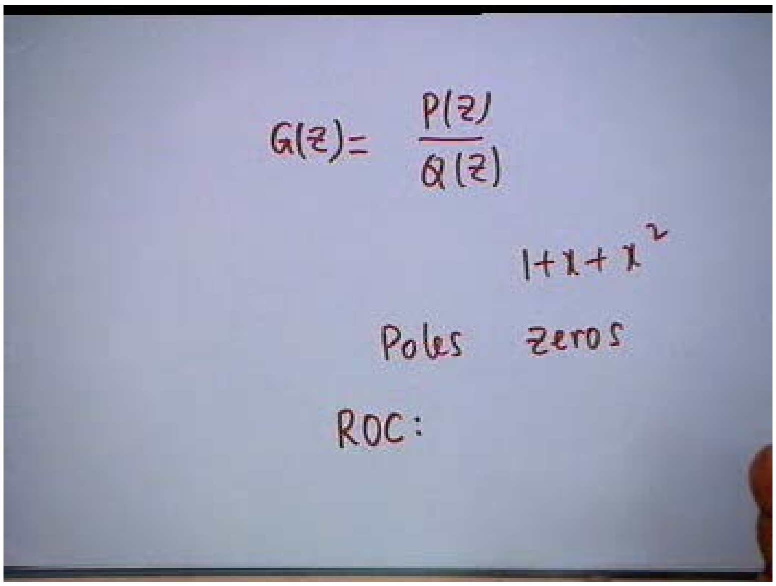

For an infinite sequence, the Z-transform is a rational function of the form G(z) = P(z)/Q(z). A rational function is a ratio of polynomials and a polynomial is a finite series containing only integral positive powers of the variable. P(z) and Q(z), in fact, we write as polynomials in z^{-1}. The variable is z^{-1}, and if z is the variable then P(z) is not a polynomial. So $1 + x + x^2$ is a polynomial but $1 + x^{-1} + x^{-2}$ is not a polynomial in x but it is a polynomial in x^{-1}. From this rational function, we define what Poles and Zeros are and we said that the ROC is bounded by the pole at the largest distance from the origin, for a special type of sequence, viz. Causal Sequence. On the other hand, if it is an anti causal sequence, then the ROC is bounded by the pole which is closest to the origin. For a Causal Sequence, ROC is outside a circle and for an Anti Causal sequence, ROC is inside a circle.

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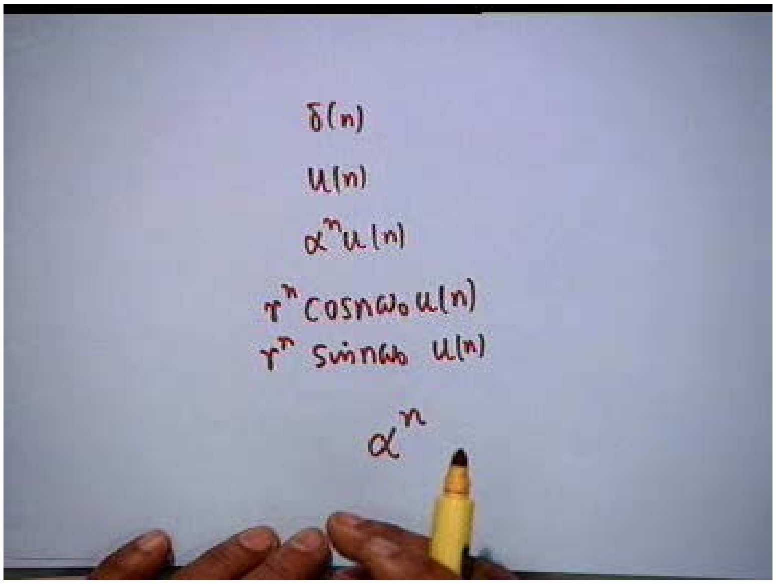

We took some examples, and made a table of the Z-transforms of some basic sequences, viz. delta(n), u(n), alpha^n u(n) and r^n cosine (n omega$_0$) u(n) or r^n sine (n omega$_0$) u(n). If you know these transforms, then you can solve almost any problem in Z-transforms provided you are aware of the ROC, the region of convergence. In Fourier Transform, there is no such complication. In Z-transform there is this complication of ROC, which is extremely important. In fact we will show that the same Z-transform can represent different sequences. We also showed that there are sequences for which Z-transform does not exist. For example, for alpha^n, the Z-transform does not exist because it consists of two parts a right sided sequence and a left sided sequence and the two ROCs do not have an overlapping region. The only overlapping region is a circle of radius mod alpha but on that is located the pole and therefore there is no ROC. It is important to understand this.

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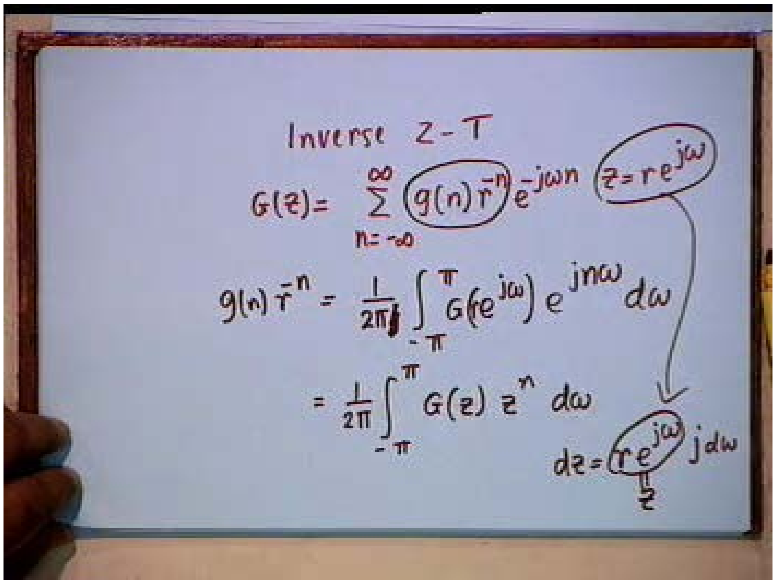

Now we talk of the Inverse Z-transform. You recall that the Z-transform G(z) = summation g(n) z^{-n} but z, in general is r e^{jw}, a complex quantity so G(z) is summation g(n) r^{-n} $e^{-jomega\ n}$ where n = – infinity to + infinity. This can also be a looked upon as the Fourier transform of the sequence g(n) r^{-n}, so the inverse Fourier Transform relationship should hold. In other words, g(n) r^{-n} sequence can be recovered from the Fourier Transform by taking the integral [1/(2pi)] integral (– pi to pi) G(r e^{jomega}) $e^{jnomega}$ d omega. Now I can transfer r^{-n} to the right hand side inside the integral because the variable is omega. Therefore I can write this as g(n) = [1/(2pi)] integral (– pi to pi) G(z) z^nd omega. We must change the variable because our function is that of z = r e^{jomega}. This means that d z = r (r is a constant and omega is the variable). e^{jomega} j d omega. And therefore we can replace d omega by d z/(jz).

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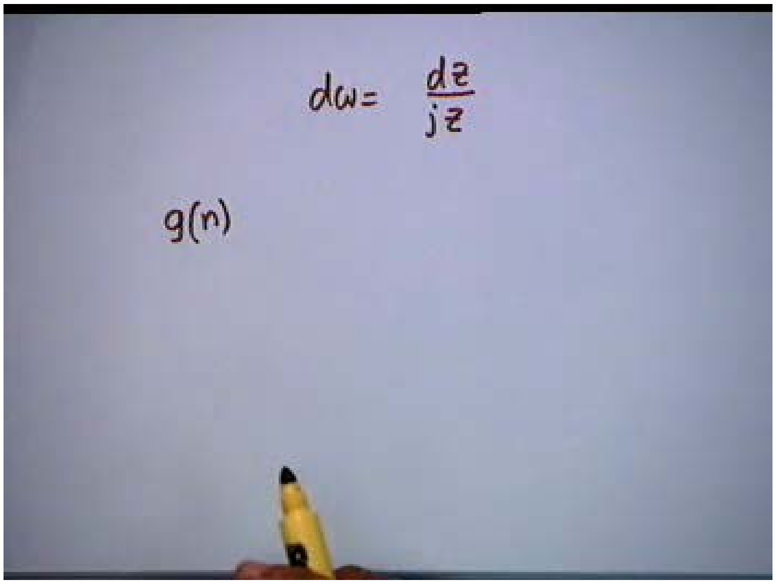

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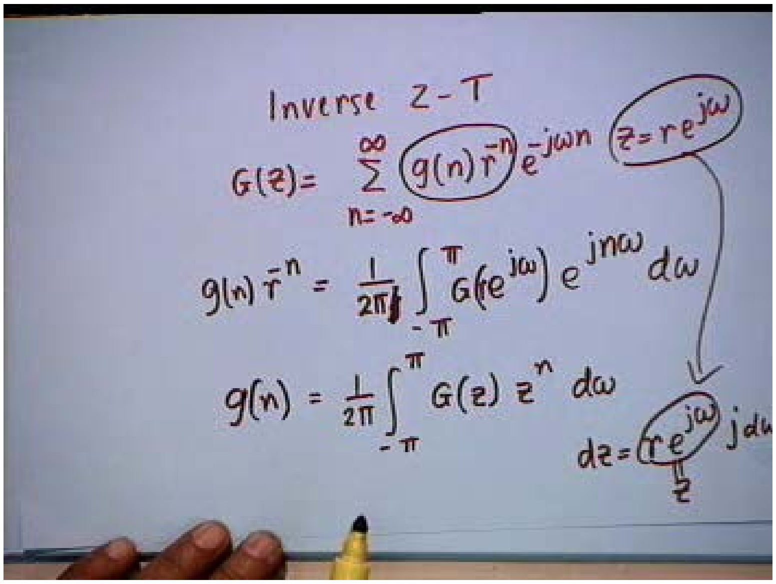

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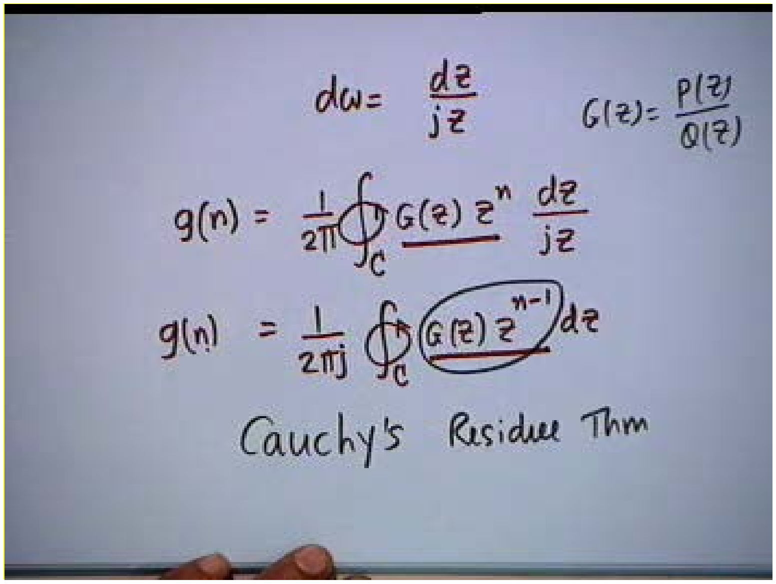

Therefore g(n) = [1/(2jpi)] integral G(z) z^{n-1} dz. Now what should be the limits of the integral? z is a two dimensional plane; it goes from – infinity to + infinity in a complex manner. On the other hand, what was the idea in integrating over omega? Omega goes over a circle from – pi to + pi. In other words, it makes a closed contour. Therefore when you go from the unit circle to the total complex plane, the integration becomes a contour integration.

In other words, what you should do is to choose a contour. In contour integration, we go in the anticlockwise direction because for omega we have gone from omega = 0 in the anticlockwise direction; so the direction must be the same. But the contour must be such that it does not pass through any of the singularities of the function G(z) z^{n-1}. And one of the singularities is at z = 0, because if you put n = 0 here then z^{n-1} becomes z^{-1}; therefore there is a singularity for n = 0 at z = 0. Therefore, the contour is to be such that it is outside the poles and does not encounter any pole including the pole at the origin. A closed contour must be around z = 0 and therefore any contour c traversed in the anticlockwise direction which includes the point at the origin and all poles of G(z) inside it is good enough; usually we choose this to be a circle with center at the origin. This is the simplest thing to do. However for useful DSP, we do not have to evaluate this contour integration.

Contour integration evaluation is not a routine job; it has to be done with lot of care. We do not have to do it because our function G(z) is always rational = P(z)/Q(z) and for a rational function, there are better alternatives. But if you have to evaluate this contour integral, the gentleman Cauchy comes to rescue. Cauchy gave a theorem, called Cauchy's residue theorem, which says that this integral is equal to sum of the residues at the poles inside the contour. Fortunately, we do not have to apply Cauchy's residue theorem.

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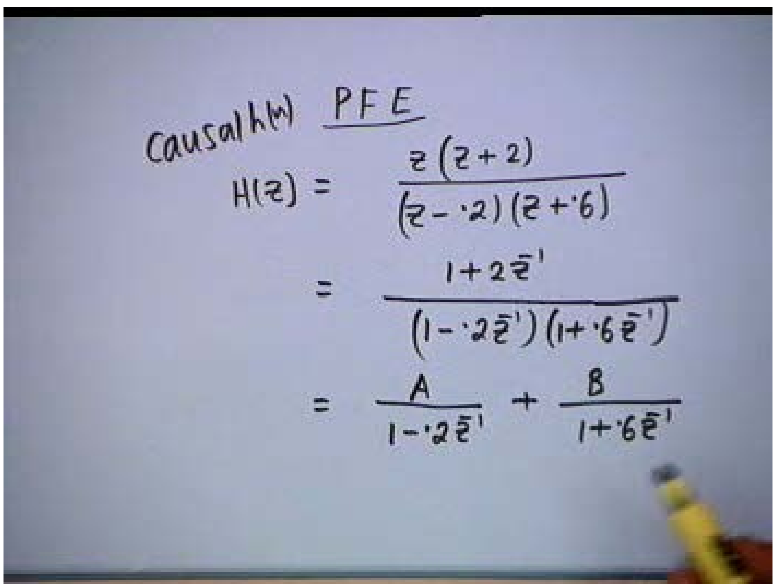

We have better alternatives for rational functions that we are concerned with. One method is that of partial fraction expansion. I told you that if you know the Z-transform of delta n, u(n), alpha^n u(n), r^n cosine (n omega 0) u(n) and r^n sine (n omega 0) u(n), then you can perform any Z-transform or its inverse also. We will take several examples, one by one. First let us find out a causal h(n) such that its Z-transform H(z) = z (z + 2)/[(z – .2) (z + .6)]. The first thing you do when we encounter such an expression is to express it as a rational function in z^{-1}; therefore you divide by z^2 both numerator and denominator. Then you get $H(z) = (1 + 2 z^{-1})/[(1 - .2 z^{-1}) (1 + .6 z^{-1})]$. Next, you expand in partial fractions. That is, you write this as $= A/(1 - .2 z^{-1}) + B/(1 + .6 z^{-1})$. Why do you write it in this form? It is because you already know that the Z-transform of $\text{alpha}^n u(n) = 1/(1 - \text{alpha } z^{-1})$. So if you can express it in this form, then your job is done and A and B can be found in the usual manner.

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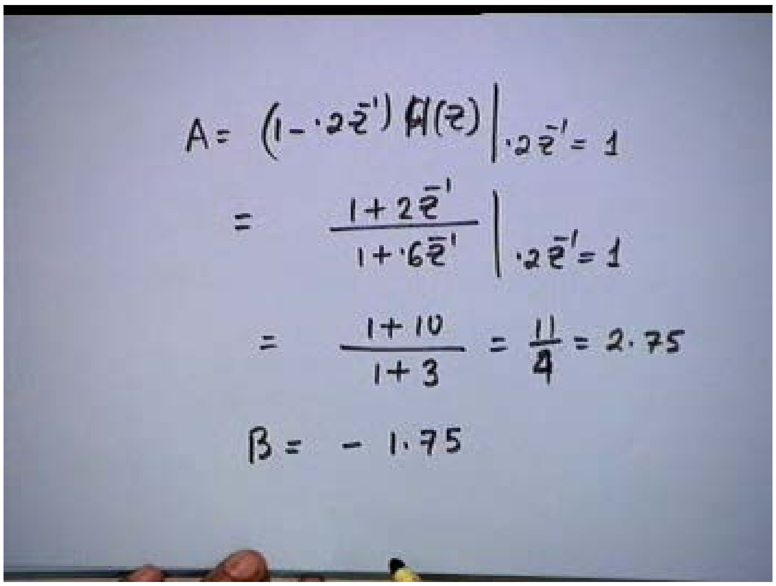

For example $A = (1 - .2 z^{-1}) H(z)$ under the condition $.2 z^{-1} = 1$ or $z^{-1} = 5$. This evaluates to 2.75. Similarly you can find B as – 1.75.

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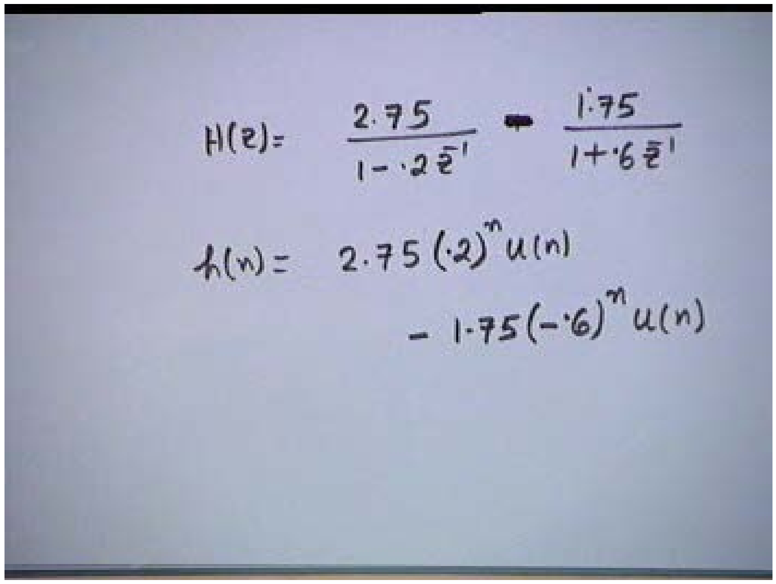

And if you substitute this you get H(z) = 2.75/(1 – .2 z^{-1}) – 1.75/(1 + .6 z^{-1}). Since z - transform is a one to one transformation, we invert H(z) term by term and get h(n) = $(.2)^n$ u(n) – 1.75 $(– .6)^n$ u(n). This is the inverse transform, because we wanted a causal sequence; if it was not, then we would have a problem. Now go back, and look at the original problem.

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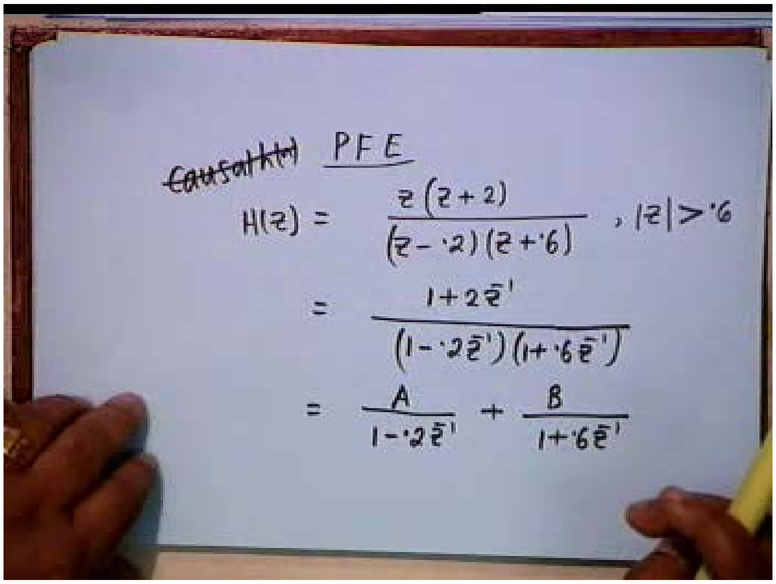

The ROC is not specified, but an equivalent specification is given because you are asking for causal h (n). So if this specification on h(n) was not given, then the answer to the inverse transform would not have been unique. The answer we worked out will be correct only if it is stated that |z| is greater than 0.6; so causal sequence and |z| > 0.6 are equivalent statements. The ROC outside a circle of radius 0.6 specifies that the inverse transform shall be causal. One of the two must be given. Again, go back to the original problem; how do you know that these two terms are adequate for partial fraction expansions. On the other hand if you make a partial fraction expansion of z (z + 2)/ (z –.2) (z + .6), there shall be a constant. The constant shall be equal to the value of z = ∞; it is 1. There is no constant here because the numerator is 1° less than the denominator. If the numerator also had another linear factor, then there would have been a constant term.

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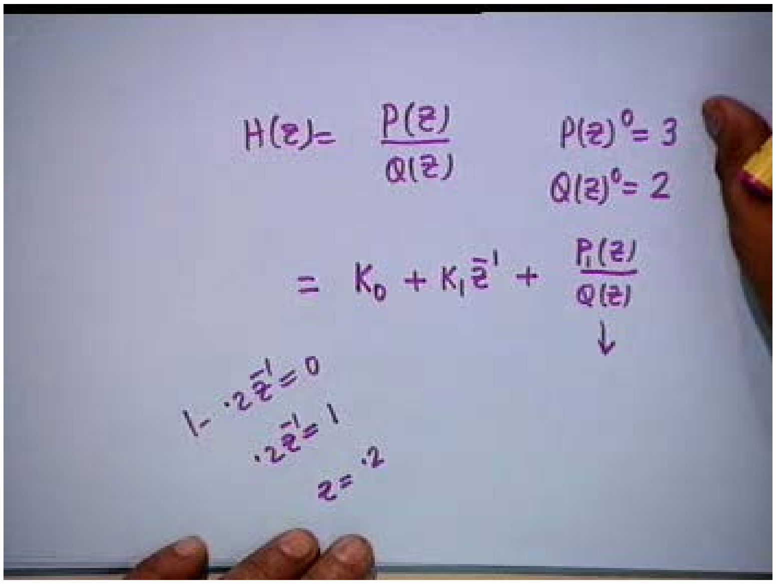

Let us take another example; suppose we have H(z) = P(z)/Q(z) where P(z)° = 3 and Q(z)° = 2. Then the partial fraction expansion of this shall have first the components $K_0 + K_1 z^{-1}$. If you take this out by making a long division of P(z)/Q(z),then the remainder $P_1(z)/Q(z)$ will be such that the degree of Q = 2 but the degree of P_1 = 1. Now you expand this into the sum of two rational functions. You must look at the given function very carefully. Inversion of a K_0 is not a problem. It is K_0 delta n and the inverse transform of $K_1 z^{-1}$ is K_1 delta (n – 1). Now, there is another simplifying feature about this problem that there were no repeated poles; the poles were distinct at + .2 and - .6. If there are repeated poles, then what do you do?

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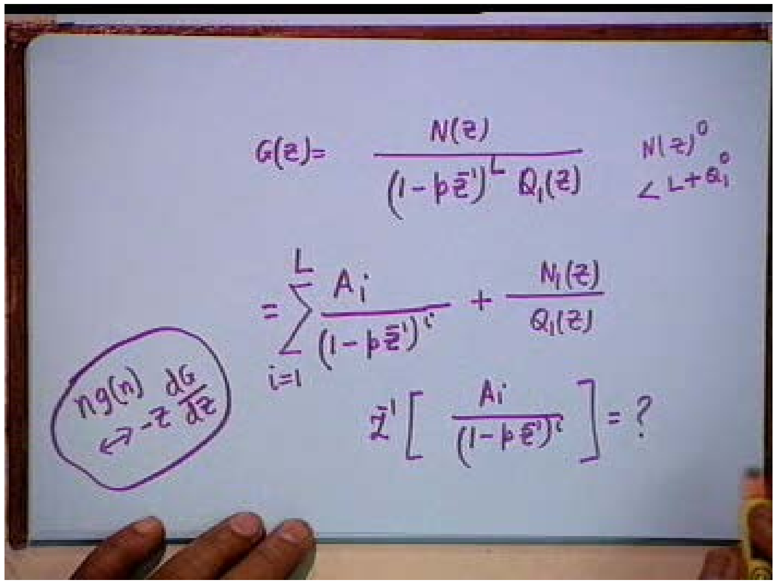

We take a general case. Suppose G(z) is some numerator $N(z)/[(1 – pz^{-1})^L. Q_1(z)]$; the pole at z = p is repeated L number of times. $Q_1(z)$ contains the other poles; it does not contain the pole at z = p. Then provided $N(z)°$ is less than L + degree of Q_1 i.e. the numerator degree is less than the denominator degree, the partial fraction expansion will include summation $Ai/(1 – pz^{-1})^i$ where i goes from 1 to L. That is, the repeated roots would give rise to a partial fraction expansion $A1/(1 – pz^{-1}) + A2/(1 – pz^{-1})^2 +…$ up to $AL/(1 – pz^{-1})^L$. So there are L number of constants and then the rest of the function will be a reduced polynomial $N_1(z)/Q_1(z)$, which you can expand by partial fraction expansion. And then you invert term by term; for the inversion of $(A_i)/(1 – pz^{-1})^i$, take help of the fact that the Z-transform of n times g(n) is given by – z dG/dz. If you take account of this, then you can find a general formula for the inversion of this. How do you find the A_I?

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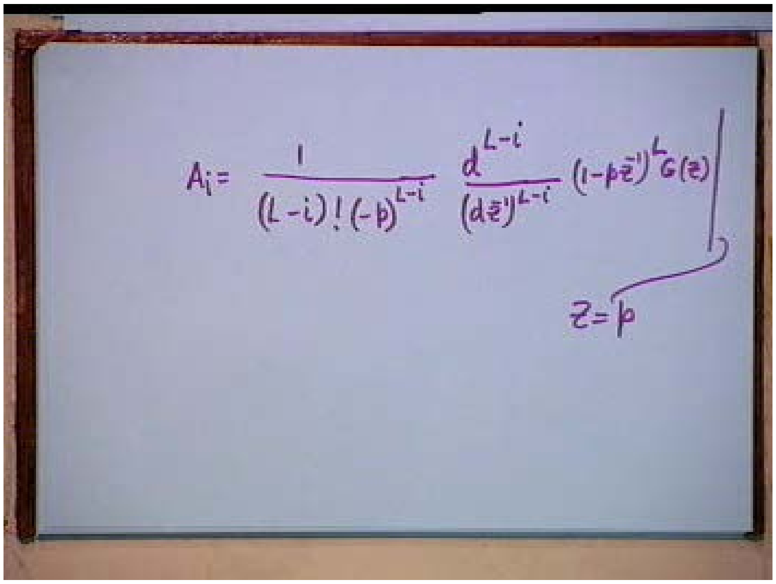

All textbooks give this formula: $A_i = \{1/[(L-i)!\,(p)^{L-i}\} \frac{d^{(L-i)}}{(dz^{-1})^{(L-i)}} (1-pz^{-1})^L G(z)$ evaluated at z = p. The general formula looks horrible. You have to differentiate L – i times and at any step, you can make a mistake. What I do is slightly different. Let us take an example to illustrate what I do. The scope of making mistake in my procedure is much less.

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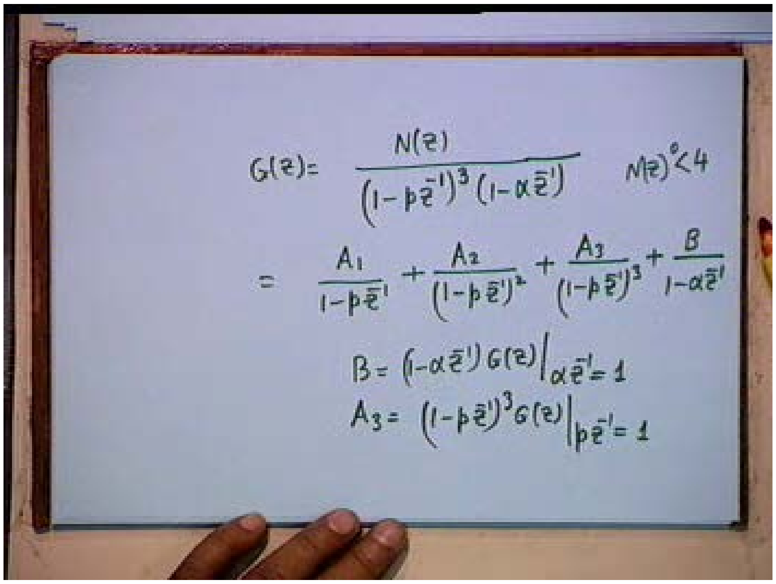

Let us say I have a $G(z) = N(z)/[(1 - pz^{-1})^3 (1 - \text{alpha } z^{-1})$; here I have taken a pole repeated three times represented by $(1 - pz^{-1})^3$ and it is assumed that $N(z)^{\circ}$ is less than 4. If the numerator degree is less than the denominator degree then we call it a proper rational function. There is nothing improper about the other rational functions in which the degree of numerator exceeds or is equal to that of the denominator, but then this term has gone into the literature and we shall use it. Now the partial fraction expansion of this would be $A_1/(1 - pz^{-1}) + A_2/(1 - pz^{-1})^2 + A_3/(1 - pz^{-1})^3 + B/(1 - \text{alpha } z^{-1})$. We can find out B very easily; you multiply G(z) by $(1 - \text{alpha } z^{-1})$ and put alpha $z^{-1} = 1$. Do not bother about finding the value of z because alpha z inverse = 1 is good enough, z^{-1} is 1/alpha. A_3 can also be found out easily: $A_3 = (1 - pz^{-1})^3 G(z)$ with $pz^{-1} = 1$. So A_3 and B are known. All that you have to do now is to find A_1 and A_2. Obviously if you want to apply the formula you should differentiate it once and find A_1; then you have to differentiate again to find A_2. Instead of doing that what I do is to use two specific values of z^{-1}.

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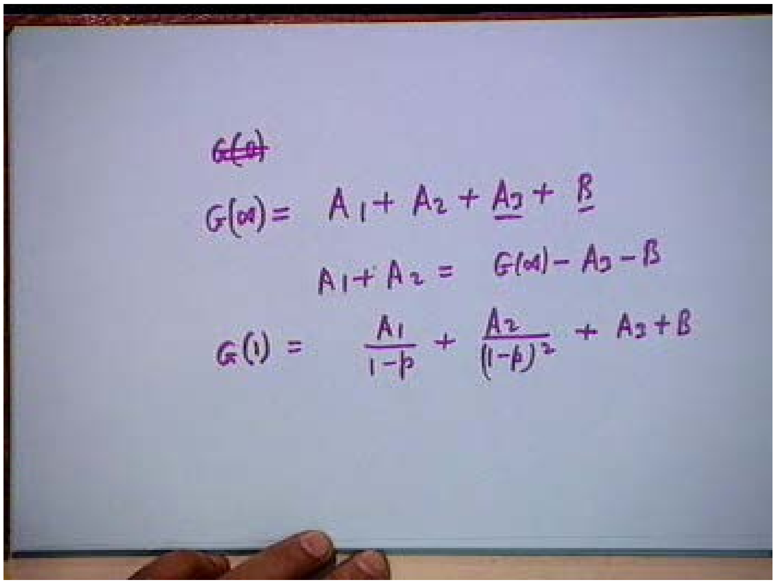

Let us put $z^{-1} = 0$ then z would be infinity. What is G(infinity)? It is $A_1 + A_2 + A_3 + B$ and A_3 and B are known. So I have got the value of A_1 + A2 = G (infinity) – A_3 – B; this is one of the equations. The other equation is obtained by using any other value of z^{-1} you like e.g. 1 or– 1. If I put $z^{-1} = 1$, then $G(1) = A_1/(1 – p) + A_2/(1 – p)^2 + A_3 + B$. Once again I get an equation in two variables A_1 and A_2 because A_3 and B unknown. Now I have two simultaneous equations and I can solve them. This procedure uses a little bit of algebra, but it is much simpler and not prone to mistakes. I would suggest that you follow this.

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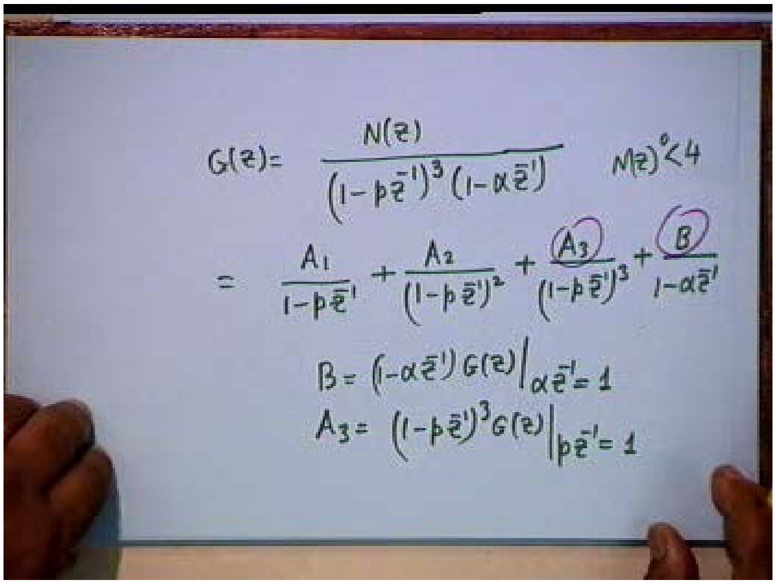

The other thing that one can do is to simplify the partial fraction expansion to a rational function with the same denominator, as that of G(z). So I get in the numerator a polynomial in z inverse with the constants A_1 A_2 A_3 and B. A_3 and B are known, so A_1 and A2 can be found by equating corresponding coefficients. But I find the previous method more convenient. This is how we handle the case of repeated poles.

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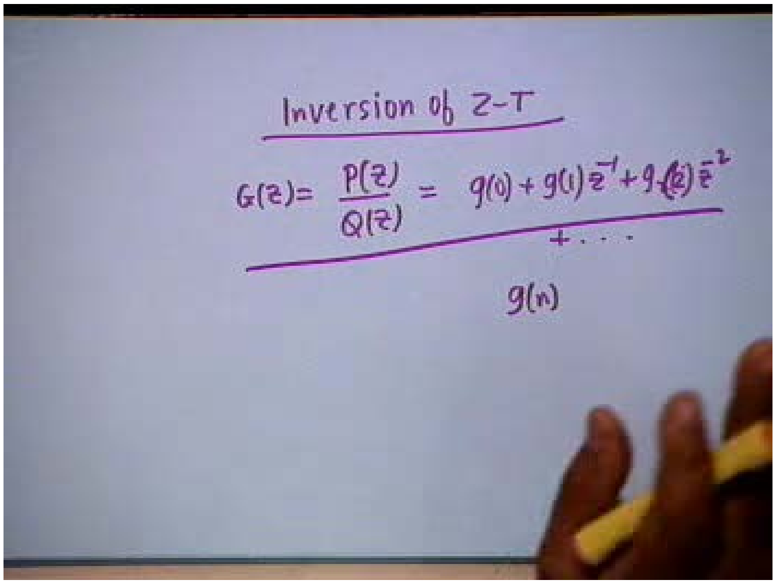

The Inversion of Z-transform can also be done by long division. All you have to do is to express G(z) = P(z)/Q(z) in the form g(0) + g(1) z^{-1} + g(2) z^{-2} and so on. You divide P(z) by Q(z) in long division and take the quotient, which is a polynomial in z^{-1}. The coefficients will give you the sequence. The difficulty is that you may not be able to find a Closed Form formula. You may not be able to find in general a formula for g(n) but you can find the individual coefficients. But we will use this to illustrate that the ROC is an extremely important attribute of Z-transform.

Consider u(n) whose Z-transform is (1 – z^{-1}). But suppose the nature of the sequence is not specified and you are only told that G(z) is (1 – z^{-1}), find the inverse transform. Obviously there can be multiple answers. The answers shall not be unique unless the ROC is specified. For example, I could write this as z/z – 1 and if I make a long division then the quotient would be 1 + z^{-1} + z^{-2} and so on. Obviously the sequence is {1, 1, 1 …}, starting at n = 0; therefore the required sequence is u(n). Now this is one of the answers. What is the ROC of this? Obviously this series converges if mod z is greater than 1 so the ROC is outside the unit circle.

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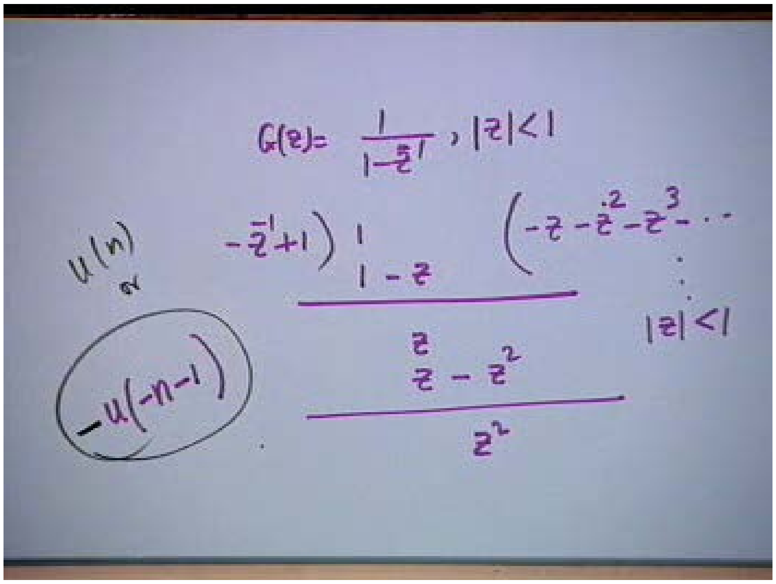

On the other hand, if I carry out the long division as it is, that is G(z) = 1/1 – z^{-1}) and divide 1 by $-z^{-1}$ + 1, I get the quotient as – z – z^2 – z^3… and so on. Thus the sequence is – 1 – 1 – 1 to infinity, starting at n = 1, which is clearly – u(– n – 1), a legally valid candidate. You must admit this also as one of the answers and this obviously converges for mod z less than 1. Therefore if G(z) is specified with ROC mod z less than 1 then you know its corresponding sequence will be the anti causal sequence – u(– n – 1). The ROC is therefore extremely important.

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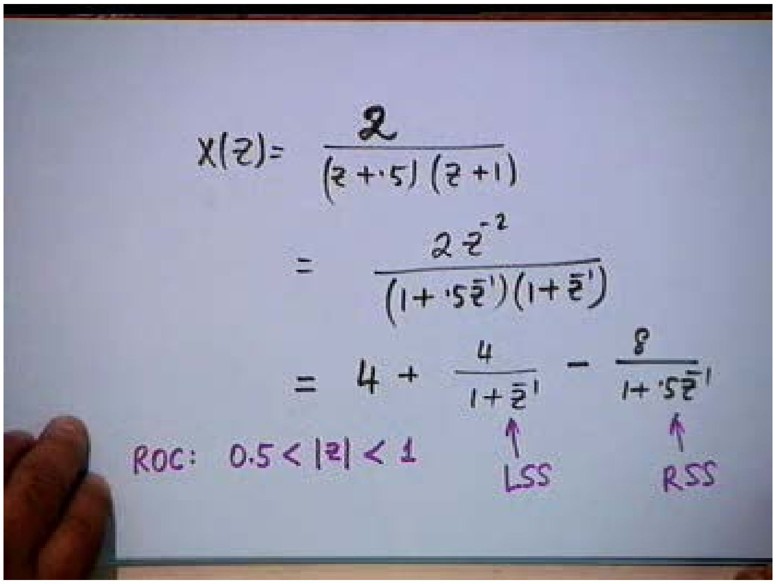

We take another example. Suppose I have X(z) = 2/(z + .5) (z + 1) and the ROC is not specified and I have to find the inverse transform. Obviously there will be multiple answers and the first thing I do is to write it is as a rational function in z^{-1}. I divide by z^2 and get X(z) = 2 z^{-2}/[(1 + .5 z^{-1}) (1 + z^{-1})].

Now you see here the numerator degree was less than the denominator degree but in a rational function in z^{-1} the numerator degree is equal to denominator degree. So in the Partial Fraction Expansion we have to take out a constant and this constant obviously would be 4. (Allow z^{-1} to go to infinity). The rest of the terms are, 4/(1 + z^{-1}) and – 8/1 + .5 z^{-1}). Now in the inversion, we have a problem. ROC has to be bounded by poles. The poles are at – .5 and – 1 therefore we can have an ROC which is the annular region between the two poles. We can also have ROC as |z| <. 5; this will give you a left sided anti causal sequence, while ROC |z| > 1 will give you a right sided, causal sequence. For 0.5 < |z| < 1 as ROC, the inversion of 4/(1 + z^{-1}) will give an anti-causal sequence, which that of –8/(1 + 0.5z^{-1}) will give a causal sequence.

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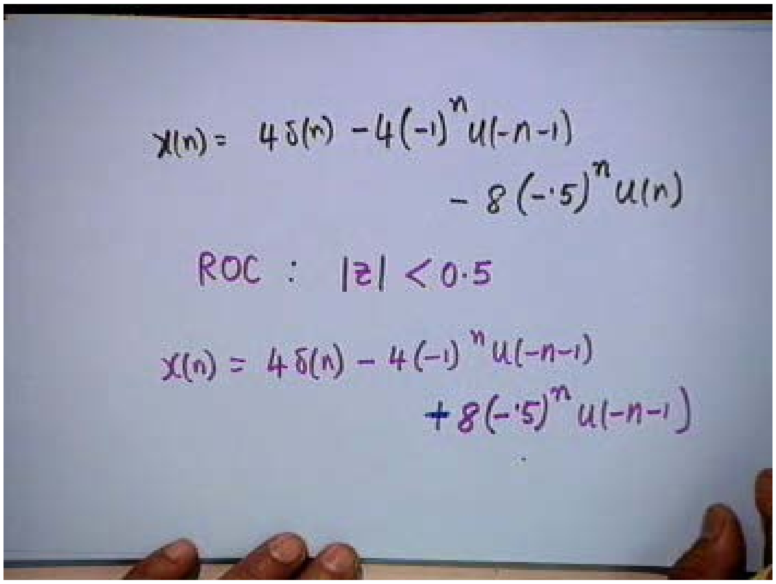

Correspondingly, you have the inverse transform as 4delta (n) – $4(-1)^n$ u(– n –1) – $8(-.5)^n$ u(n) if the ROC is 0.5 < |z| < |. On the other hand, if the ROC is simply mod z less than 0.5, then the total signal would be anti causal and therefore x(n) = 4 delta(n) – $4(-1)^n$ u(– n – 1) + $8(-.5)^n$ u(– n – 1).

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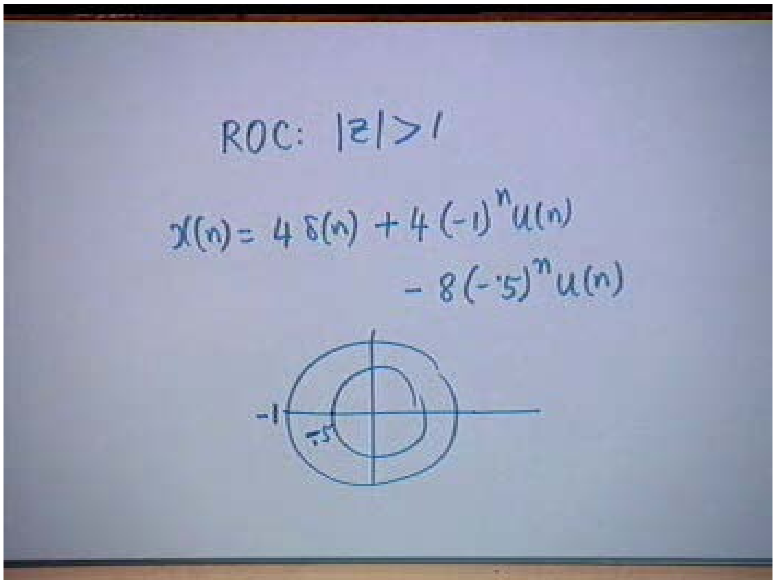

Now, the third possibility is mod z can be greater than 1. Correspondingly, x(n) would be 4 delta (n) + 4 $(-1)^n$ u(n) – 8$(-.5)^n$ u(n). Thus, there are three possibilities. Unless the ROC is specified or unless the nature of the sequence is specified, the answer will not be unique. If the sequence is totally causal, then you have one answer, totally anti causal has another one answer, but if it is the mixture of the two, then you have a third answer. This example illustrates the importance of the ROC in specifying a Z-transform.

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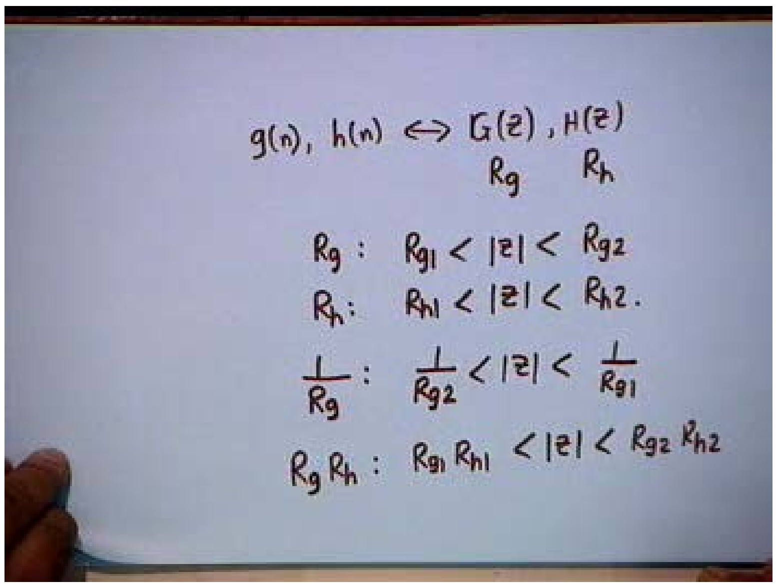

Now we talk about some properties of Z-transforms, exactly like those of the Fourier Transform. Consider two sequences g(n) and h(n) whose Z-transforms are G(z) and H(z) with ROC's specified as R_g and R_h. R_g stands for, in general, mod z lying between Rg_1 and Rg_2. It is the annular region between two circles of radii Rg_1 and $Rg_{2.}$ Similarly R_h is mod z lying between R_{h1} and $R_{h2.}$ We shall also use the symbol 1/Rg: it shall mean that mod z lies between $1/Rg_2$ and $1/Rg_{1.}$ We shall also use the symbol R_g R_h: this will stand for R_{g1} R_{h1} less than mod z less than R_{g2} R_{h2}. We shall use these symbols to discuss the properties. There are no such complications in Fourier Transform but Z-transform has this complication.

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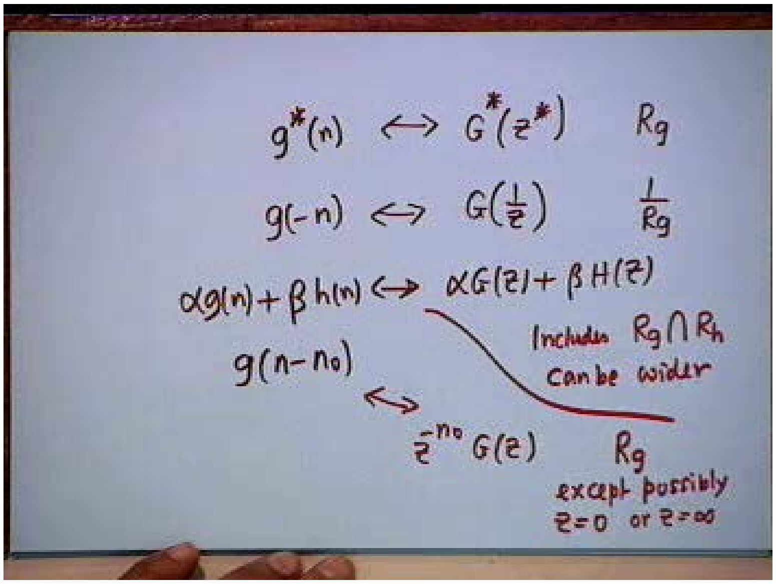
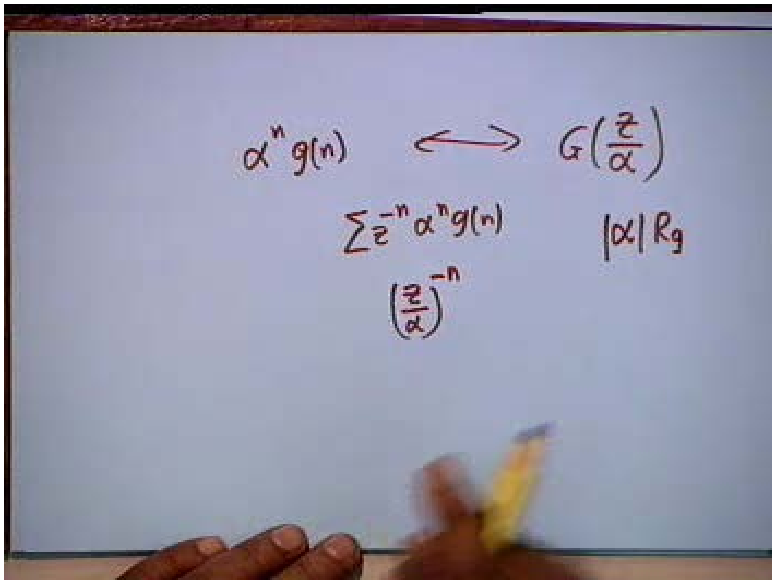


If you take the conjugate of a sequence g(n), then by applying the definition you can show that its z – transform is G*(z*). What do you think its region of convergence would be? Mod z and mod z* are of the same value, and therefore ROC shall be the same as R_g. Then if you take g(–n), its Z-transform will be G(1/z). And what shall be the region of convergence? It is 1/Rg because the argument has changed from z to 1/z. Z-transform obeys linearity: alpha g(n) + beta h(n) shall give rise to alpha G(z) + beta H(z). What is the ROC? It shall include the overlap, that is R_g intersection R_h, but that is not the total story, it can be wider than this.

Suppose G(z) = z^{-1} and H(z) = (1 – z^{-1}); alpha and beta are 1. Then the sum is 1 therefore the whole of z plane is included. Therefore the ROC includes the intersection but can be wider, because of the possibility of cancellation. Suppose G(z) has a pole at gamma and H(z) also has pole at gamma, the combination alpha and beta can be so chosen that the numerator has a 0 at gamma; then the pole and zero cancel. And then that pole is unable to bind the region of convergence because it is unobservable and the region of convergence becomes wider. The next sequence is g(n – n_0) whose z - transform is simply z^{-n0} G(z). The region of convergence is R_g with, the factor z^{-n0} taken into account. If n_0 is a positive integer then the point at the origin shall be excluded. On the other hand, if n_0 is a negative integer then the point at infinity is to be excluded. Hence the ROC is Rg, except possibly z = 0 or z = infinity. You must find the ROC very carefully.

(Refer Slide Time: 55.26 - 55.58)



If you multiply g(n) by $alpha^n$ then in the definition, z^{-n} $alpha^n$ would be $(z/alpha)^{-n}$. Therefore the Z-transform would be G(z/alpha) and the ROC is mod of z/alpha = R_g, which means mod z = alpha R_g. You must find the ROC very carefully, I repeat.