**Foundation of Wavelets and Multirate**
**Digital Signal Processing**
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**Lecture Number 3**
**Module No. 2**
**Ladder of Subspaces**

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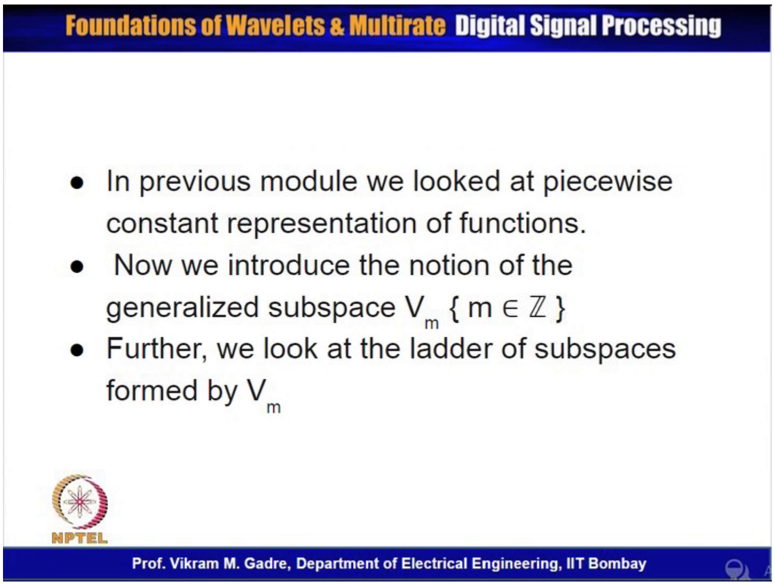


So therefore, this set of functions that we talked about a minute ago is indeed a linear space.

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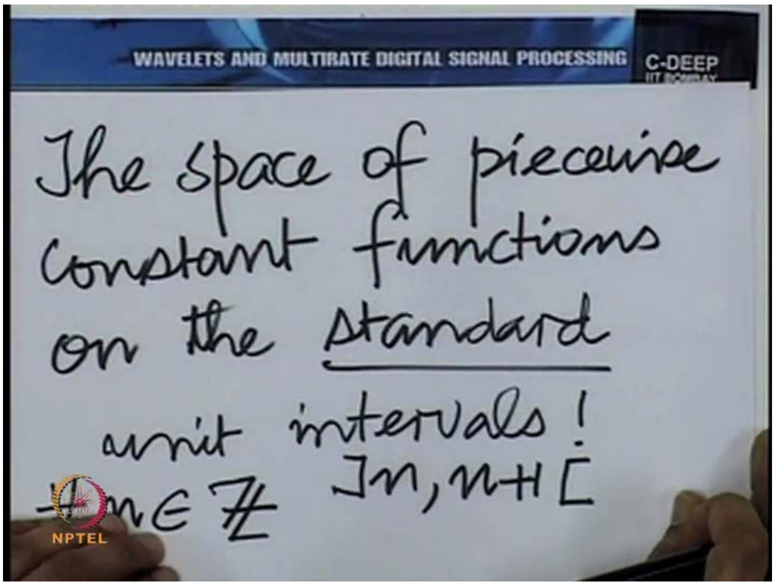

That is why I have called it space here. And we will give that space name, so we will call that space V 0. So V 0 is a set, now I am going to write the mathematical notation. V 0 is the set of all x of t such that 2 things happen, x of t.

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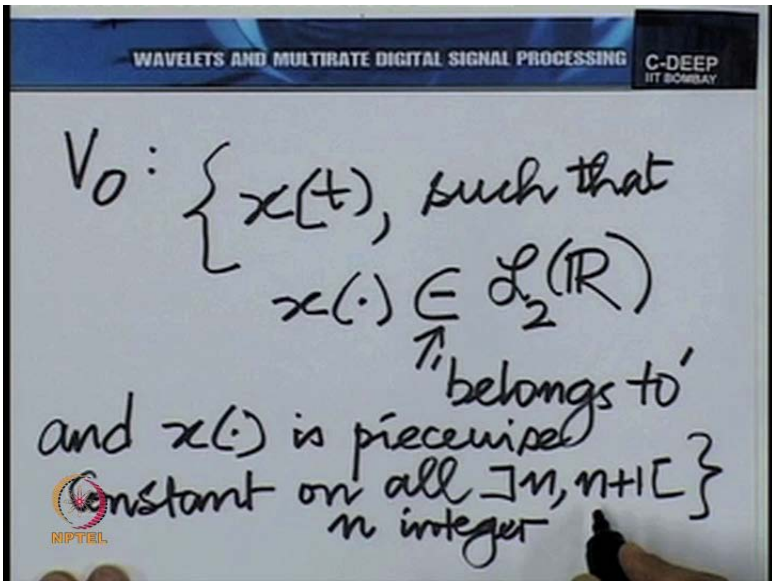


Now you know x is a function, so when I write like this, what I mean is, I am suppressing the explicit value of the independent variable. But I recognise that there is an independent variable here, I am treating the whole thing as an object. It is the function and I am treating the whole function as an object. And this object belongs to L 2 R, recall that L 2 R is the space of all function which are square integrable and this stands for belongs to.

Such that x belongs to L 2 R and x if piecewise constant on all intervals of the form n n + 1 n integer. Now, once we have talked about V 0 in fact, the reason for giving the subscript 0 here is that we are talking about 2 to the power of 0 as a side of the interval. That is important enough I think to make a noting. So, we say V 0 because of piecewise constancy on intervals of size 2 raise to 0 which is one.

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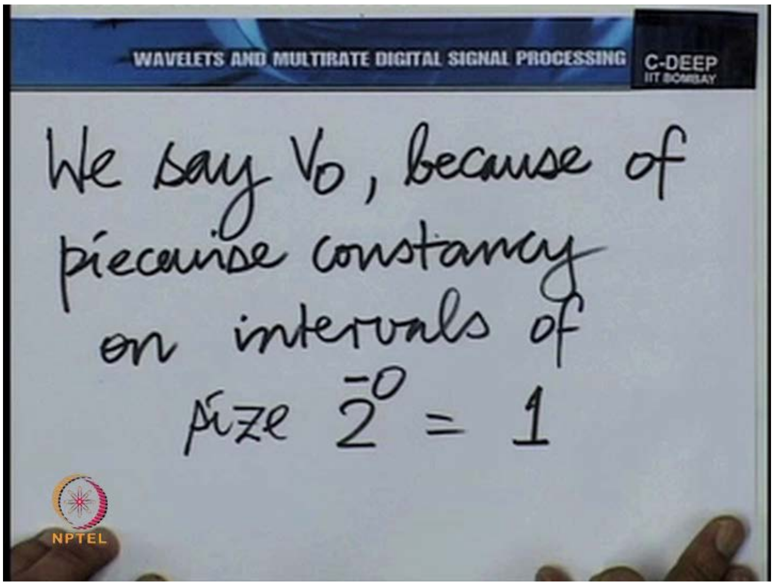


And similarly therefore, in fact you know you could call it 2 raise the power 0, you can call it 2 raise the power of - 0, we will prefer to use 2 the power - 0 because it would be consistent in future. So we could similarly have V 1 then. Similarly, V 1 is a set of all x, let us define the set of all x.

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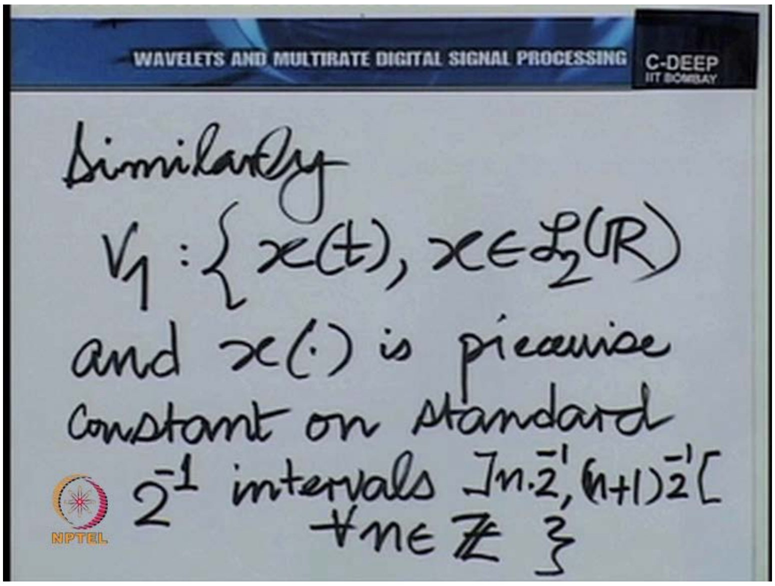


x belongs to L 2 R 2 and X is piecewise constant on standard 2 raise the power - 1 intervals. That is intervals of the form n into 2 raise to - 1, n + 1 into 2 raise to - 1 for all n integers.

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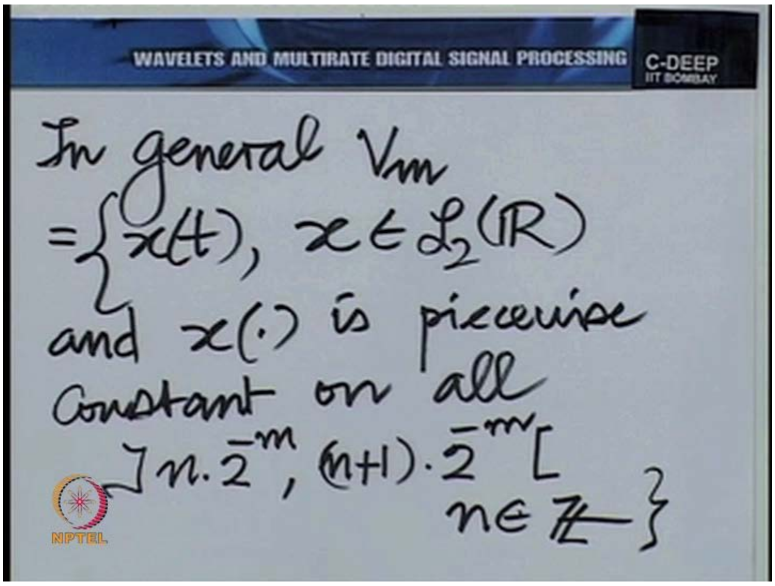


And in general, we have V m, the set of all x t. For completeness, we should write down the definition properly and x is piecewise constant on all open intervals of the form simple enough. To fix our ideas, let us catch a couple of examples. So let us take an example of x belonging to V 2, it would look something like this.

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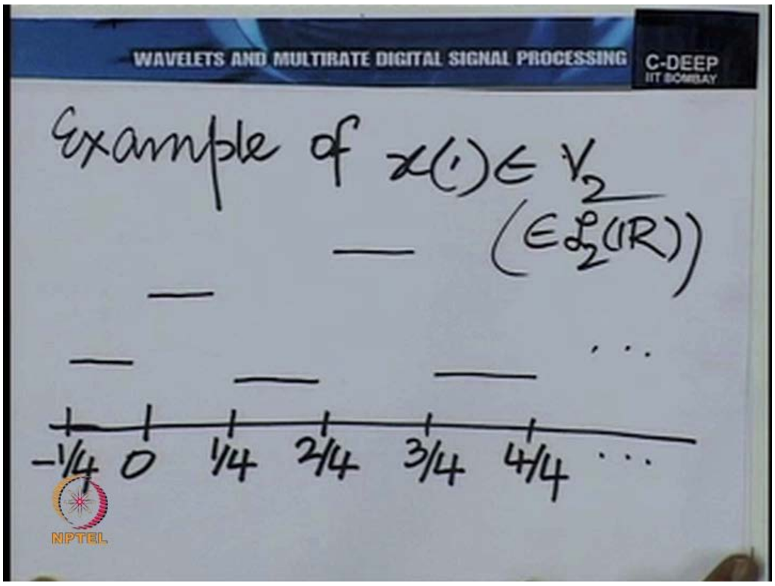


If you would have intervals of one fourth here and in fact to be complete, we should also include intervals before 0, hence so on.

And we have piecewise constant on this and so on there. And please remember x is also in L 2 R, so when you say it is in V 2, it is automatically of course in L 2 R. And that means if I take the sums square of all these constants, the sums square is going to be finite, that is an important observation. The constant that we assign here must be such that when we sum the square of all of them, magnitude square of all of them, that sum must converge.

This observation is so important that I think we should make a note of it. So we are saying the sum square, the absolute squared sum of the piecewise constant values in all these V m must be convergent.

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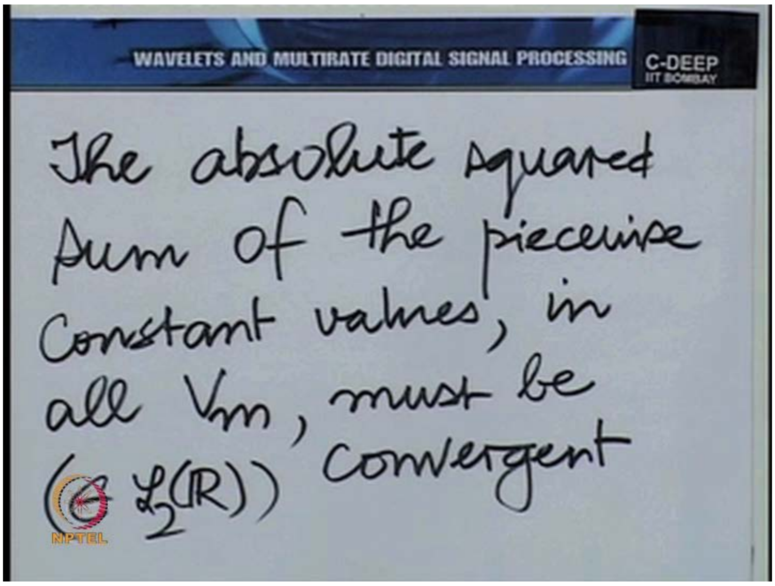


And this follows from belonging to L 2 R. Let us also take an example of a function belonging to V - 1. So - 1 means intervals of size of 2, 2 raise the power - of - 1. So interval of size 2 and so on there and so on there and we have piecewise constant there.

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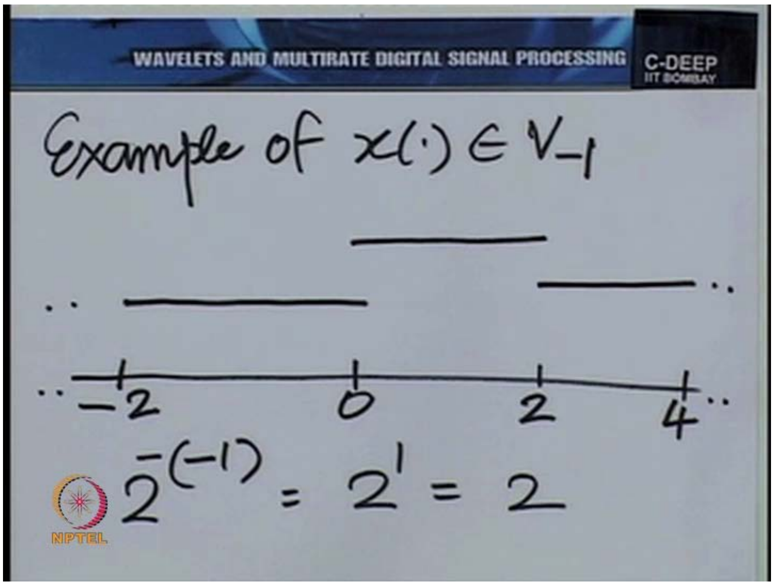


And so on here and so on there. So now we get our ideas fixed, what we mean by the space is the m. Now the moment we put down these spaces with these examples so clearly, we see a containment relationship. So the relation between these spaces, they are not arbitrary, they are not just totally destroying and unrelated.

In fact you can notice that if a function belongs to V 0 for example, which means that it is piecewise constant on the standard unit intervals, it is also going to be piecewise constant on the standard half intervals. And for that matter, if a function belongs to V 1, which means that it is piecewise constant on the standard half intervals, it is automatically going to be piecewise constant on the standard one fourth intervals.

To exemplify this, let me go back to this example of x belonging to V - 1 that I have here. Notice that this function is piecewise constant on the standard intervals of size 2.

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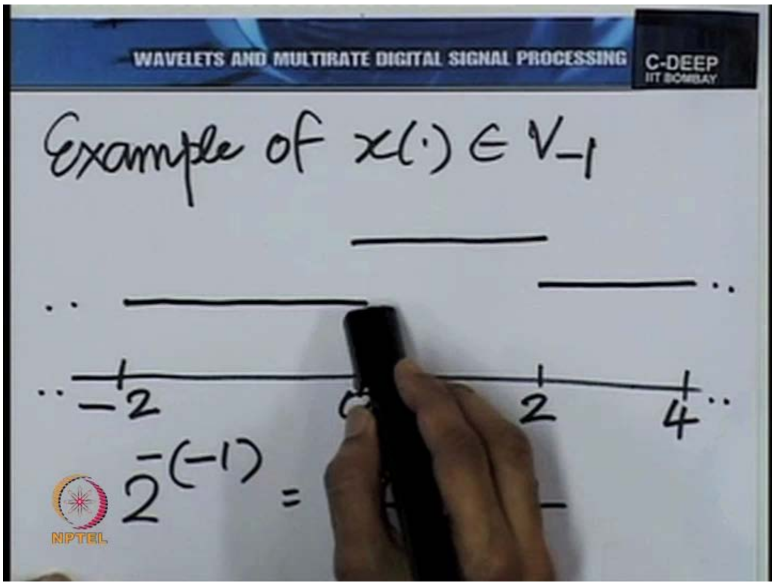


So obviously if you take the standard intervals of size 1 for example, 0 to 1, 1 to 2, 2 to 3, 3 to 4, - 2 to - 1, - 1 to 0 and so on, the function is still piecewise constant. So therefore a function that belongs to V - 1 automatically belongs to V 0, a function that belongs to V 0, automatically belongs to V 1.

And therefore there is a ladder of subspaces that is implied here. What is that ladder?

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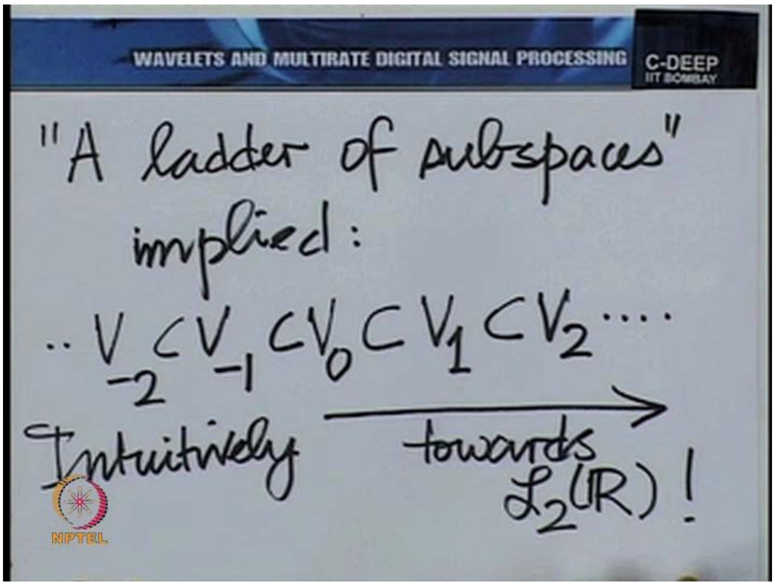


The space V 0 is contained in V 1, the space V 1 is contained in V 2 and so on this way. And of course the space V - 1 is contained in V 0, the space V - 2 in V - 1 and so on. And we expect intuitively as we move in this direction, we should be moving towards L 2 R. Of course it is an important question, what happens when we go in this direction that is interesting.

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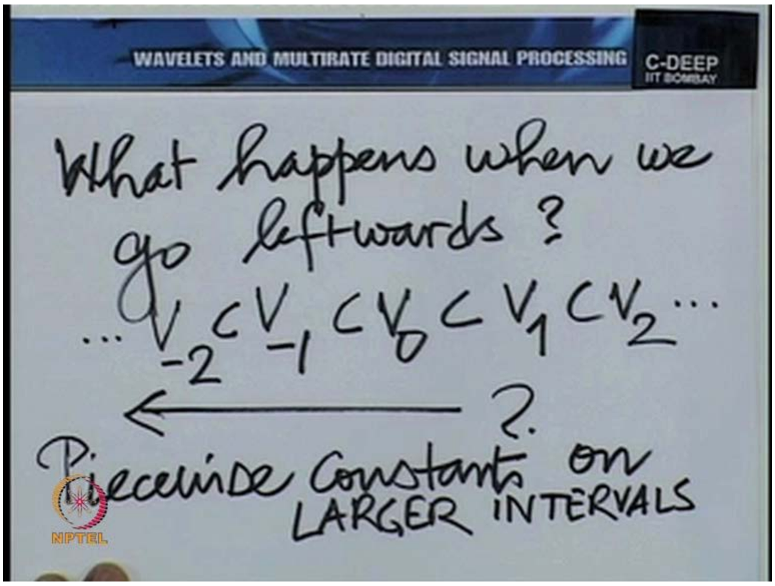


We spend a minute now and reflect on that. So you see, what happens when we go leftwards? What I mean by that is quotient of V 0 contained in V 1 contained in V 2 and then V - 1 contained I mean contained in V 0 yes, and so on here and so on there. What happens when we go this way? What do we think should happen? What are we doing?

We are taking piecewise constant function on larger and larger intervals. Let us write that down, so piecewise constant functions for larger intervals. Now you see what is the L 2 norm of functions as you go leftwards?

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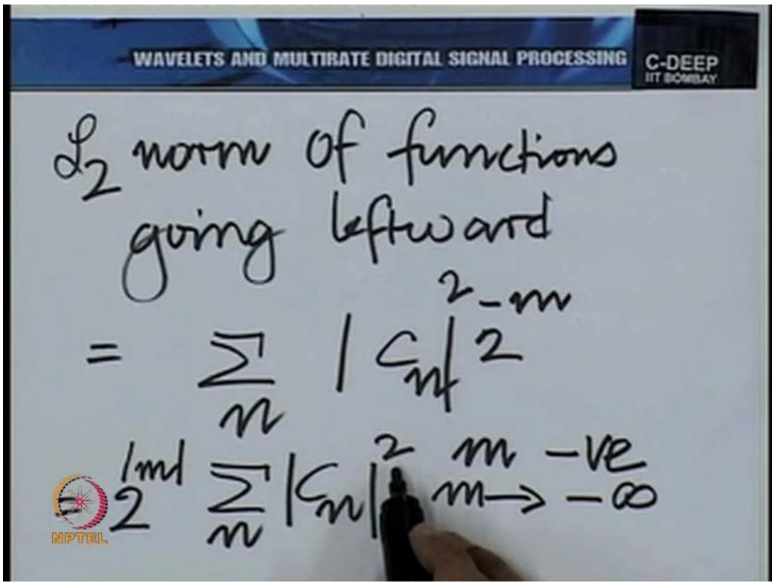


What kind of form will it have? It is going to have a form like this, submission on n, now you see remember the L 2 norm is the integral of the absolute square of the function and please remember that the function is piecewise constant.

So you have one constant, let us call it C n on the nth interval. And the interval is of size to raise the power - m. So this is essentially, you know you are talking about integrating mod C n square, it is a constant over an interval of 2 raise the power - m, and please remember m is negative. And m goes towards - infinity as you go leftwards. That is the same thing as 2 raise, now you see 2 raise the power - m is 2 raise the power of mod m in the context of negative m.

And submission on n mod C n square. Now you see this subtle point is that if this needs to be finite irrespective of how large m is, we have no control on this except that this part must be finite. But then when we say finite, if it is nonzero and if we allow m to grow without bound, this is going to diverge. So the only way in which this can converge no matter how large I mean large in the sense, large in magnitude.

How large in magnitude m is, no matter how large in magnitude m is if this is to converge, then this must be 0, a very important conclusion. So we are saying that if 2 raise the power mod m submission n mod C n square must converge no matter how large or how negative.

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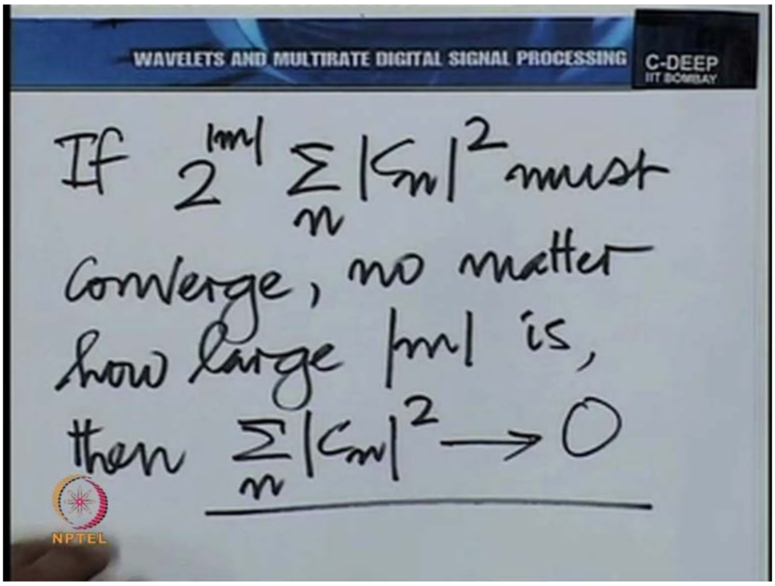


Then we must have submission over n C n square tending to 0. So essentially what we are saying is, every mood leftwards, we are going towards the 0 function.

A point that takes a minute to understand but is not as difficult as you can see. So now we have very clearly an idea of our destination as we move up this ladder towards + infinity and as we move down the ladder towards - infinity and we can formalise that. What we are saying is moving upwards, now you know one has to use proper notations. We would have attempted to say something like limit as m tends to infinity or + infinity or something like that.

But you see it is not really correct to talk about limits of sets, so we need to use that notation that is appropriate in the context of sets, namely union. So when we take a union of 2 sets and if one set is contained in the other, we are automatically taking the larger set, so moving upwards is attained by using union. In other words we are saying, the union of V m, m over all the integers should almost be L 2 R that is whether little catch is.

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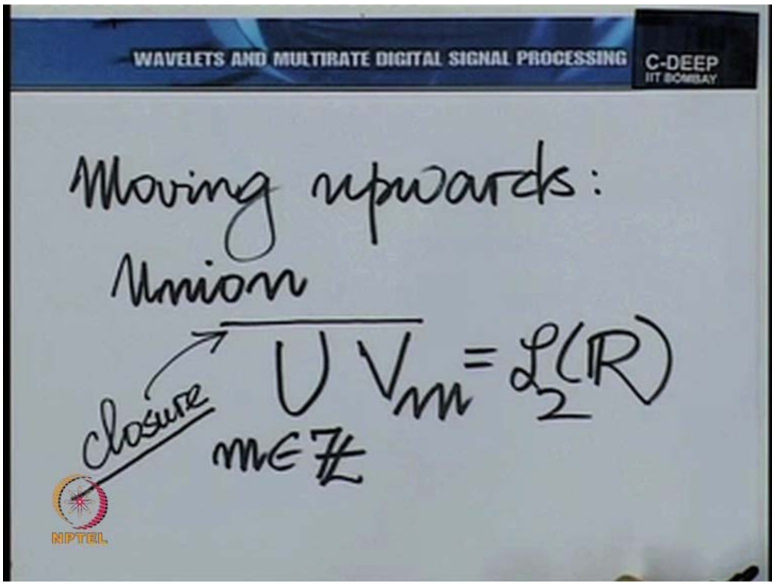


I mean, we would have been happy to write is equal to L 2 R. But, you know we need to make a little detail here, we need to put something called a closure. I will explain what I mean by a closure, you know suppose you have to visualise L 2 R to be like an object with a boundary. Suppose, this is where L 2 R, just notional and this is a boundary of L 2 R, so it is a space you know.

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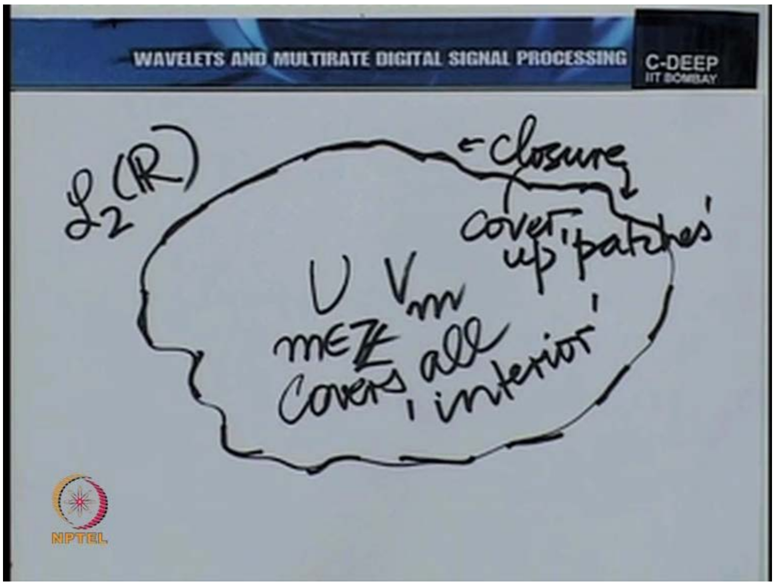


Now what we are saying is, as we go in union that is union m over all integers of V m, it would cover all the inside, it covers all the interiors. But then, it might leave out some peripheral things on the boundary, so it may also cover some part of the boundary. Now of course do not ask me at this stage what we mean by boundary and interior.

Save that, you know you are talking about situations you know boundary now informally when you say boundary, you are talking about functions where moving in a certain direction does not remain L 2 R moving in the other one does. So you know it is since the boundary and the interior at the moment needs to be understood only informally.

But what we are saying is, as far as this union evolves, it can take you almost all over L 2 R, it covers all the interiors, it may also cover quite a sizeable part of the boundary, but it might leave some patches of the boundary untouched. And therefore when we do a closure, we are covering up those patches. What we just did was covering up those patches. So closure means cover up boundary patches.

So this is a small detail and we need not spend too much of time in reflecting about this idea of closure and so on, but to be mathematically accurate, we do need to note that it is after closure that the union over all m integer of V m becomes L 2 R, otherwise it is almost L 2 R, which means that when you take this union that is when you take piecewise constant approximation on smaller and smaller and smaller intervals, you can go as close as you desire to a function in L 2 R.

So you can reduce the L 2 norm of the function to 0. So if you look at it that is what we mean by that is what mean that is what is implied by boundary. You know, you can you can go as close as you like to a certain function. You can make the L 2 norm 0, but it still wouldn't quite reach there, so you know you could just visualise that you might just be a teeny-weeny bit inside that boundary, but not quite on the boundary.

And how teeny-weeny, as small as you like, the touch where the union takes you. That is the subtle idea of closure. Anyway, as I said that we do not need to spend too much of time in talking about disclosure, but we should be aware of this idea because when we read literature on wavelet of that matter, when we really wish to put down the axioms of multiresolution analysis properly, we must be aware that this closure is required so much so.

Anyway, now let us take the second (()) (21:33) moving downwards so to speak. So how would we move downwards?

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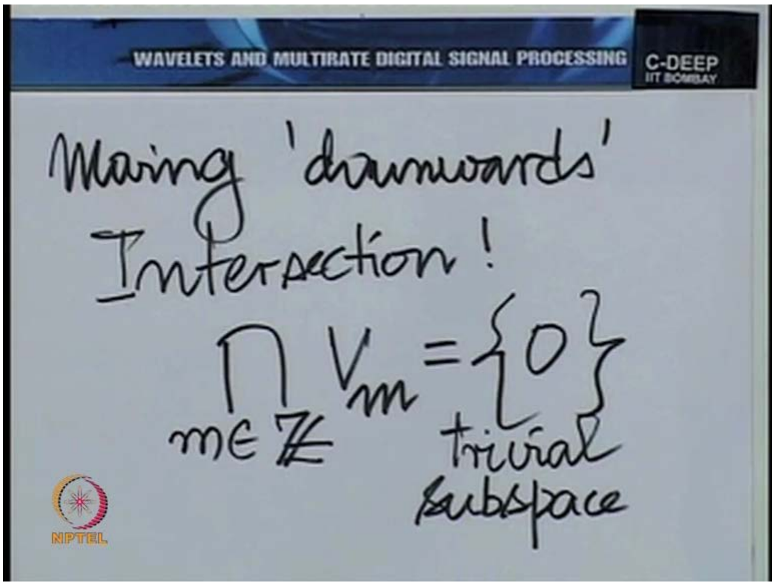


Just as union takes you upwards, interaction takes you downwards. So if you take an intersection, on all m belonging to Z of V m, there I do not need to worry about closure or anything of that kind. I can simply put down, this is a trivial subspace essentially the subspace of L 2 R with only the 0 function included, and this is called a trivial subspace.

Now again I must make an observation here to clarify. The trivial subspace is not the same as the null subspace; the trivial subspace has only the trivial 0 element in it. The null subspace does not have any element, so that is the subtle distinction and we must bear in mind that we are talking about the trivial subspace. Now, you know yesterday I told you that there is this beautiful idea about just one function Si t, it is dilates and translates going all over to capture incremental information.