Foundation of Wavelets and Multirate Digital Signal Processing

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Lecture Number 3

Module No. 1

Piecewise Constant Representation of a Function

Warm welcome to the 3rd lecture on the subject of Wavelets and multi-trate signal processing. Let us spend the minute on what we talked about in the second lecture. We had introduced the idea of wavelets in the second lecture. And we had done so by using the Haar wavelets. Essentially where piecewise constant of approximations are defined in steps by factors of 2 at a time.

In today's lecture, we intend to build further on the idea of the Haar wavelet by introducing what is called a multiresolution analysis or an MRA adult is often referred to in brief. So let me title today's lecture, we shall title today's lecture as the Haar multiresolution analysis and in fact let me also put down here the abbreviation for multiresolution analysis MRA.

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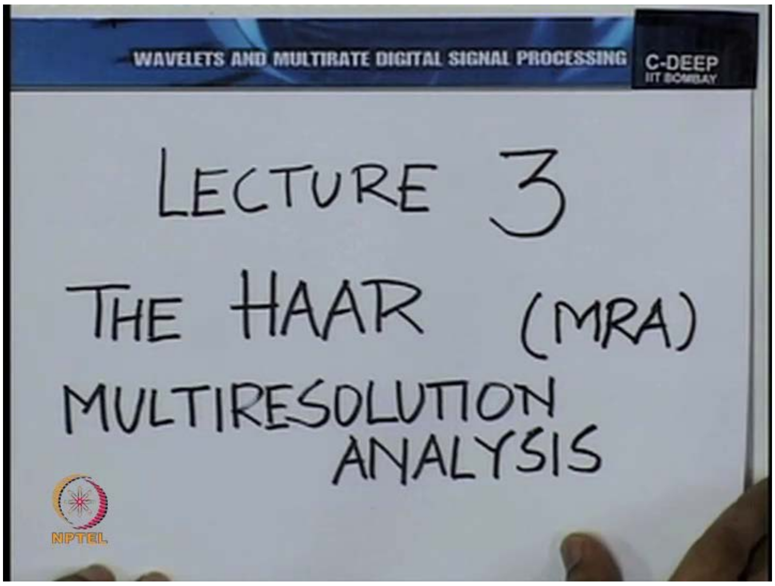

You see, in the whole idea of multiresolution analysis has been briefly introduced in the context of piecewise constant approximation. So recall what we said that the whole idea of the wavelet is to capture incremental information. Piecewise constant approximation inherently brings in the idea of representation at a certain resolution. We took the idea of representing an image at different resolution.

In fact we use the term resolution when we represent images in the computer. 512 cross 512 is a resolution lower than 1024 cross 1024 and one way to understand the notion of wavelets or to understand the notion of incremental information is to ask, if I take the same picture, the same two dimensional scene or same two dimension objects and represent it first at the resolution of 512 cross 512 and then at a resolution of 1024 cross 1024.

What is it that I'm additionally putting in to get that greater regulation of 10 24 cross 1024 which is not there in 512 cross 512? The Haar wavelet captures this. So in some sense, you may want to think of the Haar wavelet as being able to capture the additional information in the higher resolution and therefore if you think of an object with many shells. So this is a very common analogy.

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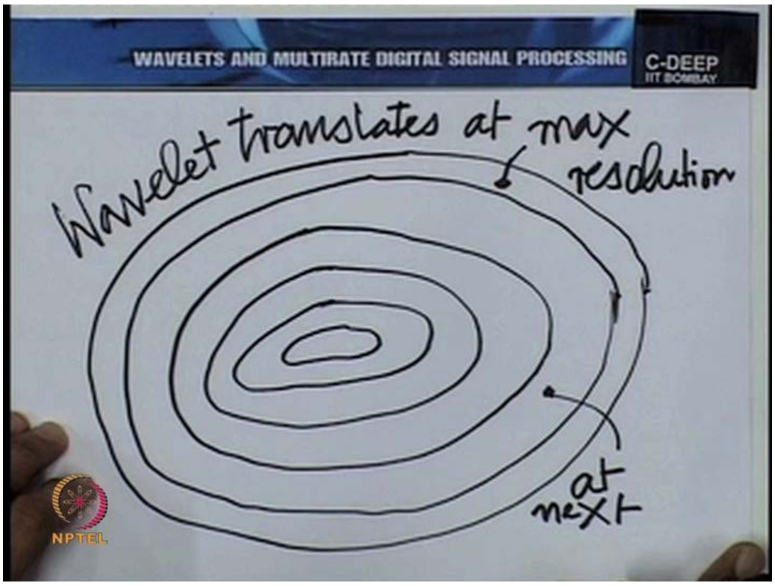

You know, if you think of the maximum information, maybe as a cabbage or an onion informally. And if you visualize the shells of this cabbage or the onion like this, then the job of the wavelet is to take out a particular shell. So the wavelet at the highest resolution, wavelet translates at highest resolution, at max resolution would essentially take out this. At next resolution, it would take out this shell and so on.

So when you reduce the resolution, what you are doing is to peel off shell by shell. In fact this idea is so important that we should write it down.

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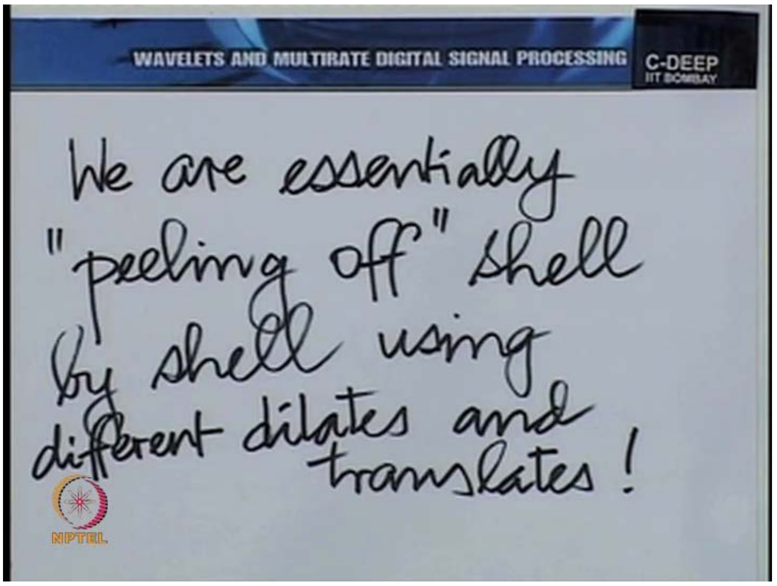

We are essentially peeling off shell by shell using different dilates and translates of the Haar wavelets. And there again a little more detail, different dilates correspond to different resolutions, and different translates essentially take you along a given resolution.

That is the relation between peeling off shells and dilates and translates. Now all this is an informal way of expressing this, we need to form (())(06:17) and that is exactly what we intend to do in the lecture today. Again we would now like to talk in terms of linear spaces. So, without any loss of generality let us begin with a unit length for piecewise constant approximation.

I say, without the loss of generality because after all what you consider as unit length is entirely your choice; you can call 1 meter unit length, you can call 1 centimetre unit length. Or if you are talking about time, you can talk about 1 second as unit length or unit piece and so on. So unit on the independent variable is our choice and that since without any loss of generality.

Let us start, make the focal point, piecewise constant approximation at a resolution with unit input. So let us write it down formerly. Piecewise constant, you know the (())(07:33) focal point is piecewise constant approximation on unit interval.

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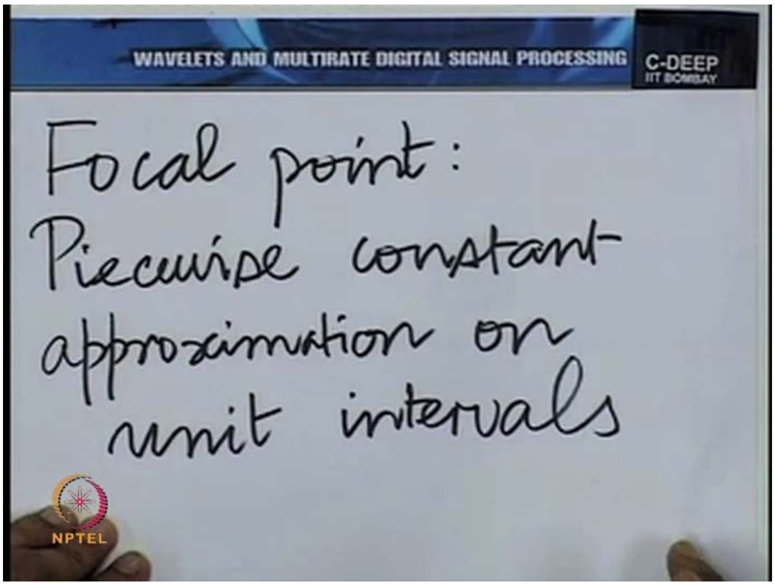

And let us sketch this to explain it better.

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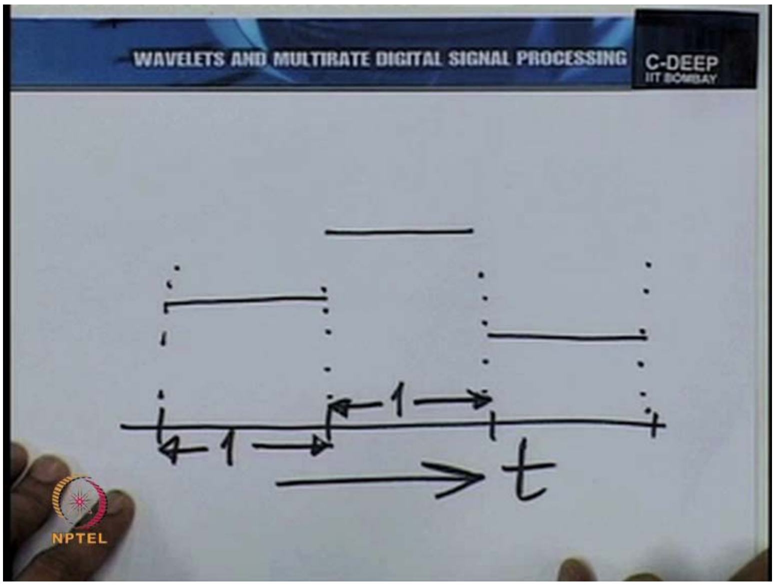

So what we are saying is, you have this independent variable again without any loss of generality, let that independent variable be t.

You have unit interval on this and on each of these unit intervals, you write down a piecewise constant function. Essentially corresponding to the average of the original function on that interval. So this is the average of the function on this interval, this one on this interval and this one on this interval. Now, how can we express this function mathematically with a single function and its translate?

So essentially we want a function, let us call it 5 of t.

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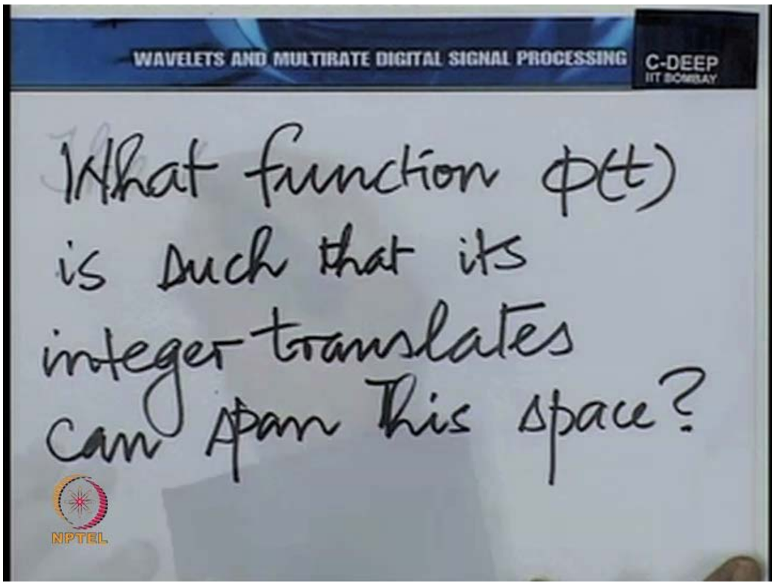

So what function 5 of t is such that its integer translate can span this space, what space?

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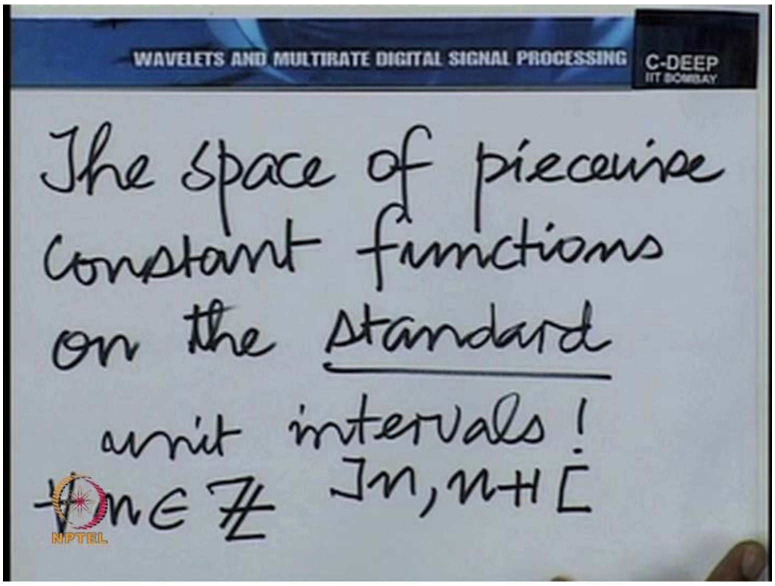

1st the space of piecewise constant function on the standard unit intervals. What are the standard unit intervals? The standard unit intervals are the open intervals in n + 1 for all over the set of integers. Now I wish to slowly start using notations which is convenient.

So this notation script Z would in general in future refer to the set of integers. So I think we should make a note of this.

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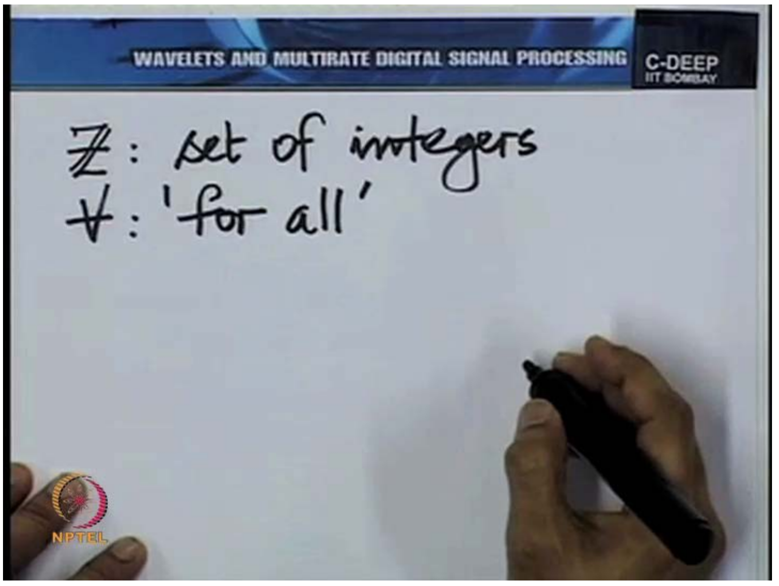

Script Z is a set of integers and this refers to for all. So what are we saying here, let us go back, we are saying, we have the space.

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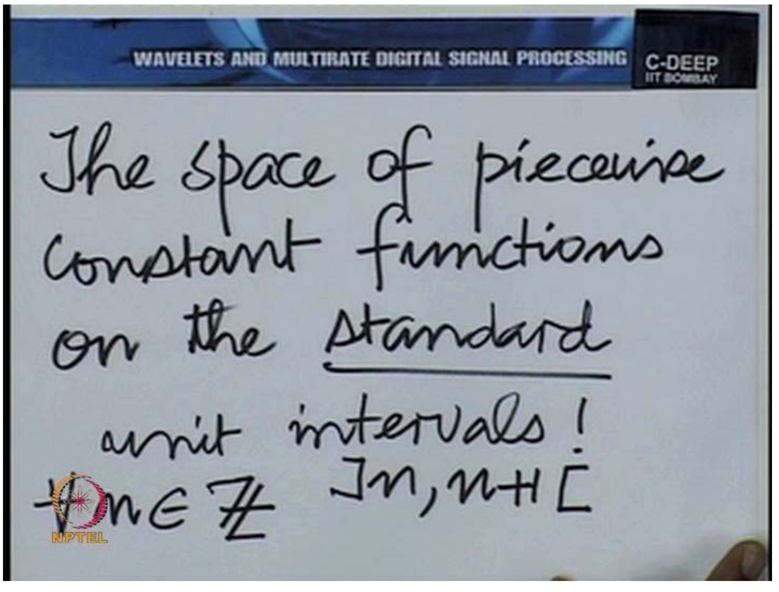

Now again I must a capital it recapitulate the meaning of space linear space. A linear space of functions is a collection of functions and a linear combination of which comes back into the same space.

So if I add to functions, it goes back into the same space. If I multiply a function by a constant, it goes back into the same space. If I multiply to different functions in that space by different constants and add up these resultants, it would still be in the same space. In general, we would say any linear combination, so we say the set a set of functions forms the space, a linear space if it is closed under linear combination.

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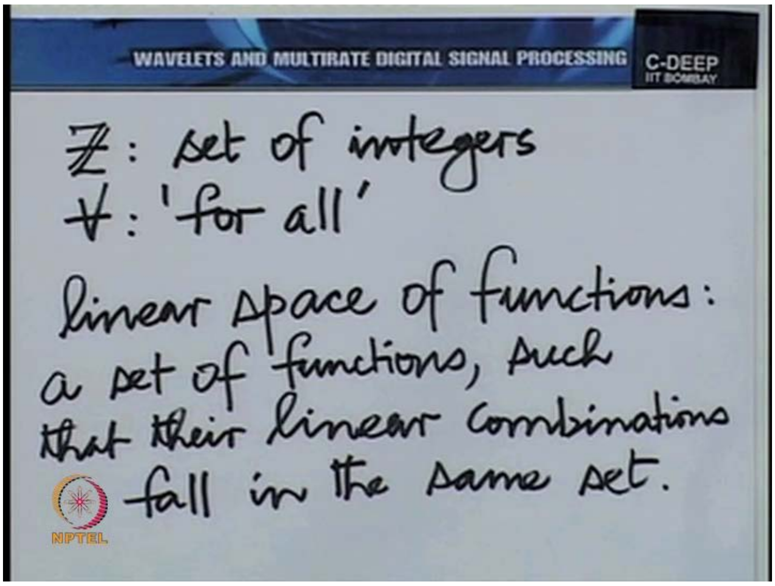

So we say a linear space of functions is a set of functions such that linear combinations fall in the same set. By making it a point to write down certain definitions and derivations in this course and there is an objective behind that, I believe that a course like this is best learned by working with the instructor. So, although one could just listen and try and remember, that does not give the flavour in a course like this.

It does require in depth reflection and thinking and therefore I do believe that the student of this course would do well to actually note down certain things and work with the instructor. So it is then that the full feel of derivations and the full feel of concepts would dawn upon the story. Anyway, with that little observation and instruction, let us go back to our pedagogy.

So you see a linear space of functions is one in which any linear combination of functions in that set fall back into the same set. Now here there is little bit of clarification required. You say in general if you consider the space or function that we talked about a minute ago.

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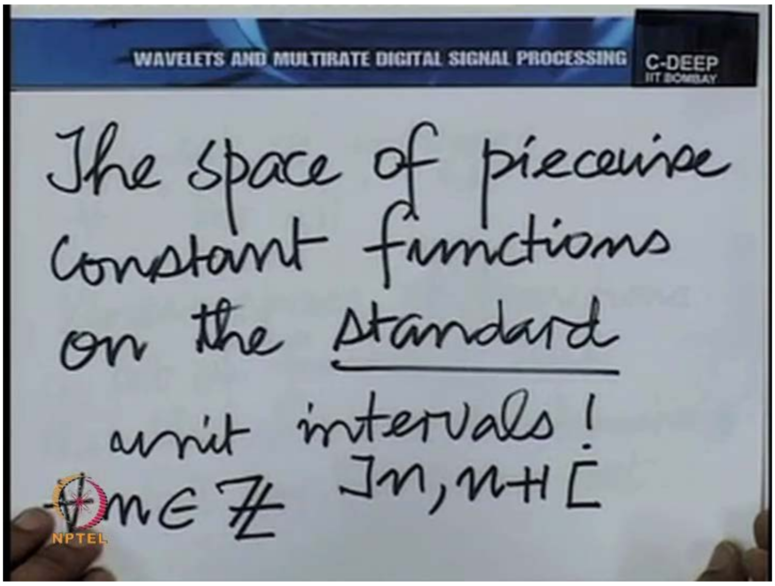

Namely, the space of piecewise constant functions on the standard unit intervals. Which is the standard unit interval? The intervals of the form, the open intervals of the in n for all n over the set of integers.

Then there is infinity of such functions. And naturally when you talk about linear combination, you could have finite linear combinations and you could have infinite linear combination. Now, for this point in time when we talk about linear combination, we are essentially referring to finite linear combinations. That is just the little clarification for the moment. While the idea could be extended to infinite linear combinations 2, but I don't want to into those niceties at this point in time, they would carry us away from our primary objective.