Foundations of Wavelets and Multirate Digital Signal Processing
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Lecture - 3
Module - 3
Scaling Function of Haar Wavelet

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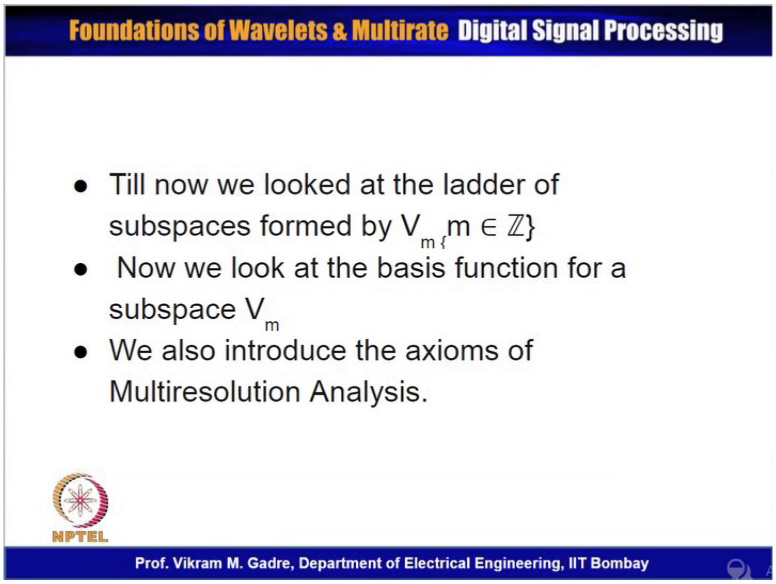

Now we need to state that formulae too, but in order to move in that direction we first need to bring in, as I said, another function which will span V0. So we need to bring in this idea of spanning.

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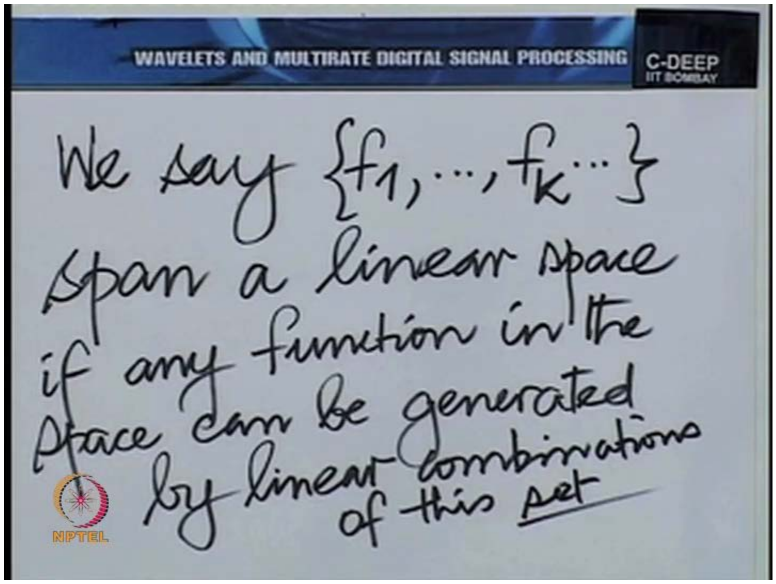

We see a set of functions, let’s say f1, fk, & so on, span a linear space, if any function in that linear space can be generated by the linear combinations of this set.

Now again there is a subtle distinction between the finite linear combination & infinite combinations, I don’t wish to dwell on those distinctions at the moment. But what we are saying, what we mean by span, so when we talk about the span of a set of functions, we’re talking about all the linear combinations of that functions & therefore the set of space, in fact the space of function generated by all linear combinations of that set.

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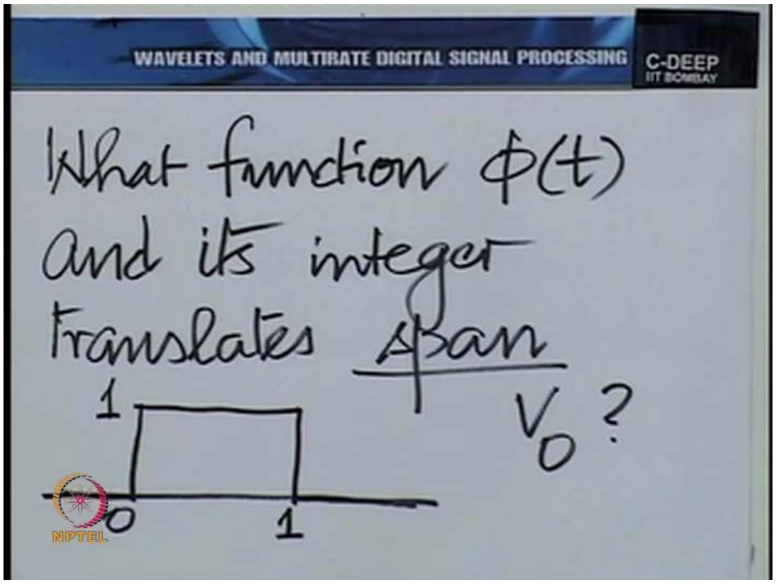

Now we ask a question which should make our life easy. What function, we ask this question before but now we answer it, what function, suppose we call it Pit & its integer translates, span V 0. And the answer is very easy, in fact if you were to visualize a function which is 1 in the interval from 0 to 1 & 0l's, lo & behold! You have the answer.

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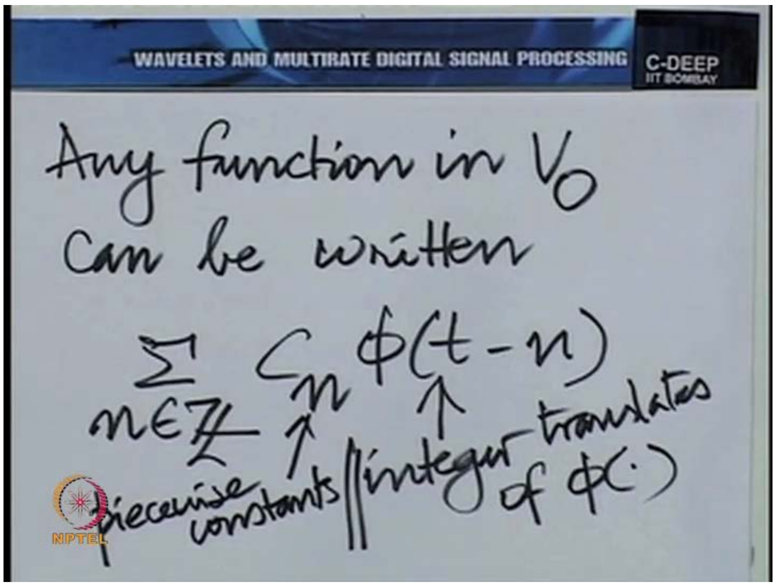

So what we are saying is any function in v0 can be written like this. Summation n over all the integers, cn Pit - n. So essentially integers translates of Pi. And these are the piece wise constants here.

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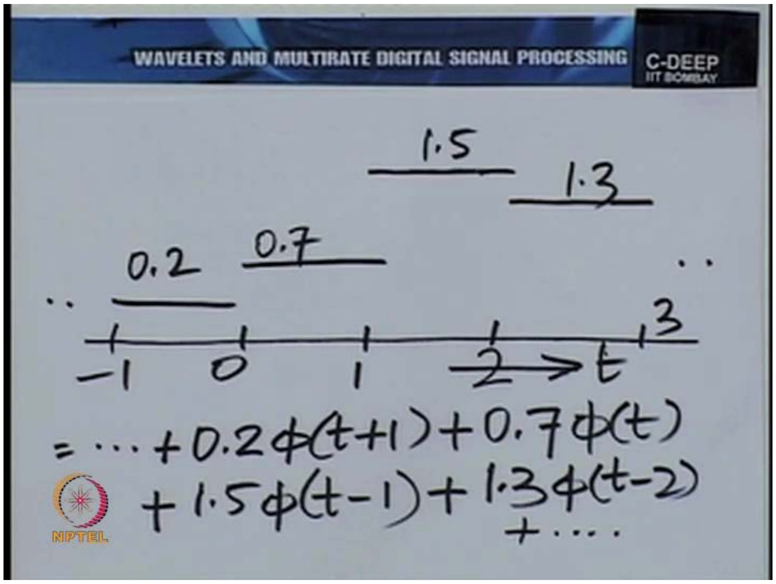

Just to fix our ideas, let us take an example here. So what we are saying is for e.g. suppose I have this example of a function in v0, so I have 0,1,2,3 & so on, the value here is, let us say 0.7, the value here is 1.5, the value here, between 2 & 3, is 1.3, the value between -1 & 0, let us say, is .2 & so on, this could continue. Then this fuction can be written as ...+0.2 times Pit+1+0.7 times Pit +1.5 times Pit-1 + 1.3 times Pit-2 +… & so on. Simple enough not at all difficult to understand.

So we have this function Pit, whose integer translates span v0. Now the subtle point is that if you were to go to any space vm, so the same thing would carry forward.

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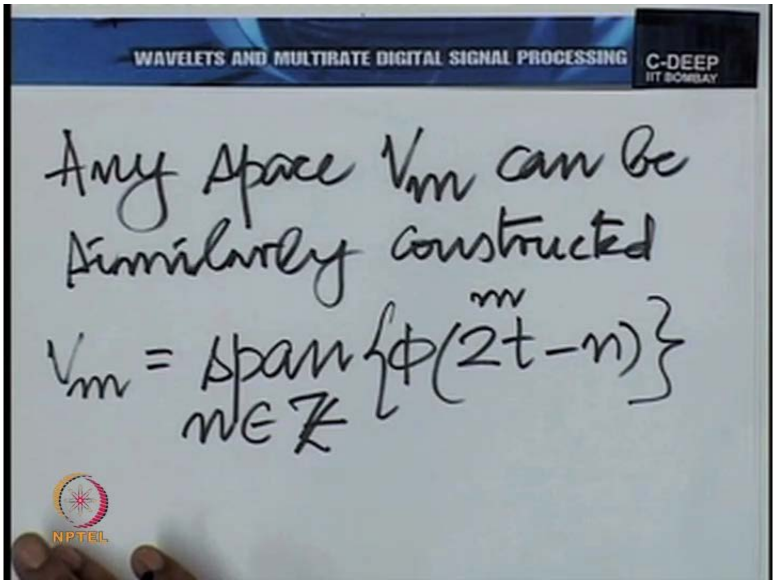

So it is very easy to see that any space Vm can be similarly constructed. In fact we can be more precise, we can write down Vm is essentially the span, overall n belonging to z of Pi 2 raise the power of m t - n. So just as we looked at the wavelength yesterday & said it's this single function which can allow details to be captured, we now have this function Pit which captures representation at a resolution. It's a very powerful idea, you think about it.

If you want to capture information at a resolution, at a certain level of detail, i.e. all the information up to that resolution, if have the function Pi. If you wish to capture the additional information in going from one sub space to the next, you have the function Psi, the wavelet. Now we need to give this function Pit a name.

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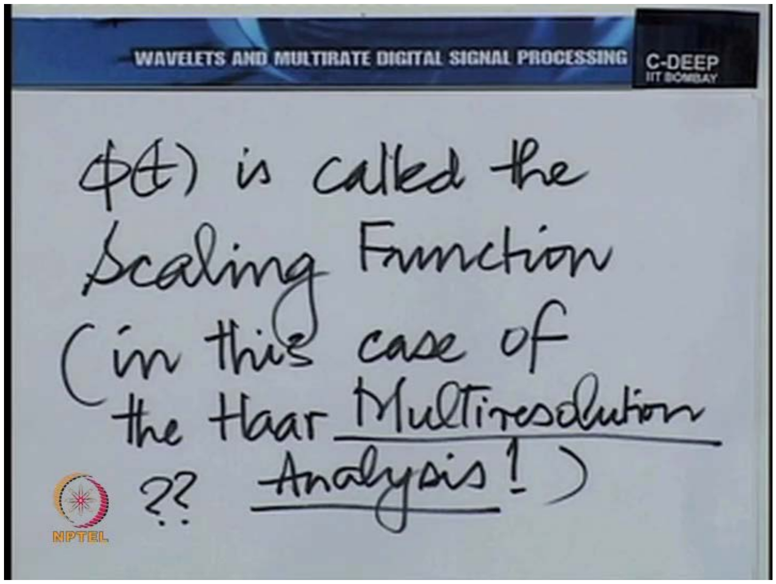

We shall call it a scaling function. Now of course here the Pit that I've drawn in this context is the scaling function of the higher wavelength, or the higher multi resolution analysis. Now what is this multi resolution analysis, I've suddenly brought in this word. What is this?

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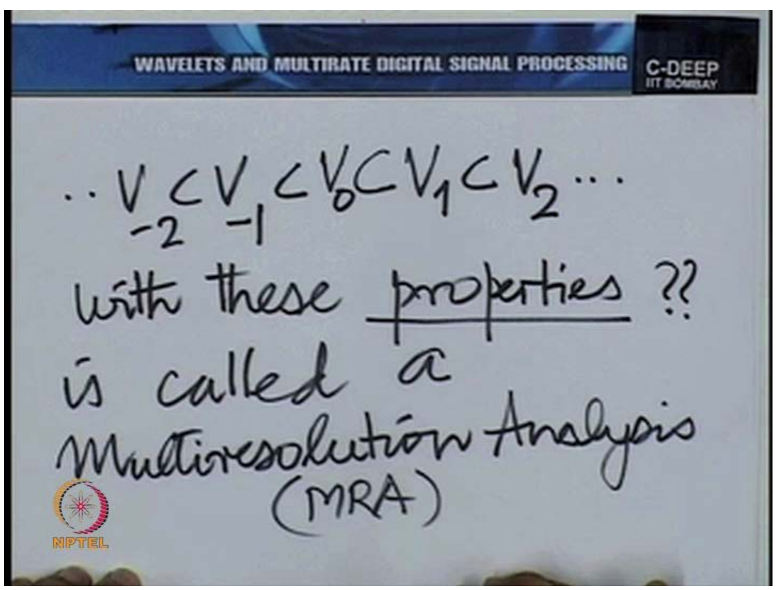

Well, this ladder of sub spaces, that we are talking about here, with these properties, is called a Multiresolution Analysis, or MRA for brief. Of course in this case the Haar higher multi resolution analysis. Now what properties, we need to put them down once again, we introduced two of them & one subtly, but we now need put down the axioms very clear.

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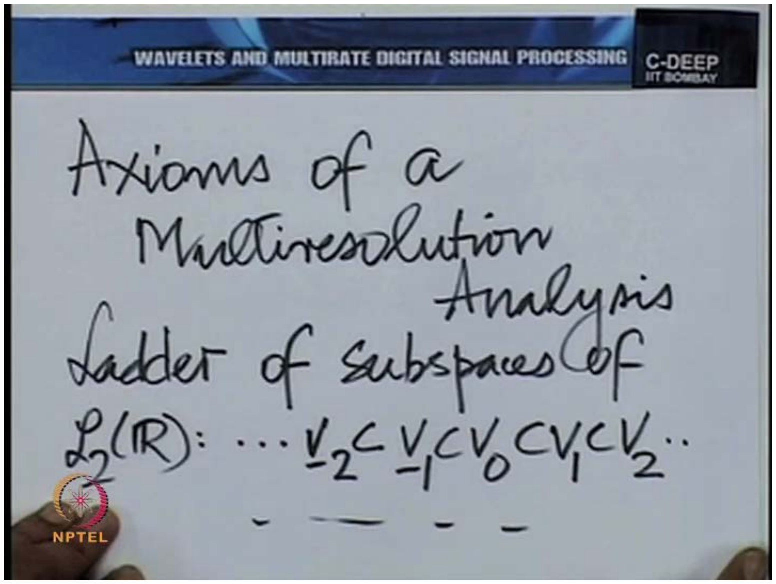

So let's put down what we call the axioms of a multi resolution analysis. So the first axiom is the ladder of sub spaces of L2 r, & we know what that ladder looks like.

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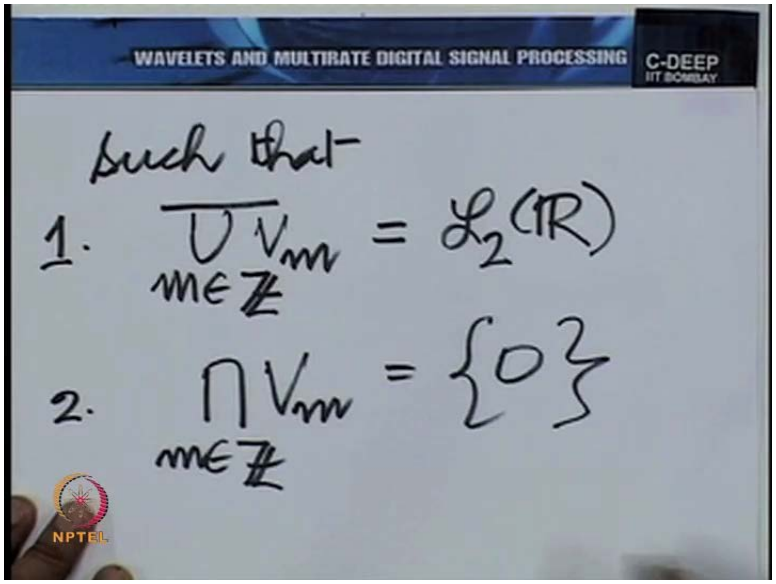

Such that axiom no.1 union over all integers closed Vm is equal to L2r, intersection over all integers Vm is the trivial sub space with only 0 element. These are not all, further…

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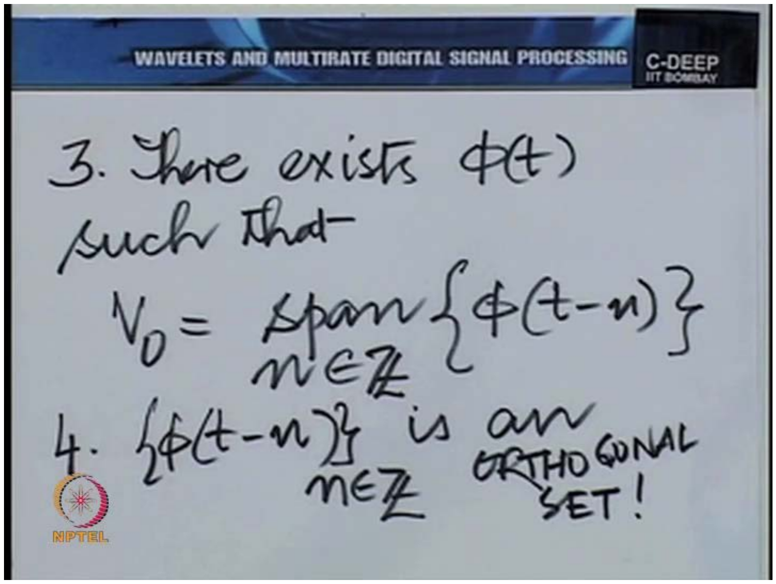

There exists a Pit such that V0 is the span over all integers n of Pit-n. Point four, in fact it's not just span, there's something more. These Pit-n over all n is an orthogonal set. This is a deeper issue here.

Now we will explain in more detail the notion of orthogonally in the next lecture, but for the moment let us be content to out this down as an axiom.

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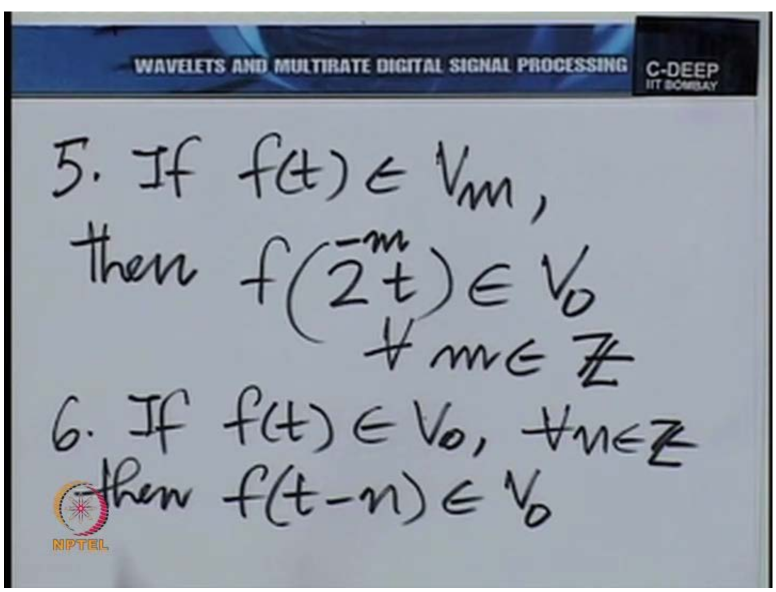

Next if ft belongs to Vm then f2 raise then power of mt, or 2 raise the power of –mt belongs to V0. So for example if ft belongs to V1 then ft by 2 or f2raise the power of -1t belongs to v0. For all m belonging to z & if ft belongs to VM, to V0, then ft-n also belongs to V0 for all integer n. So these are the axioms of a multi resolution analysis. That means, this is what constitutes a multi resolution analysis.

And here we've taken the Haar Multiresolution Analysis to bring the idea up. But the whole abstraction is that we can have several different files & then to end this lecture the corresponding Psi. Where does this Psi come in. it comes in, what is called, the theorem of Multiresolution Analysis.

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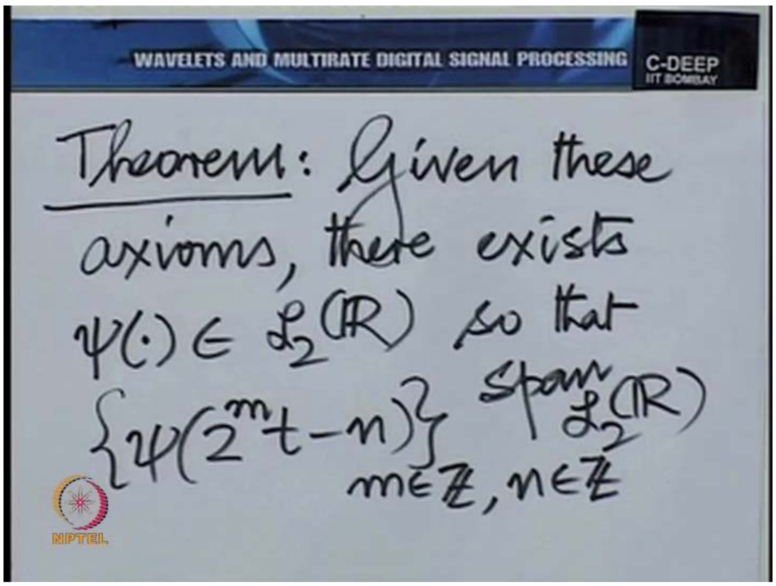

Given these axioms there exists a Psi belonging to L2r so that Psi 2 raise the power of mt-n for all integers m & all integer n span L2r. This is a very significant idea. In other words, this is exactly what we said yesterday, take dyadic dialates & translates of the function Psi & you can cover all the function, or go arbitrarily close to any function in L2r as you desire. We built this idea up from the Haar example, but in the next lecture we shall try & built a little more abstraction into what we've done & proceed further from there. Thank you.

"Professor - Student conversation starts"

Hello! Everyone, I'm Nikunj Patel. I'm TA for this course. You can ask any questions or doubts if you have regarding the material covered in the class.

Hi! Nikunj can you tell us about the analytical group of ladders of subspaces.

I'm not giving the exact proof but I can prove that V0 is a subset of V1 & you can further extend it by induction.

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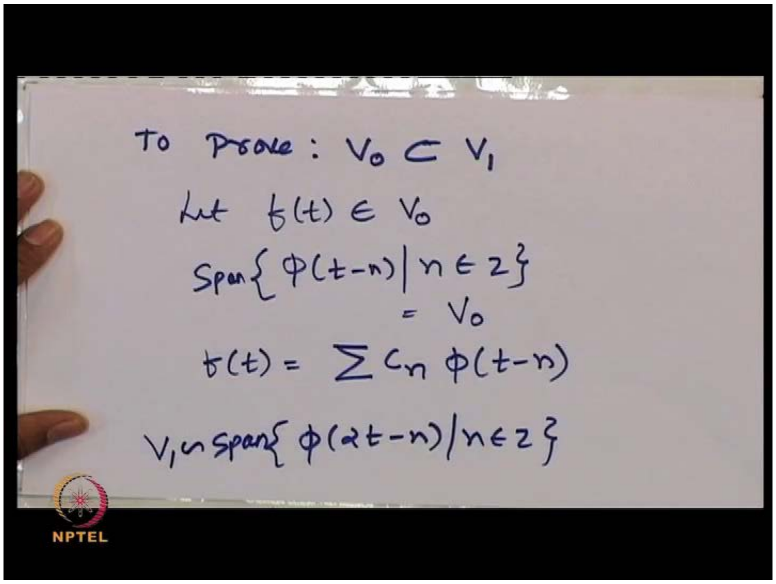

Here we will prove analytically that V0 is a subspace of V1. Let us take any function ft that belongs to V0. Basis functions for V0 are Pit-n for n belongs to z. Hence ft can be written as a linear combination of basis functions of V0. The basis functions of V1 are Pi 2t-n, n belongs to z.

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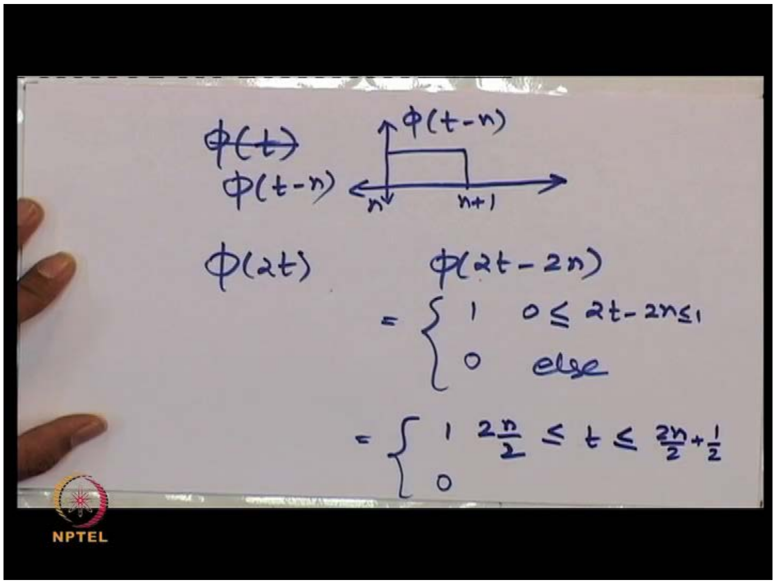

Let us try to write the basis functions of v0 i.e. Pit in terms of basis function of v1, this is Pit, Pit-n. What are the basis functions of v1, Pi2t? Let us try to draw how Pi2t-2n looks.

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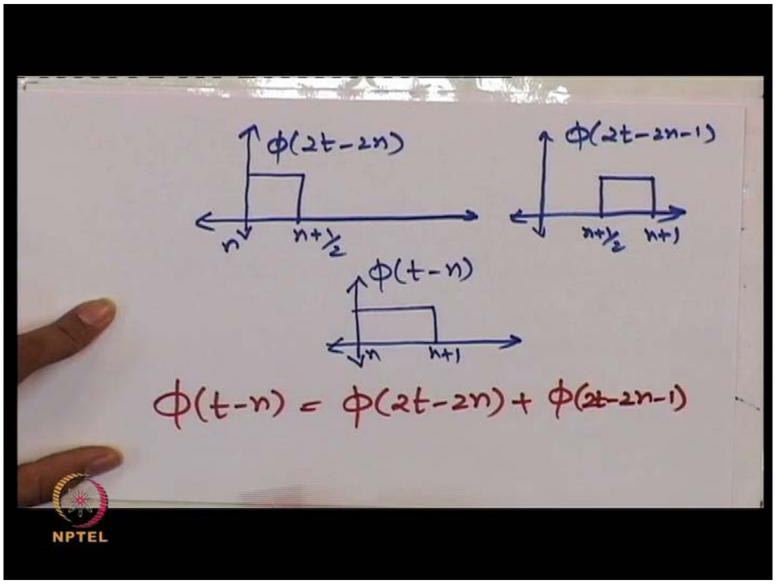

Now Pit will be look like this, Pi2t-2n, n to n + 1 by 2. Now let us plot Pi2t-2n-1. If we derive analytically then it will be from n+1 by 2 to n+1. This is nothing but Pit, Pit-n. Hence we can write Pit-n is equal to Pi2t-2n+Pi2t-2n-1. We will replace Pit-n in this equation 1.

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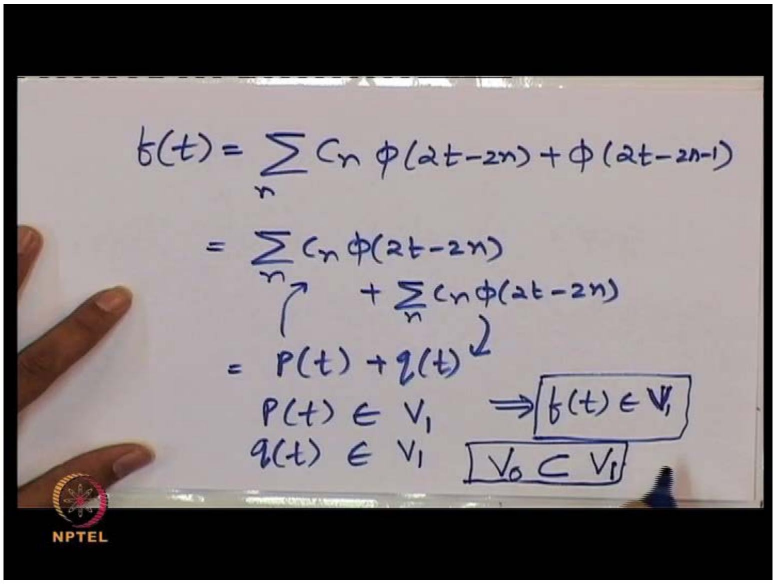

Then ft is equal to sigma Cn Pi2t-2n+Pi2t-2n-1 is equal to sigma nCnPi2t-2n+sigma ncnPi2t-1. Let us call this pt & call this qt. Now since pt is represented as a linear combination of basis functions of V1 we can say that pt belongs to V1, also qt is represented as linear combinations of basic functions of V1, hence we can also say that qt belongs to V1. Since ft is linear combination of two functions which belongs to V1 we can say that linear combinations pt & qt belongs to V1. It implies ft belongs to V1.

Here we proved that first we take a function ft that belongs to V0 & analytically we proved that ft also belongs to V1. Hence we can say that V0 is a subspace of V1.

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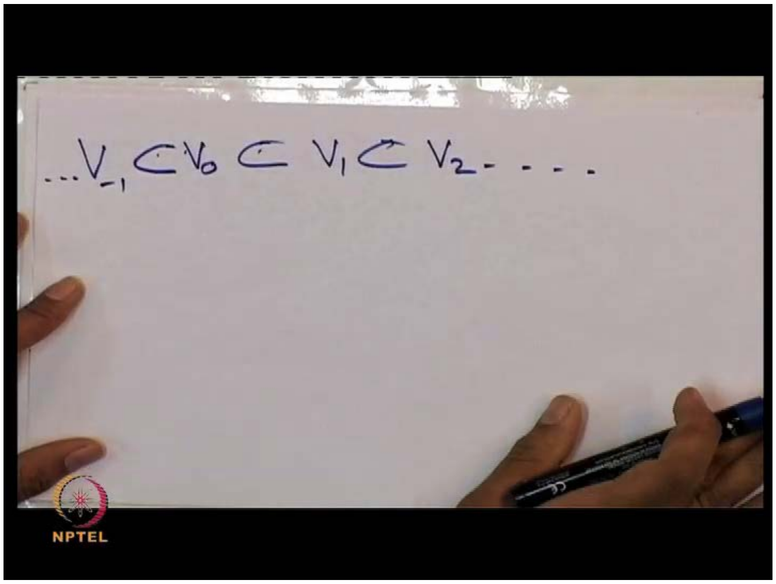

You can derive this by induction, the whole ladder of subspaces which is V0 is a subset of V1 is a subset of V2, so on. You can also go backwards like this V-1 & all. That’s the prove that the ladder of subspaces.

I wanted to ask that is L1r a subset of L2r?

It is generally not the case, means all the functions of L1r does not belong to L2r, I can give you a counter example of a function which belongs to L1r but doesn’t belong to L2r.

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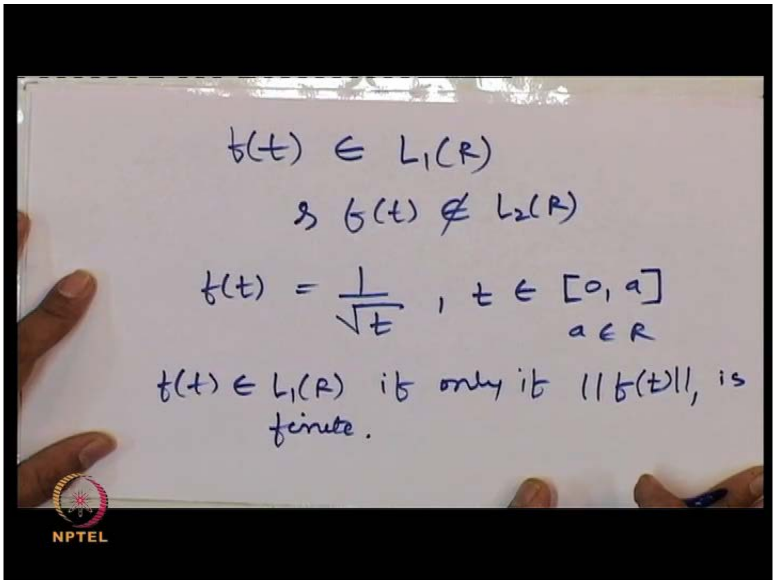

Here I will give a counter example of a function which belongs to L1r but not belongs to L2r. Let us take ft as 1 upon root t. t belongs to 0 to some constant a. a is some finite constant belongs to r. Now we can see that ft belongs to L1r if and only if L1 norm of ft is finite.

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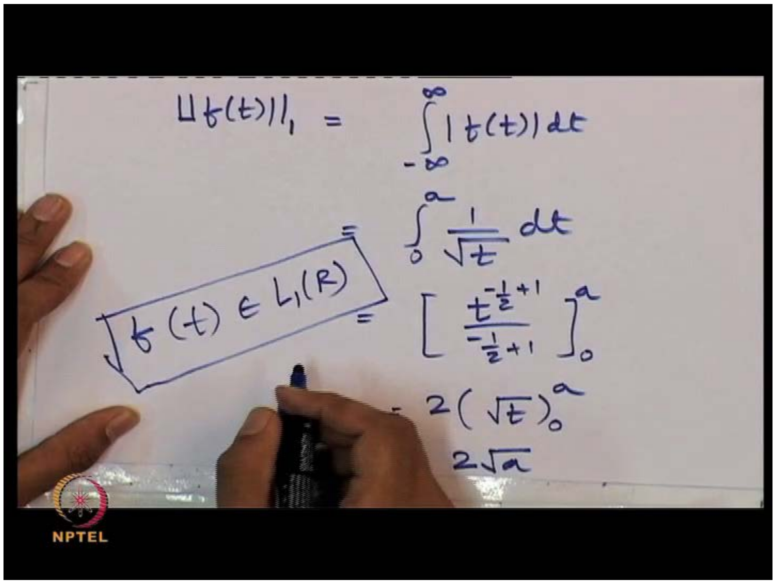

We will compute the L1 norm of ft. is equal to ft dt- infinity to infinity. Since ft exist only for 0 to a, we can apply the limit. Is equal to t power -1by 2+1 upon 1-1 by 2+1 0 to a. It is equal to 2 times root t, 0 to a, is equal to 2 times root a. since a is some finite number L1 norm of ft is a finite, hence a is some finite number L1 norm of ft is finite; hence we can say that ft belongs to L1r.

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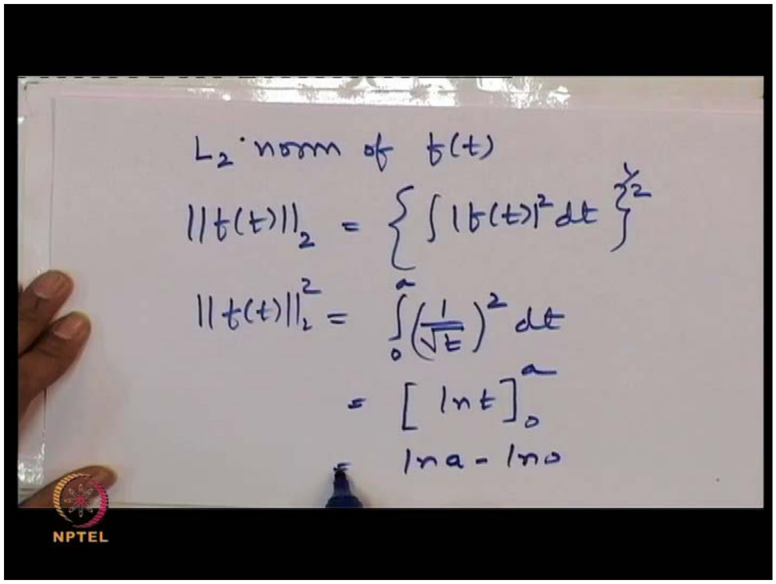

Now we will compute the alto norm of ft & see that it is finite or infinite. Is equal to square dt or 1 by 2. We will square it on both the sides then we will take the square root. 1 upon root t 0 to a, the whole square dt is equal to ln t 0 to a, it is equal to ln a-ln0 which is indefinite.

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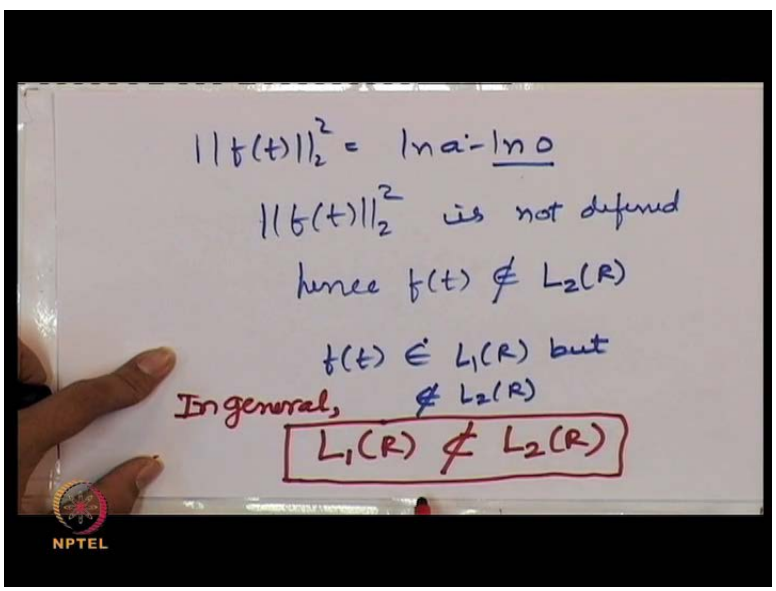

Is equal to ln a-ln 0 so this is indefinite. Hence we can say that l2 norm of ft is not defined. Hence ft doesn’t belong to L2r. Here we have seen that f2 belongs to L1r, but it doesn’t belong to L2r, hence we can’t say in general that L1r is a subspace of L2r. Thank you!