Foundations of Wavelets and Multirate Digital Signal Processing

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Lecture - 2
Module - 2
Dialates and Translated of Haar Wavelet

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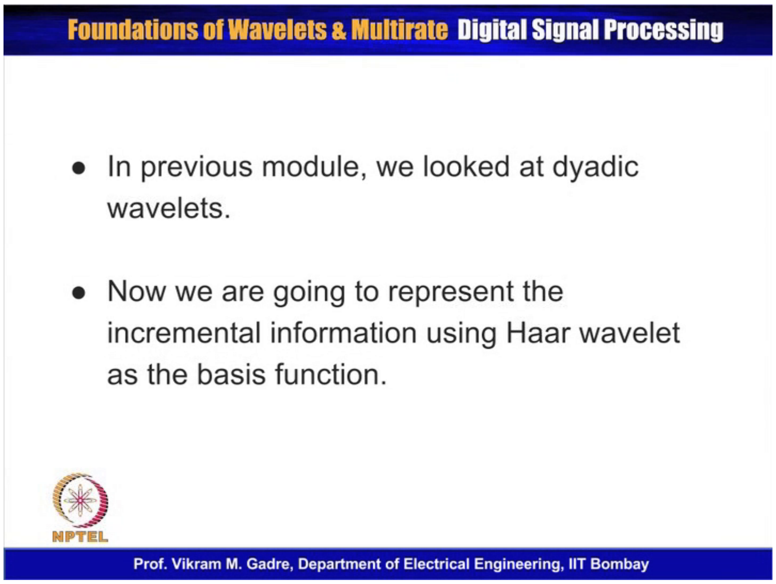

Now we would see how F2T minus F1T would look. It is very easy to see F2T minus F1T has an appearance like this.

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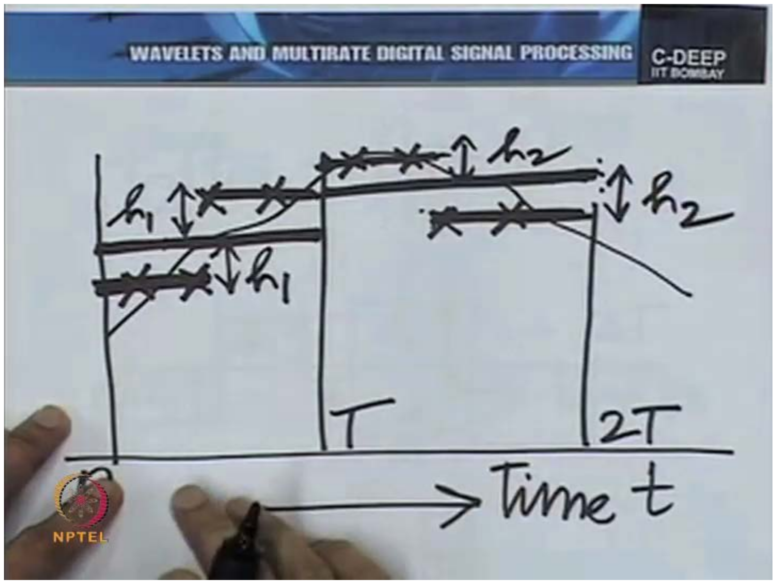

Let me flash them before you. F2T and F1T. Just for a second here so that you get a feel. This is F2 and this is F1 and visualize subtracting this from this.

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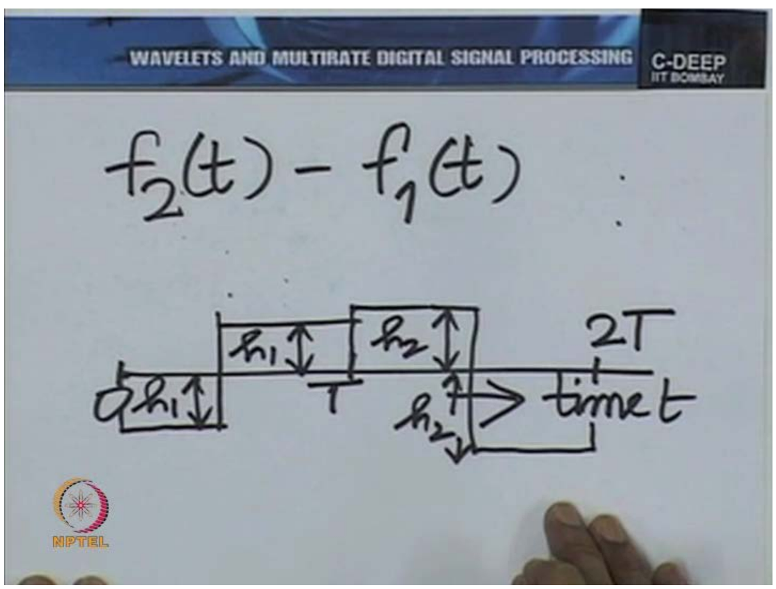

What would you get? A function that look like this. I have the time axis here so if I mark intervals of size T, there something like this. Maybe this has height H1 and this has height H2. Let me mark H1 and H2 on this diagram 2.

So this is H1 and this is H2. Of course so is this. Simple enough. Now if you look carefully we can construct all of this by using just one function and what is that function?

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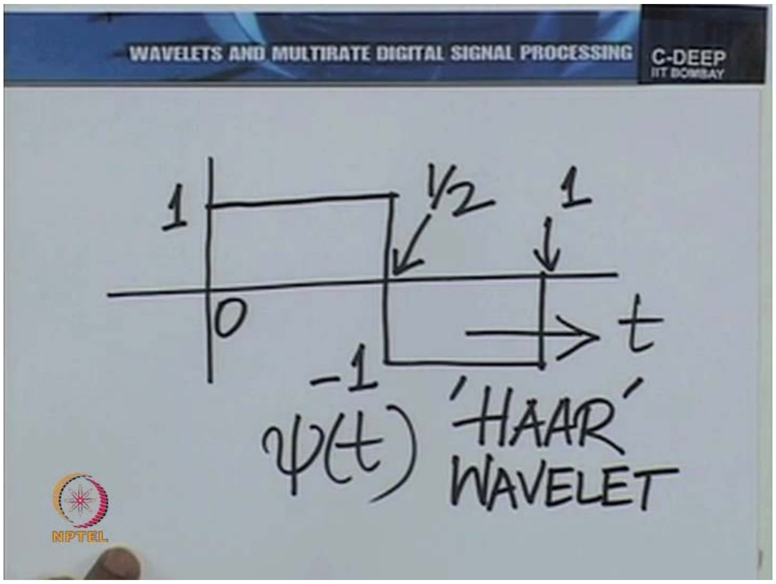

Suppose I want to visualize a function like this. 1 over the interval of 0 to 1 half and minus one over the next half interval. This is the 0.5, 0.1, 0.0,and 1 here and minus 1. And let's give this function a name, let's call it Psi of T. infect this is indeed called the Haar wavelength. Haar again the name of the mathematician.

It is very easy to see that using this function I can construct any such F2T minus F1T. Indeed if I went to take this function, stretch it or compress, whatever might be the case, depending on the value of capital T, dialated, and dialated is the more general word. So if I were to dialated this function to occupy an interval of T and bring it to this particular interval of T.

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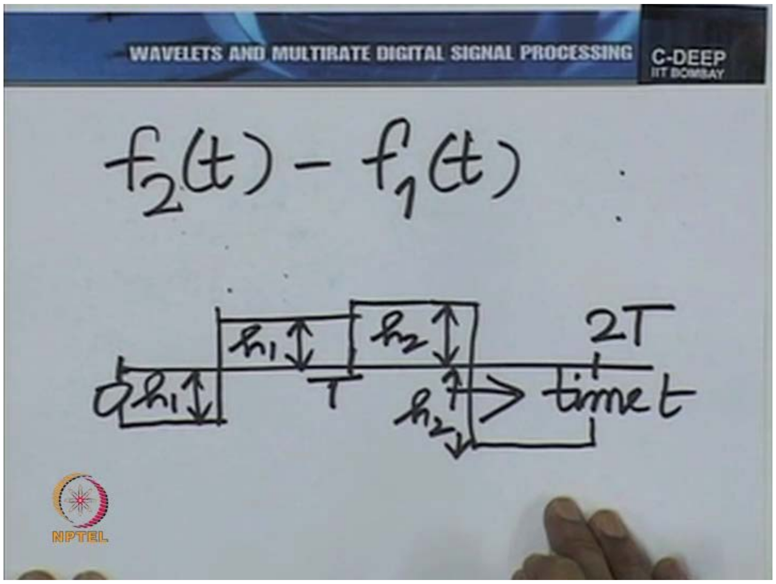

So I dialated that function Psi T and bring it to this interval of T. and then I multiply Psi T so dialated by the constant H1. Ofcourse H1 should be, is an algebraic constant. It should be given a sign.

Here for example H1 should be given a negative value because we started Si T with a positive minus 1, here. Similarly H2 has a positive value here. So in other words, this segment of F2T minus F1T, is of the following form.

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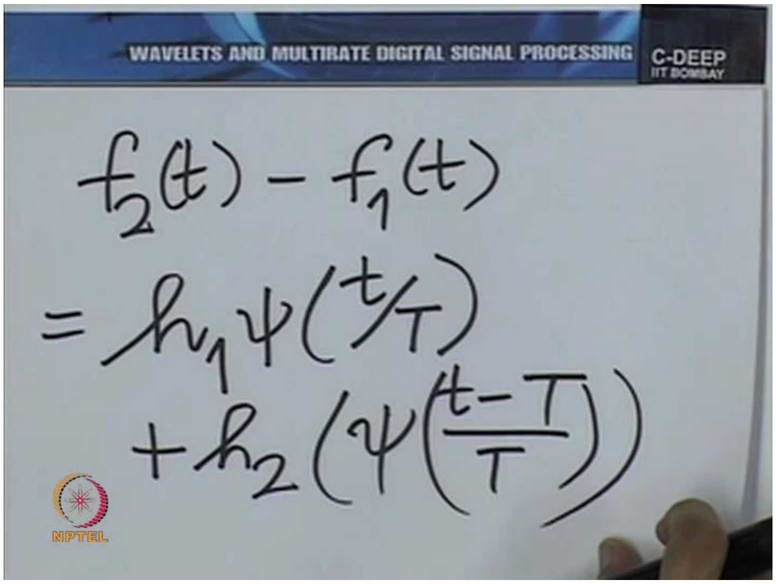

So in H1 times Psi of T dialated by T, plus H2 times Psi of T minus T dialated by T. So here this is both dialated and translated.

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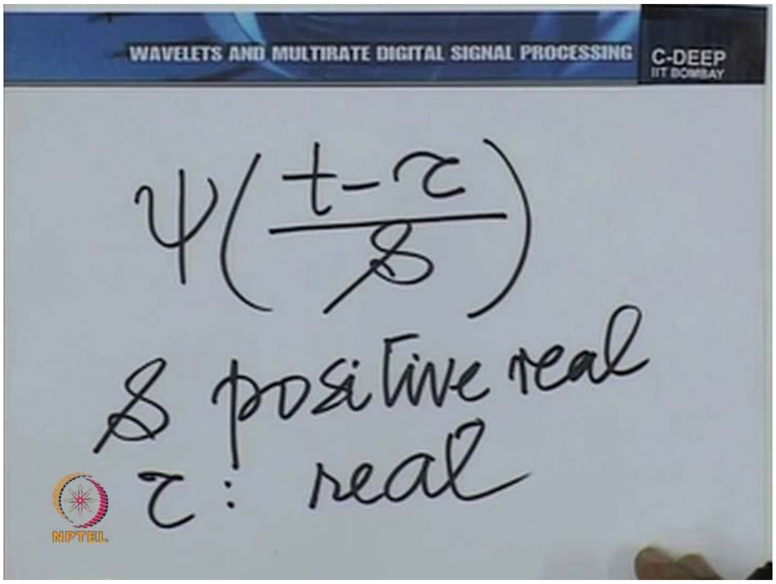

In other words in general when we start from the functions Psi of T, we are constructing functions of the form Psi T minus Tau by S. Where of course S is a positive real number and tau is real.

This is the general function that we are using as a building block different values of tau and different values of S. Of course here at a particular resolution, at a particular level of detail, the value of S is only 1. For example when we are representing the function on intervals of size T, we take S equal to T. If we were to represent the functions on intervals of Psi T by 2, then S would become T by 2 and so on. Then what we are doing in effect is dilating and translating.

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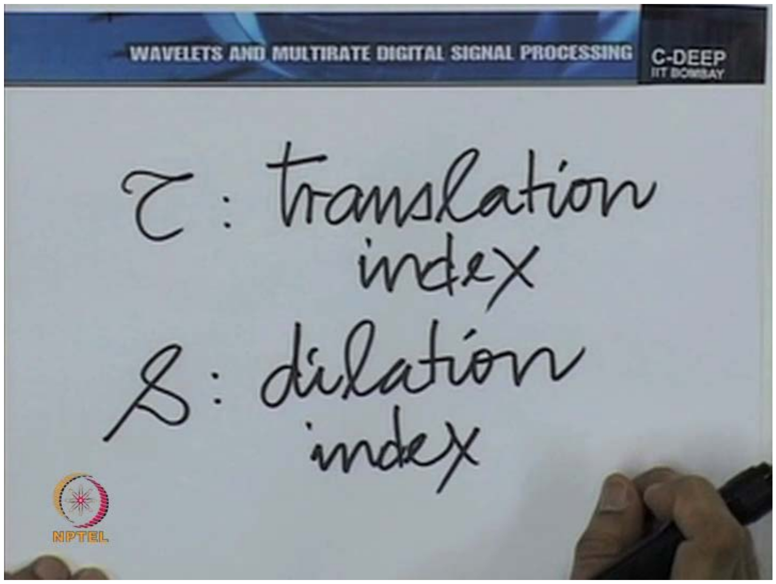

Now we introduce those terms. Tau is called a translation index or translation variable and S is called a dilation index or dilation variant and we are dilating and translating or we are constructing dialateds and translates of a basic function.

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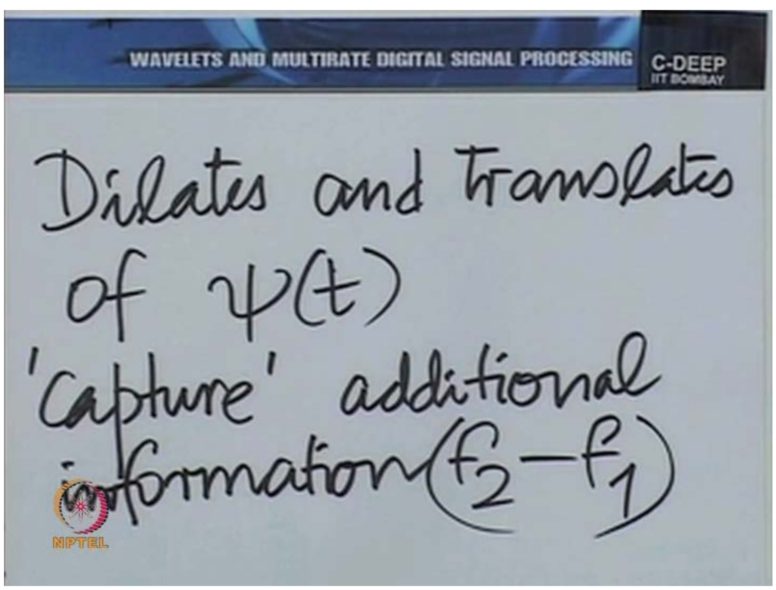

Dialateds and translates of Psi T capture the additional information. In F2T minus F1.

Let's spend a minute in reflecting about why this is so important. What have we done so far? Just looks like very simple functional analysis. Just a simple transformation or algebra of f1unctions.

What is so striking in just what we have said? What is striking is that what we have done to go from T to T by 2, can also be done to go from T by 2 to t by 4.

Not only that, what we have done to go from T to T by 2, in other words from intervals of length T, to intervals of length T by 2 all over the time axis can be done all over the time axis, to go from intervals of size T by 2 to intervals of size T by 4, and then you could go from intervals of size T by 4 to intervals of size T by 8, T by 16, T by 32, T by 64 and what have you, to a small interval as you desire.

Each time what you add in terms of information is going to get captured by these dialates and translates of the single function Psi T. a very serious statement if we think about it, deeply enough. That one single function Psi T allows you to bring in resolutions step by step to any level of detail. In fact in formal language in functional analysis we would put it something like this. You know in mathematics, these arguments of limits and continuity and so on and in some of these proof related to conversions, there is this notion of adversary and the defendant.

So here the defendant is trying to show, the one who makes the propositions is trying to show that by this process, you can go arbitrarily close to a continuous function. As close as you desire. Now as close in what sense? Well it could be in terms of what is called the mean squared error or the squared error. So let’s formulate that adversary proponent kind of argument here.

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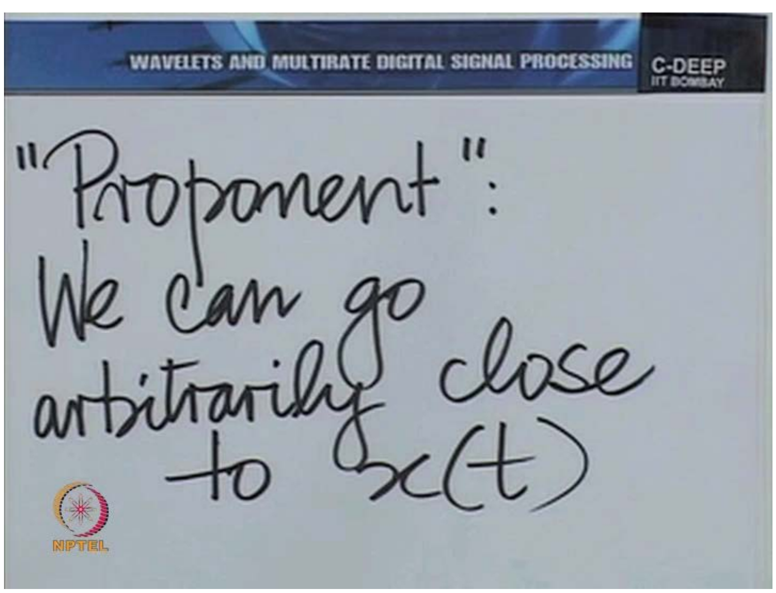

So what we are saying is, the proponent says we can go arbitrarily close to XT, to a continuous function XT by this mechanism.

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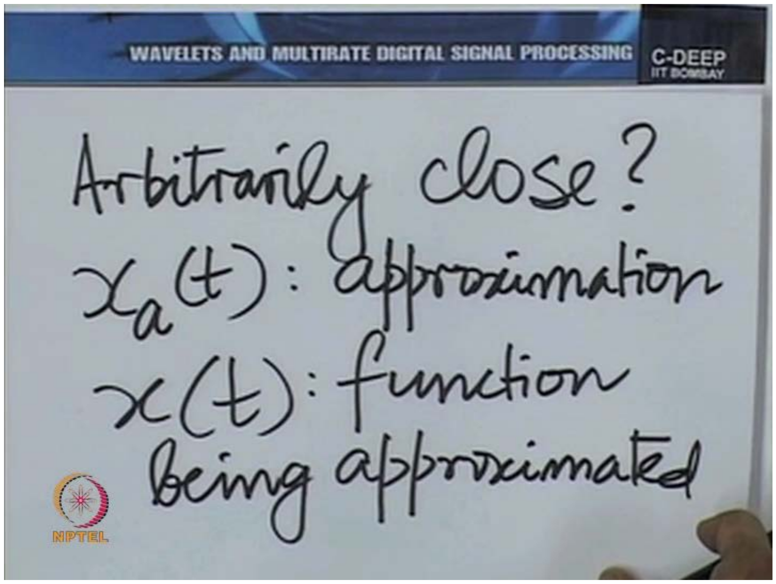

Arbitrarily close in what sense? In the sense if XA is the approximation, approximation at a particular resolution.

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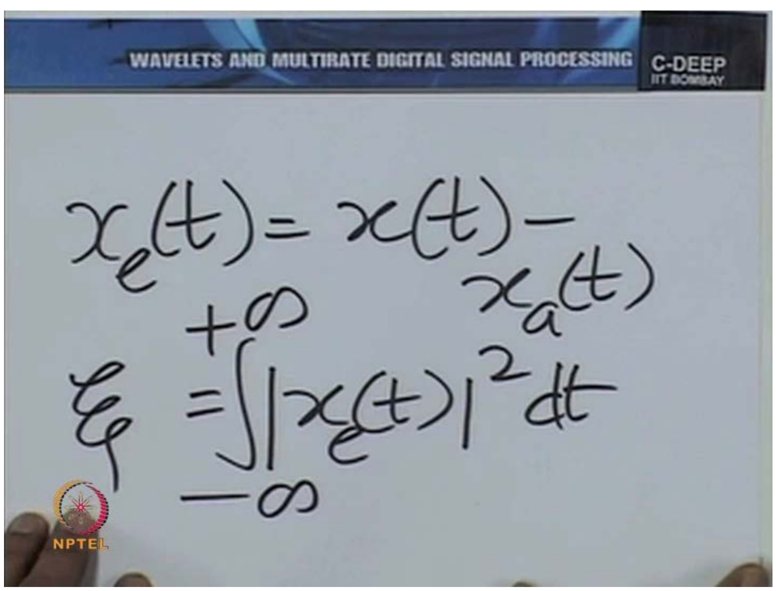

Then if we take what is called the squared error, when we look at XET that is XT minus XAT and integrate XET the whole squared. The modulus whole square actually, over all T, we call this the squared error. Script E.

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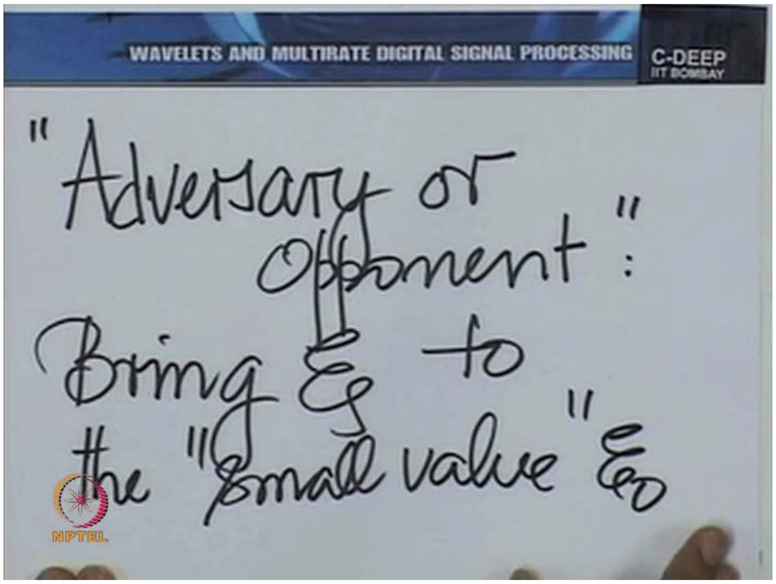

Then the adversary or opponent says, Bring E to the small value. Let's say 0.

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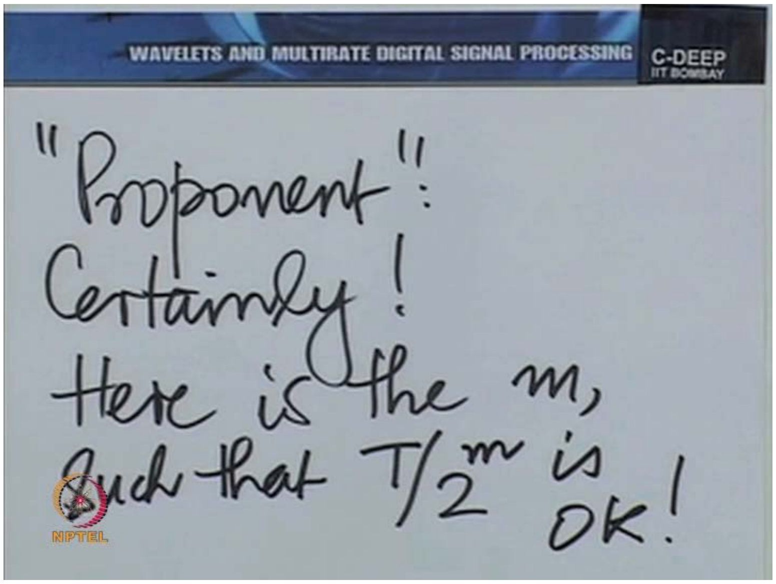

And the proponent says S certainly here is the M, Such that T by 2 raised to the power of M is OK. That is the idea of proponent and opponent here.

The adversary or the opponent gives you a target. It says I want the squared error to be less than this number E0 and the proponent says well here you are. If you make that interval of Psi T by 2 raised to the power of M, lo and behold your error is going to be less than or equal to E0. And what is striking in this whole discussion is no matter how small we make that E0, the proponent is always able to come out with an M. Such that T by 2 raise to the power of M, I mean piecewise constant approximation on the intervals of size T raised, T by 2 raised to the power of M would give you an approximation close enough for that small E0.

We need to spend a minute or two to reflect on this. It is a serious thing that we are saying. In fact lets for a moment think on how this is dual to the idea of representation of a function in terms of its Fourier series for example.

In the Fourier series what do we do, we say give me a periodic function or for that matter, give me a function on a certain interval of time. Let's say an interval of T, Psi T. If I simply periodically extend that function. That means if I take the basic function of the interval of T, I repeat it on every such interval of T, translated from the original interval. Suppose that original interval is 0 to T then repeat whatever is between 0 and T, between T and 2T, between minus T and 0, between minus 2T and minus T, between 2T and 3T and go on doing this.

So you have a periodic function. Decompose that periodic function into its Fourier series representation. So what am I doing in effect? I have a sum of sinusoids, sine waves. All of who's frequencies are multiples of fundamental frequency. What is that fundamental frequency? In angular frequency term it is 2 Pi by T, in Hertz term it is 1 by T. So in Hertz term your sine waves with frequencies which are all multiples of 1 by T and appropriate set of amplitudes and phases assigned to these different sinusoidal components, with frequencies of multiples of 1 by T, when added together would go arbitrarily close to the original function, of course the original periodic function on the entire real axis of that matter specifically on the interval from 0 to T if you restrict yourself from to a function where you started.

So not only does the 4 year series allow you to represents by using the tool of continuous functions, analytic functions, remember we talked about sine waves in the previous lecture. Sine waves are the most continuous in the some sense. The smoothest function that you can think of. The derivative of a sine wave is a sine wave.

The integral of a sine wave is a sine wave. When you add two sine waves as the same frequency that gave you back a sine wave of the same frequency. So sine waves are the smoothest functions that you deal with, and even if you had a somewhat discontinuous function in the interval from 0 to T, and you use this mechanism of Fourier series decomposition you would line up expressing a discontinuous function in terms of extremely smooth analytic functions.

What would you be doing in this Haar approach that we discussed a few minutes ago, exactly the dual? Even if you had this continuous audio pattern you would decompose it into highly discontinuous functions which are piecewise constants. Constant on intervals of size, T at the resolution T, on intervals of size T by 2 at the resolution of T by 2 and so on so forth. Now just as in the Fourier series representation you have this proponent opponent kind of argument that is for reasonably good class of functions.

Even if they are discontinues even if they have a lot of non-analytic points and so on. For a reasonably wide class of functions, remember in the Fourier series, that wide class of functions if captured by what are called the Dirichlet conditions. I won’t go into those details here. There are certain kind of conditions, very mild conditions which a function need to obey before it can be decomposed into the Fourier series or in other words before the Fourier series can do this job of representing the discontinuous function in terms of highly continuous and analytics smooth functions.

So similar set of conditions just exist even for the Haar case. I mean if one really wishes to be finicky one does need to restrict oneself to a certain sub class or function. Then again that restriction is not really serious in most physical situations. For the time being in the course you may even ignore that restriction.