Foundations of Wavelets and Multirate Digital Signal Processing
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Lecture - 2
Module - 1
Dyadic Wavelet

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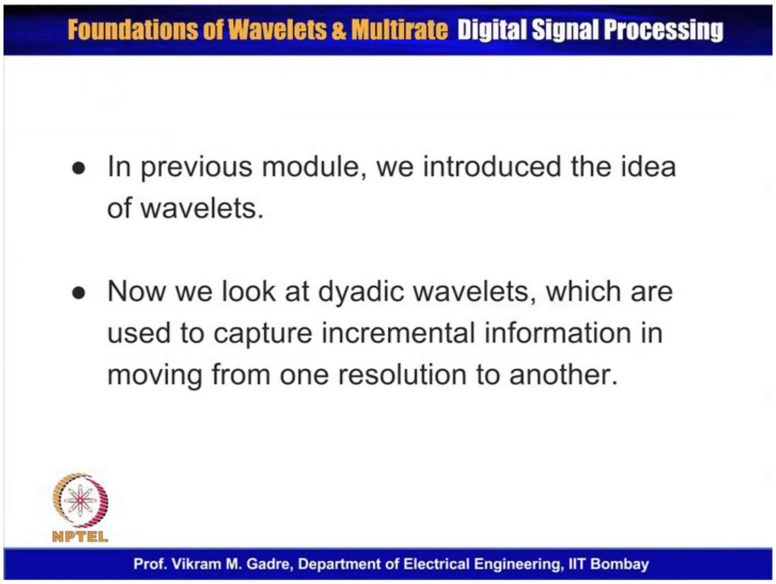

Today, we shall begin with the second lecture on the subject of wavelets and multirate digital signal processing in which our objective would be to introduce the Haar Multi resolution analysis, about which we had very briefly talked in the previous lecture. Before I go on to the analytical and mathematical details of the Haar Multi resolution analysis or the MRA as it is called for short, let me once again review the idea behind the Haar form of analysis or functions.

Recall that Haar was a mathematician or a mathematician scientist if you would like to call him that and the very radical idea that he gave was that one could think of continuous functions in terms of discontinuous ones and do so to the limit of reaching any degree of continuity that you desire. What I mean is start from a very discontinuous function and then make it smoother and smoother all the while adding discontinuous functions until you go arbitrarily close to the continuous functions that you are trying to approximate. This is the central idea in the Haar way of representing functions.

We also briefly discussed why this was something important, it seemed like something silly to do at first glance, but actually is very important and the reason why it is important is mentioned was if you think about digitally communicating, say for example an audio piece you are doing exactly that. The beautiful smooth audio pattern is being converted into a highly discontinuous stream of bits. What I mean by discontinuous is when you transmit that stream of bits on a communication channel you are in fact introducing discontinuities every time a bit changes so after every bit interval there is a change of waveform and therefore discontinuity at some level even if not in the function in its derivative or its second derivative whatever be.

Whatever it is, the idea of representing continuous functions in terms of discontinuous ones has its place in practical communication and therefore what Haar did was something very useful to us today. What we are going to do today is to build up the idea of wavelets in fact more specifically what are called Dyadic Wavelets starting from the Haar wavelet and to do that let us first consider how we represent a picture on a stream and I am going to show that schematically in the drawing here.

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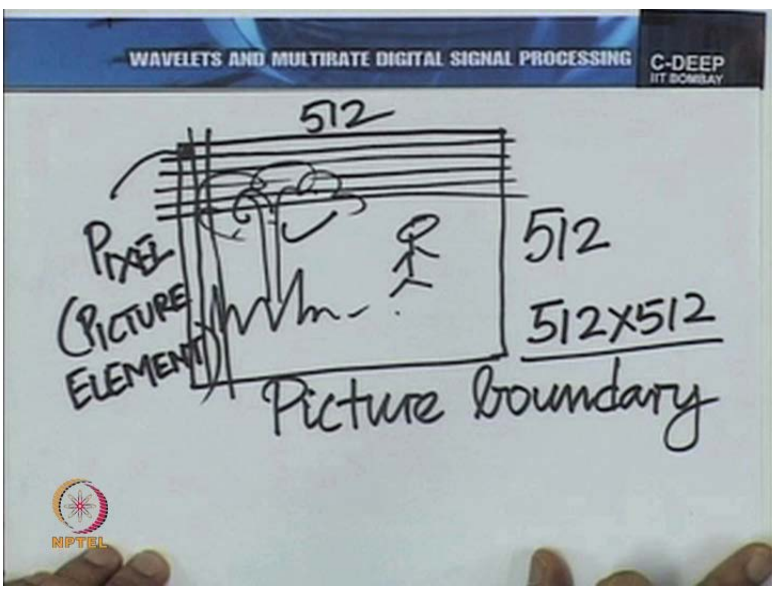

So you see let’s assume that this is the picture boundary and I am trying to represent this picture on a screen whatever that picture might be. So just for the sake of a drawing let me draw some kind of a pattern there let’s say we have a tree and some person standing there I mean forgive my drawing but put some grass may be here. Now this is inherently a continuous picture how do I represent it on the computer I divide this entire area into very small sub areas. So I visualize this being divided into tiny what are called picture elements or pixels.

So each small area here is a pixel a picture element, so to speak and there are for example suppose I make 512 divisions on the vertical and 512 divisions on the horizontal, I say that I have a 512 cross 512 image that many pixels and in each pixel region I represent the image by a constant. So the first thing to understand is there is a piecewise constant representation lets write that down.

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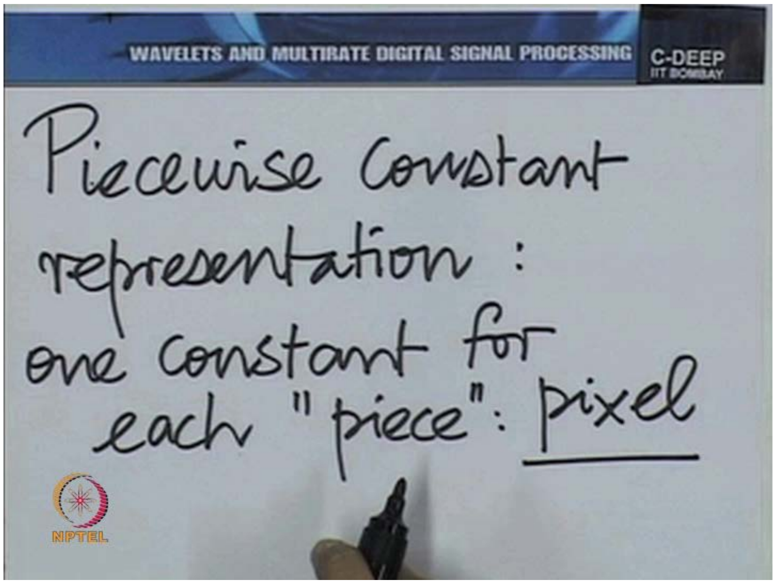

There is a piecewise constant representation of the image. One constant for each piece and the piece is the pixel or the picture element. Now suppose I increase the resolution so I go form a resolution of 512.

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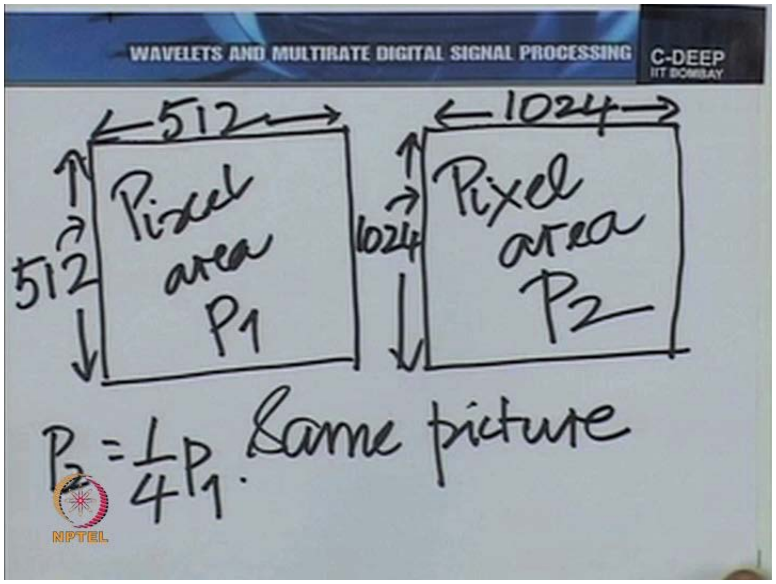

So I take the same what I mean is I take the same picture same picture. In this case I make a division 512 cross 512. In this case I make a division 1024 cross 1024. Now obviously the pixel area here let's say the pixel area here is P2 and the pixel area here is P1 it is very easy to see that P2 is $1/4^{th}$ of P1 and therefore I have reduced the area by a factor of 4.

Naturally if I use a constant to represent the value or you know the intensity of the picture on each pixel here and let's do the same thing here.

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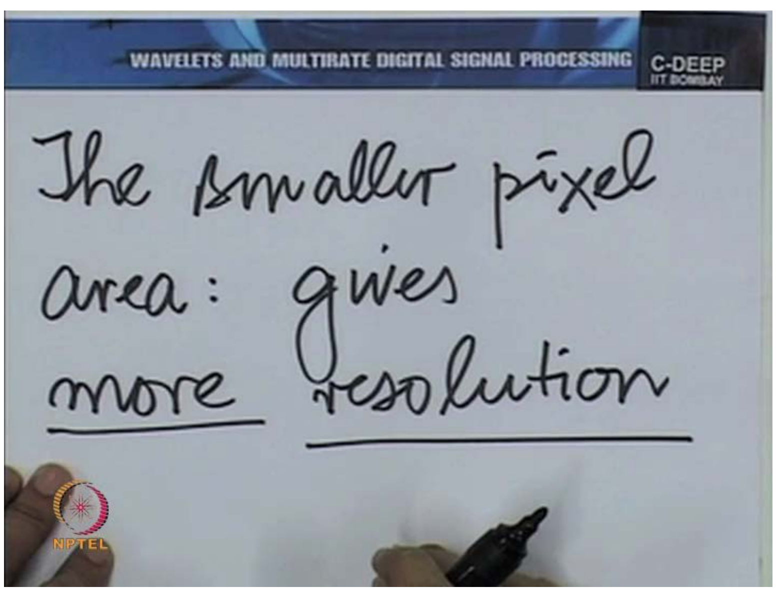

What you see in this picture is going to be closer to the original picture in some sense and what you see here so in other words we can capture this by saying the smaller the pixel area the larger the resolution. Now this is the beginning of the Haar multiresolution analysis. The more we reduce the pixel area the closer we are going to the original image.

Even though this captures the idea that we are going to build it is not quite the idea of the Haar MRA. The Haar MRA does something deeper and that is what I am now going to explain mathematically in some depth. Now here I gave the example of a 2-dimensional situation which apparently is more difficult that a 1 dimensional but it is easier for us to understand physical. We can more easily relate to the idea of a piecewise constant representation in the context of images or pictures, but the same thing could be true of audio for example. So you can visualize a situation though seemingly more unnatural where you record an audio piece by dividing the time of which the audio is recorded into small segments.

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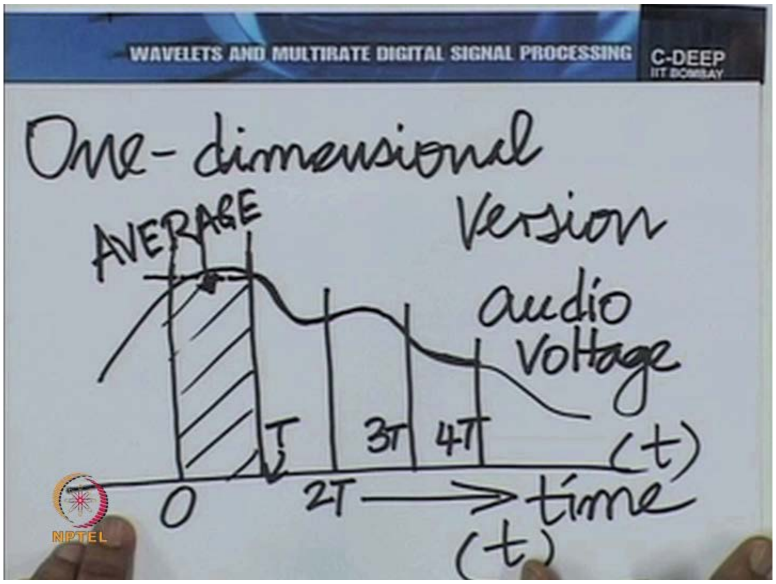

Now let me show that pictorially it would be easier to understand. So suppose for example you had this waveform here so the 1 dimensional version. So suppose I have this is the time axis and I have this waveform here assume that this is the audio waveform audio voltage recording. Let’s without any loss of generality assume, that this is the zero point in time so let time be represented by t and this be the zero point in time.

Now let me assume that I divide this time axis into small intervals of size t here so this point is t this point is 2t and so on. I make a piecewise constant approximation that means I represent the audio voltage in each of these regions of size t by one number. Now what is the most obvious number or what are the set of obvious numbers that one can use to represent this waveform in each of these time intervals.

For example in this time interval or for that matter any of the time intervals it makes sense to take the area under the curve and divide by the time interval to get the average of the waveform in that time interval and use that as the number to represent the function. So here for example you can visualize that the average will lie somewhere here, I am just showing it in dotted, so average so intuitively it makes sense to represent the voltage waveform in each of these intervals of size t by the average of that waveform in that interval. Is that right?

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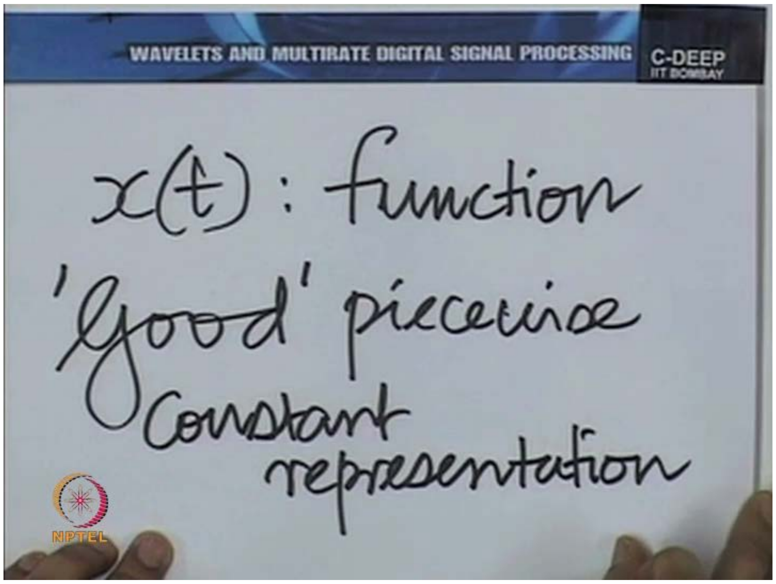

Let us write that down mathematically so what we are saying if you have a function x of time, a good piecewise constant representation is the following.

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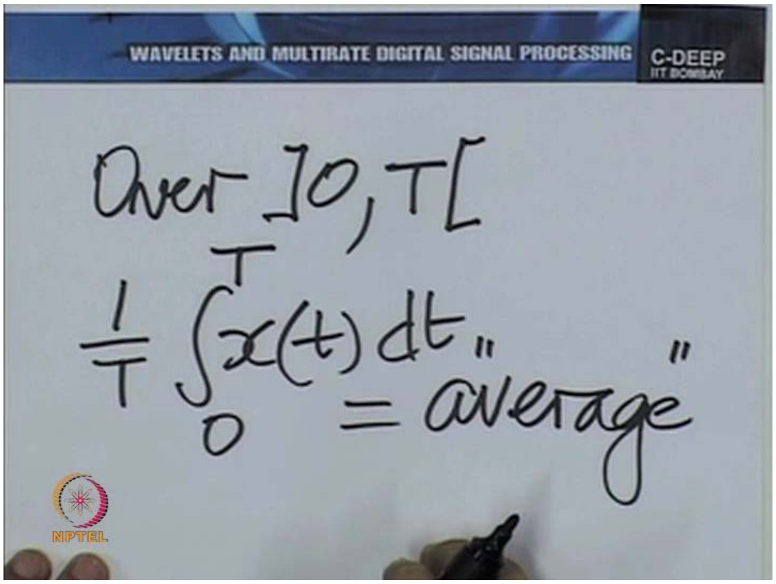

Over the interval of t over the interval form say 0 to t now you know strictly it is the open interval between 0 and t the representation would be integrate x of tdt from 0 to t and divide by t the average.

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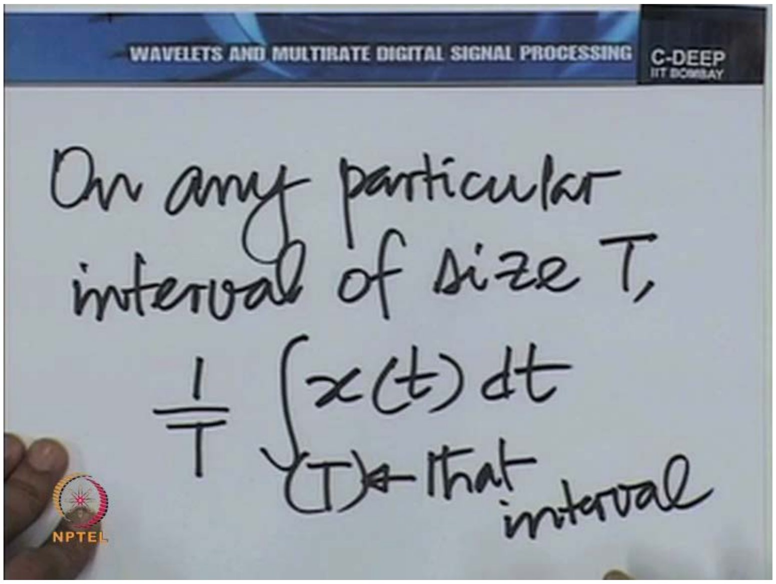

Now of course on any particular interval of t the same holds so we say that on every interval of t on any particular interval of t of size t the average would be obtained by 1 by t integral over that interval of t, when you write it like this you mean that particular interval of t integral of xt with respect to small t. This is a piecewise constant representation of the function on the interval of size t. Now the same thing could be done for an interval of size t by 2.

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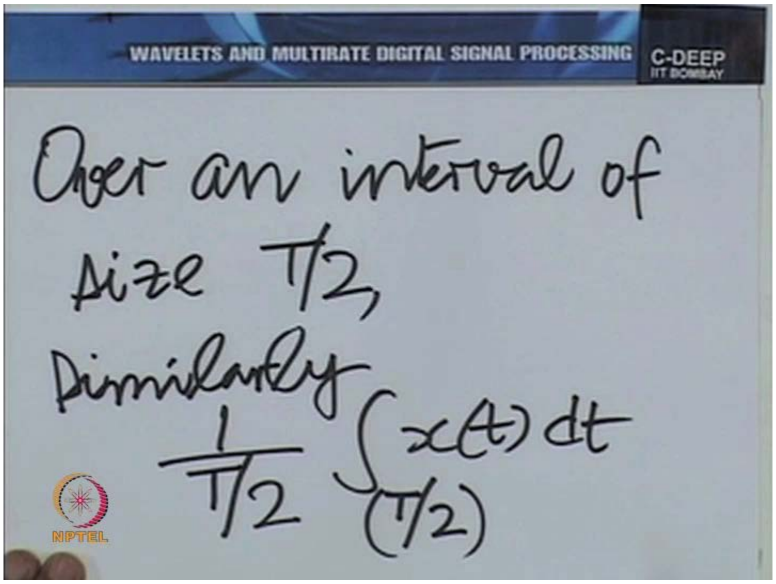

So over an interval of size t by 2 you would similarly have 1 by t by 2 integral over that interval of length t by 2 x(t)dt..

Now we are going closer to the idea of wavelets lets pick a particular interval of size t in fact again without any loss of generality, let us choose the interval from 0 to t and divide it into 2 subintervals of sizes t by 2.

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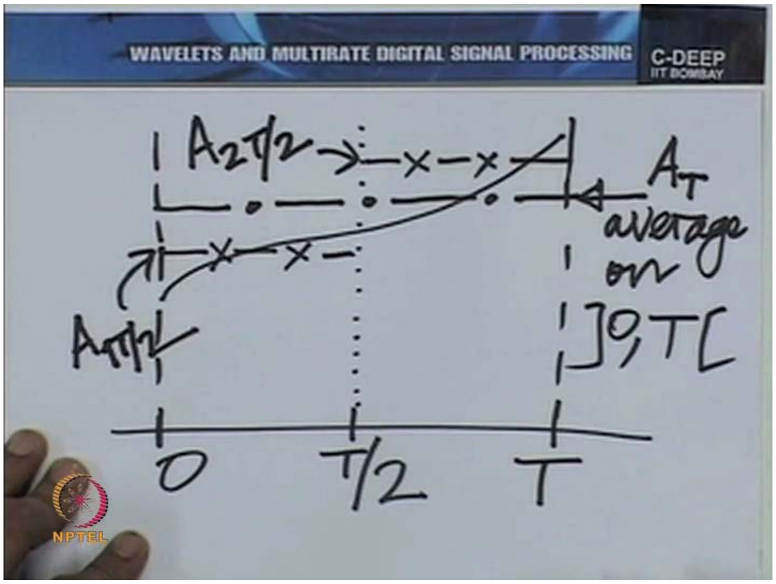

So what I mean is take interval of size t 0 to t I am expanding it so you have this function here over that interval divide this into 2 sub intervals of sizes t by 2 first take the piecewise constant approximation on the entire interval of t and I will show that by a dot and dash line. You can visualize the average would be somewhere here so this is the average on the entire interval 0 to t.

Now I take the sub intervals of size t by 2 so I have this sub interval of size t by 2 I use a dash and cross to write down the average there, so I have a dash and cross dash and cross here. You can visualize that in this sub interval the average would be somewhere here and similarly in this sub interval you could write down an average something like this. Now let us give this a name, let us call this average A on t, let us call this average a1 on t by 2 and let us call this average a2 on an interval of size t by 2 and let us write down the expressions for each of these averages.

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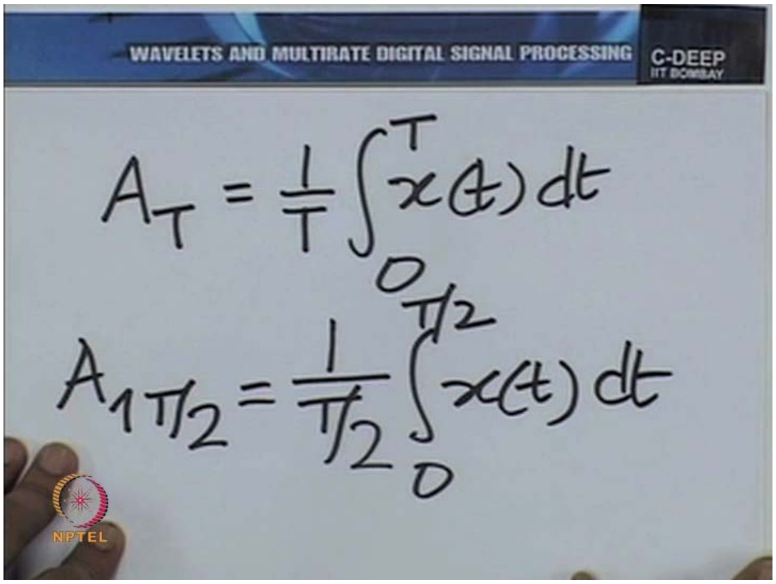

What are the expressions at is obviously 1 by t integral x of t dt from 0 to t. a1t by 2 is one by t by 2 integral from 0 to t by 2 xtdt.

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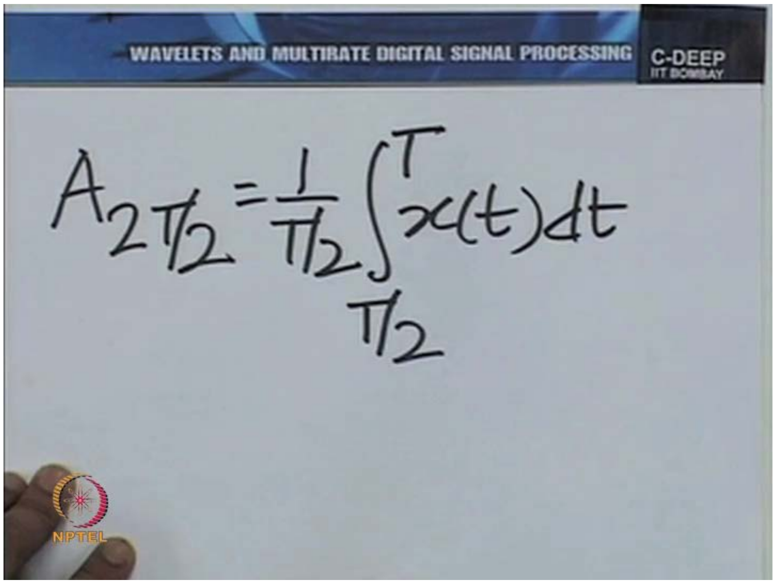

And similarly a2t by 2 is 1 by t by 2 integral from t by 2 to t xtdt. For convenience let me flash all the three expressions before you once again. AT is the average over the entire interval of t a1t by 2 the average over the first interval of t by 2 with this expression and a2t by 2 the average from t by 2 to t the second sub interval of size t by 2 with this expression and just to get our ideas straight here again is the picture. Now the key idea in the Haar multiresolution analysis is to try and relate these 3 terms so to relate at a1t by 2 and a2t by 2 and it is in that relationship that the Haar wavelet is hidden.

So that is the relationship now the relationship is very simple I mean all that we need to do is to notice that we have divided the integral from 0 to t into 2 integrals over 0 to t by 2 and t by 2 to t and then remember the slight difference in the constants associated so we have a constant of 1 by t in at and a constant of t by 2 in a1 t by 2 and in a2 t by 2. Where upon we have this very simple relationship between the 3 quantities…

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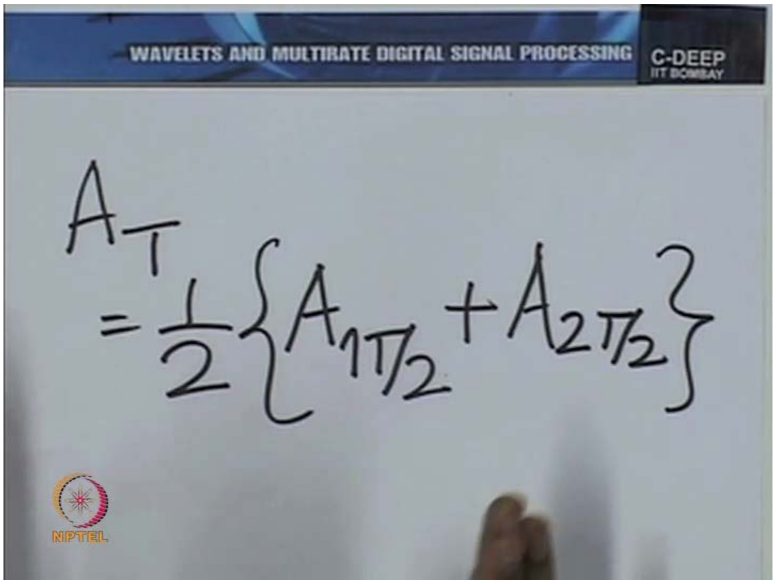

AT is half I leave it to you to verify its half of a1 t by 2 plus a2 t by 2 and how do we interpret this let me try and you know kind of focus just on this relationship in other words let’s just focus on these 3 constants and make a drawing there.

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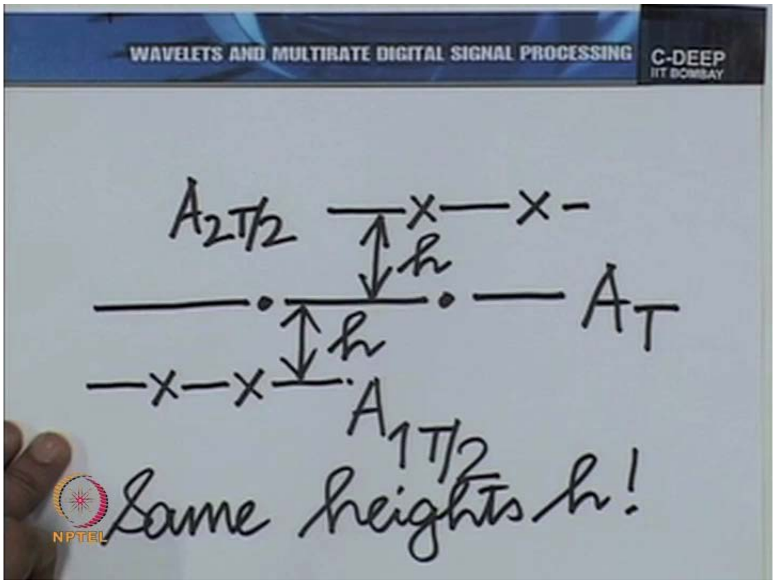

So what we are saying is something like this I have this at there I have thisa1t by 2 here, and I have this a2t by 2 there and we are saying this plus this by 2 gives you this, in other words this is as much higher above at as this is lower. What we are saying is these 2 heights are the same. That's what this relationship implies. Now another way of saying it is if I were to make a piecewise constant approximation on intervals of size t how would they look so let me just sketch them.

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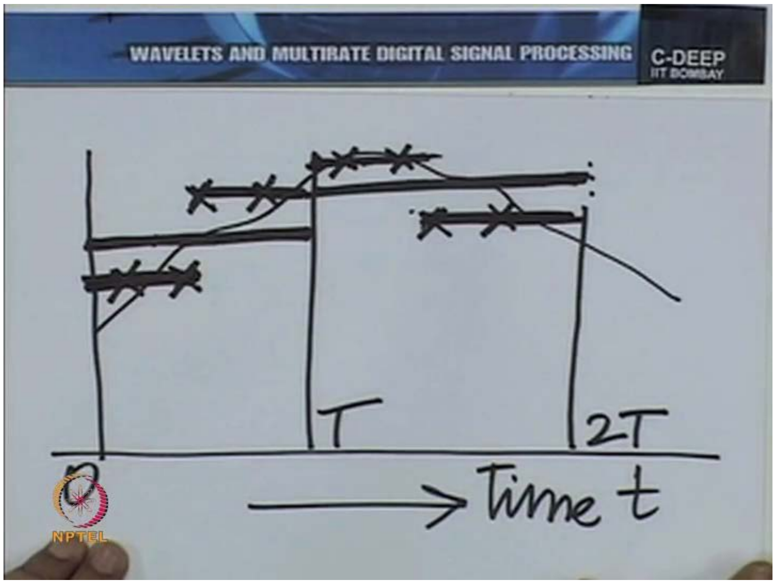

So I take this function once again here I have this function here. I have divided to sizes of intervals of size t let me show just 2 intervals for the moment. So this is how the function would look when you make a piecewise constant approximation on intervals of size t and when you do it on intervals of sizes t by 2 it would look like this, something like this. Now this is a function so let me highlight it let me darken it this is in its own righter function piecewise constant function the one which I have darkened here and this is in its own right the darkened part is in its own right an approximation to the original function here.

Similarly let me now darken this and put some other mark on it lets keep the cross so I will darken this and I will put crosses on it so this is also another function. The dark and cross function is another function that is in its own right an approximation too.

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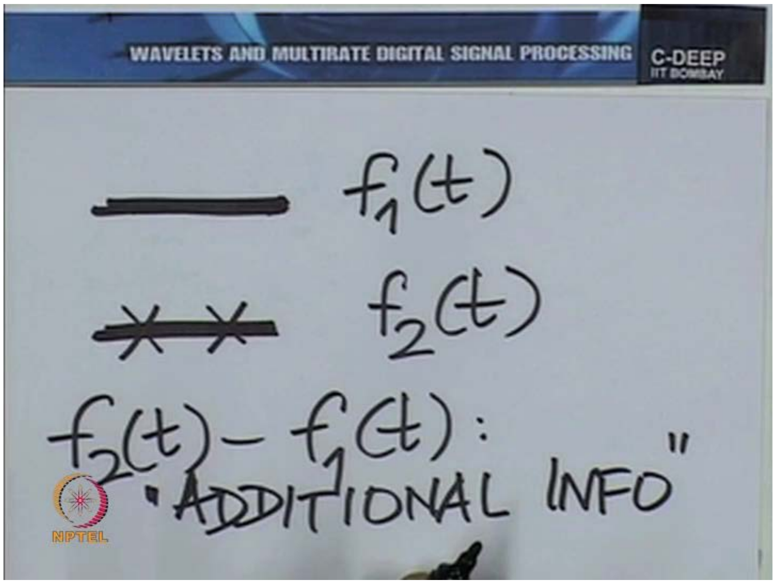

So let's give them names let's call this function just the dark one as f1t and let's call this function the one which we have shown with dark and cross as f2t. f2t minus f1t is like additional information, what we are saying is instead of a piecewise constant approximation on a interval of size t when we try and make a piecewise constant approximation on intervals of size t by 2 we are bringing in something more.

Go back to the original case of the picture, we have inherently underlined a continuous 2 dimensional picture a continuous 2 dimensional scene when we make an approximation with a 512 cross 512 resolution then we have actually brought in one level of detail, when we go to a

1024 cross 1024 representation the level of detail is 4 times more. What is the additional detail that we have got in going form 512 cross 512 to 1024 cross 1024, in effect when we take this difference f2t minus f1t we are answering that question.