

Foundations of Wavelets and Multirate Digital Signal Processing
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Module 4
Lecture No 25
Realizable Two Band Filter Bank

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Foundations of Wavelets & Multirate Digital Signal Processing

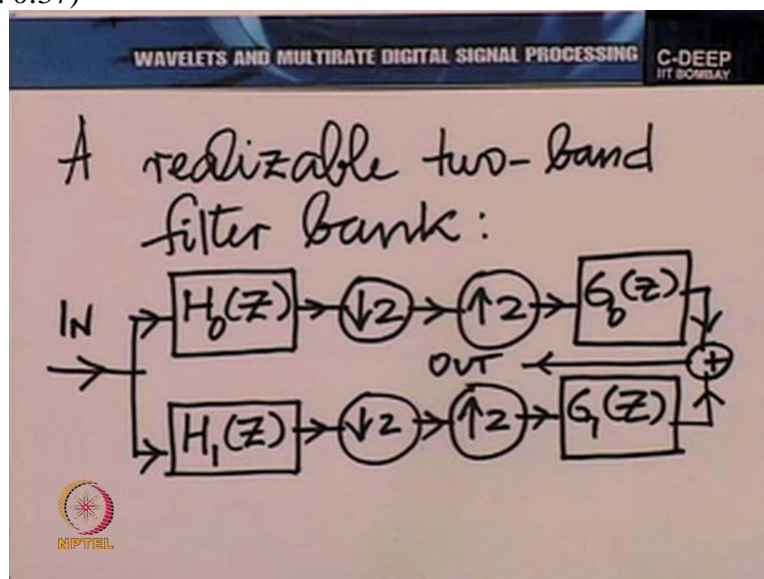
- So far we have seen that:
 1. Why are Haar filter banks not ideal and
 2. What factors are responsible for making ideal filters unrealizable.
- We also understand that even though we cannot achieve ideality but still we can go as close as possible towards it by investing more and more resources.
- The big agenda for this module would be to develop relation between scaling function, wavelet function and the filter banks.

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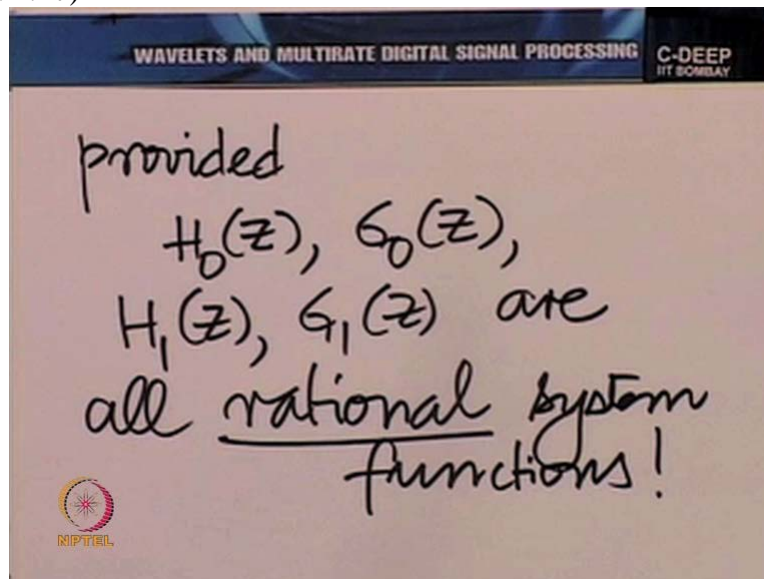
1st, what would a`realisable Two band filter bank look like, first put that. Write that down in terms of a drawing first.

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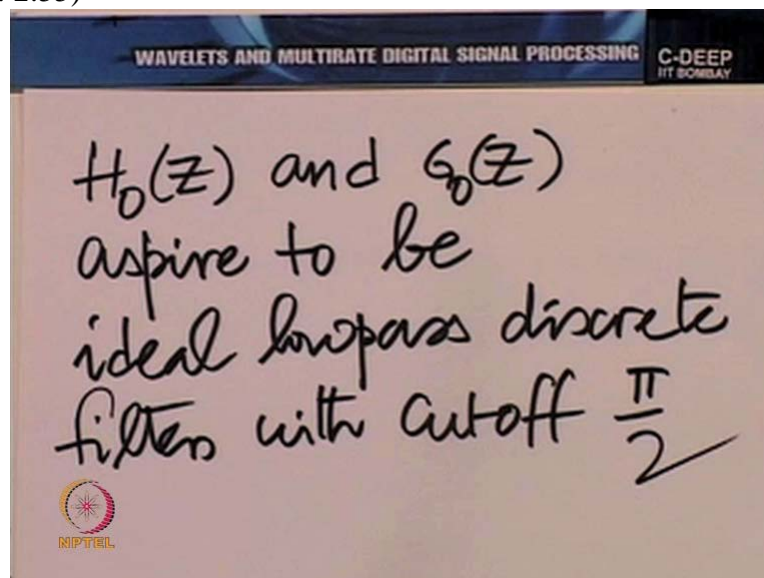
So a realisable to a band filter bank is like this. So I straightaway write it in terms of system functions here. This is a realisable Two band filter bank provided...

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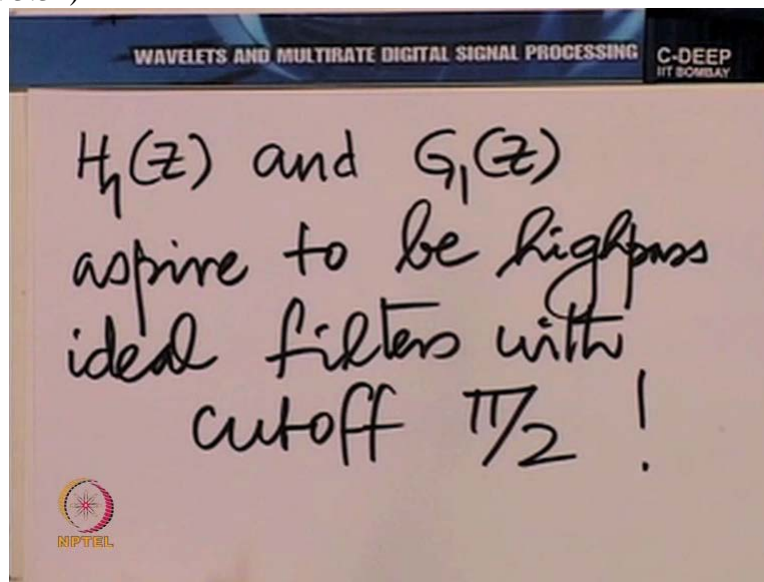
... $H_0(z)$, $G_0(z)$, $H_1(z)$, $G_1(z)$ are all rational functions, rational system functions.

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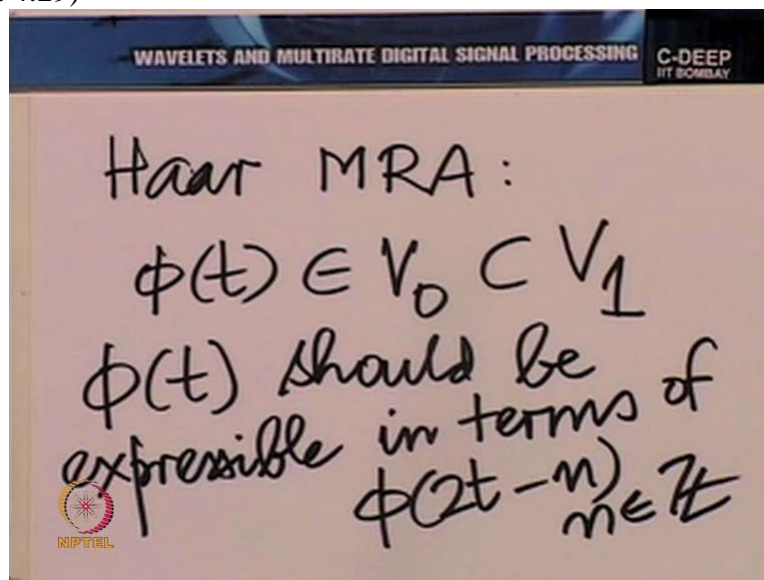
And of course, $H_0(z)$ and $G_0(z)$ aspire to be ideal low pass filters with cut-off π by 2.

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$H_1(z)$ and $G_1(z)$ aspire to be high pass ideal filters with cut-off $\pi/2$. Now, having put down the structure more generally we must now ask what is the connection between the multiway solution analysis and this filter bank that we're trying to design? So to answer that, let us look at the Haar once again. In fact, let us begin with the $\phi(t)$ in the Haar.

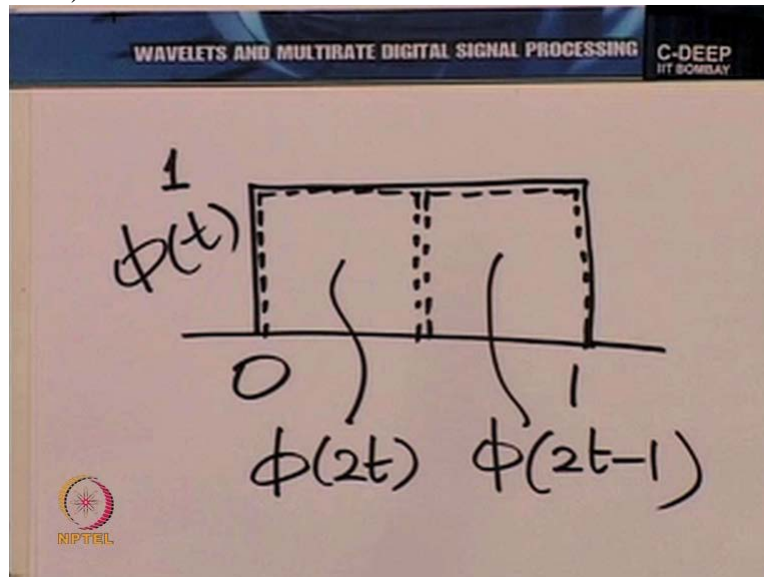
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So, for the Haar MRE, we noticed something very interesting. $\phi(t)$ belongs to V_0 which is the subspace of V_1 . So $\phi(t)$ should be expressible in terms of the basis of V_1 . And what is that basis of V_1 ? It is $\phi(2t-n)$ for integer n . It should be expressible in these terms. It is very interesting. You know, $\phi(t)$ is an element of V_0 . V_0 is a subspace of V_1 and the basis of V_1 is

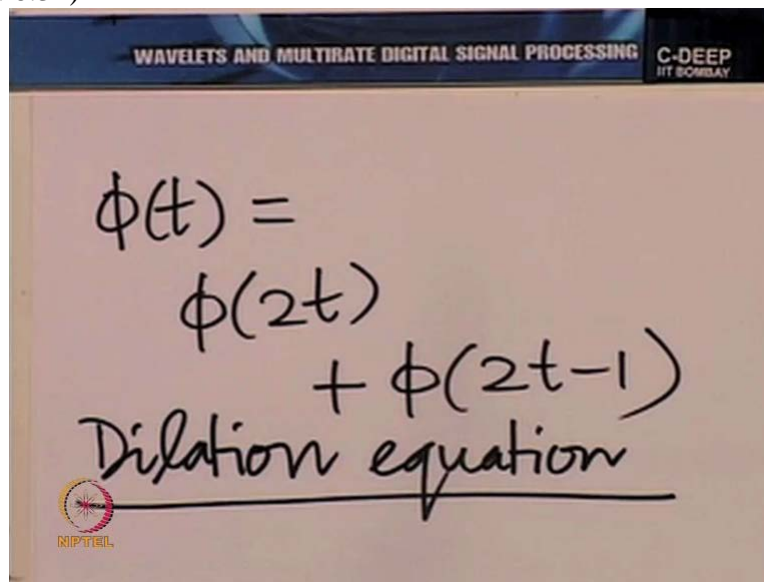
again, the dilate of ϕT by a factor of 2 and their integer translates. So ϕT can be expressed in terms of its own dilates and translates. This leads to what is called as recursive dilation equation on ϕT . And lo and behold, what is that dilation equation? That is also not difficult to determine.

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In fact, we can even say graphically. Indeed, you know if you recall, ϕT looks like this. I've kind of scaled up the drawing. And all that you need to do to get this recursive equation is to note that this can be redrawn like this. So, it has 2 components in it. And the 1st component is $\phi 2T$ and 2nd, $\phi 2T - 1$ and the whole thing is ϕT . So we have a beautiful dilation equation which governs ϕT .

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WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING C-DEEP IIT BOMBAY

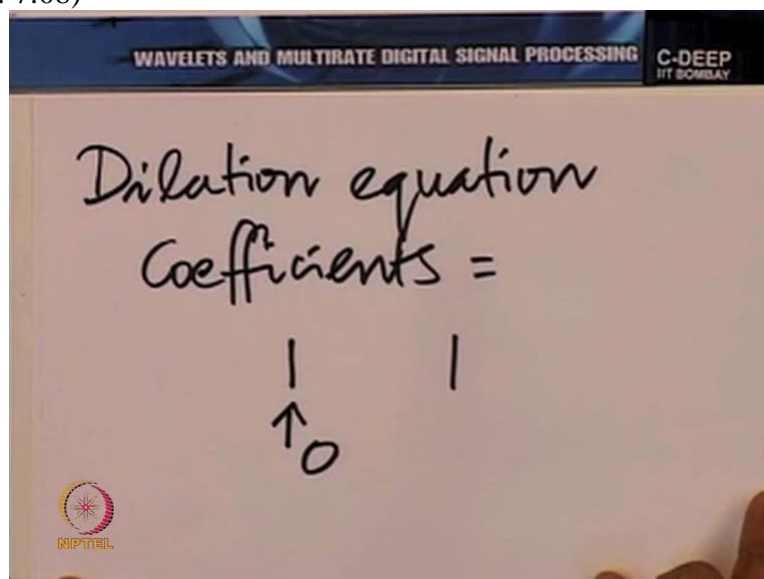
$$\phi(t) = \phi(2t) + \phi(2t-1)$$

Dilation equation

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$\phi(t)$ is $\phi(2t) + \phi(2t-1)$. A beautiful dilation equation which governs $\phi(t)$. Now let us look at the coefficients in that dilation equation.

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Dilation equation
Coefficients =

1 1

↑
0

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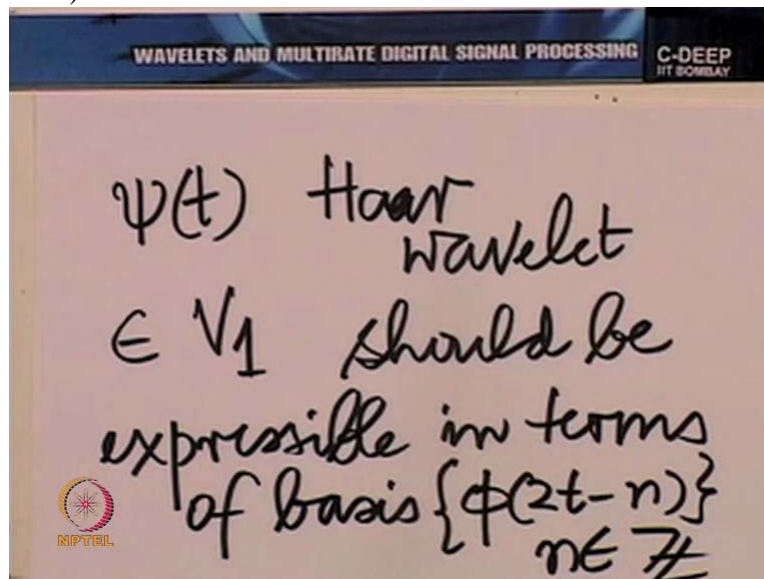
So if we put the dilation equation before you once again the coefficients are 1 and 1.

Let me try and call this a sequence. So if you agree, I will talk about the sequence which is 1 at N equal to 0 and then one subsequently as shown at N equal to 1, this is a way of denoting a finite length sequence. The number below the arrow tells us the value at the point of the arrow. And all the other numbers tell the value of the sequence at adjacent points. So this for example means

at N equal to 0, the sequence takes the value 1 and at the next point which is of course N equal to 1, the sequence will take the value 12.

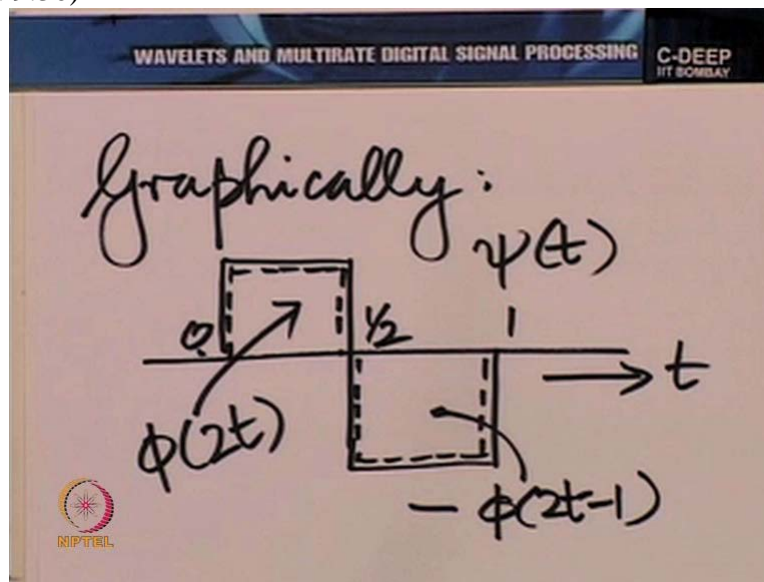
So this is the sequence corresponding with the dilation equation coefficients. Let us carry out a similar exercise for the wavelet now. So let us take the Haar wavelet and let us express the power wavelet also in terms of the basis of V_1 . And what is our ground for doing so? Let us put it down clearly.

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You see recall that $\psi(t)$ or the Haar wavelet for example also belongs to V_1 . So it should be expressible in terms of its basis. What is that basis? $\phi(2t - n)$ for integer n . And again, if we look at it graphically, it is not difficult to do.

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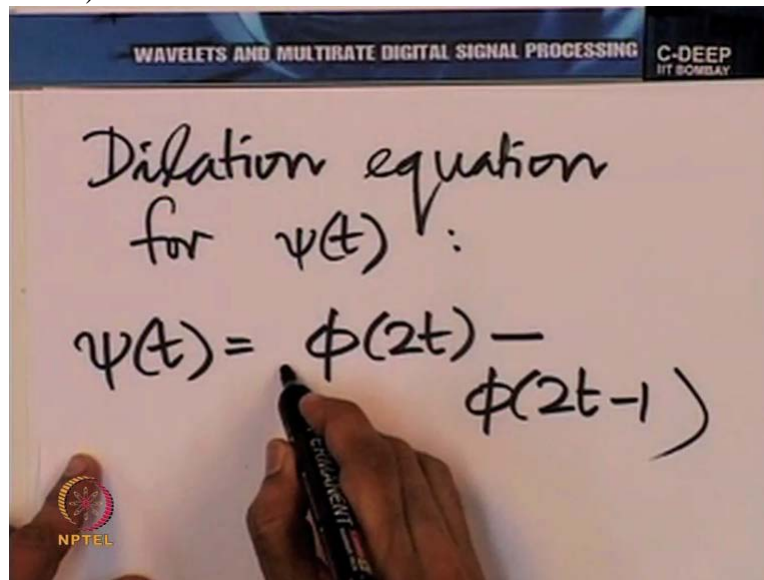


Graphically, one can sketch ϕT actually and ψT too. So this is. And you can see the ϕT s embedded in it. So you have one here and you have one there. This is easily seen to be ϕ of $2T$ and this is easily seen to be $-\phi 2T-1$. And therefore we have a very simple dilation equation for ψT . Now here, it is not recursive but it is a dilation equation all the same. So dilation equation for ψT .

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ψ of T is $\phi 2T - \phi 2T-1$. And once again, let us put down the coefficients of this dilation equation as we did previously. So you know, the coefficients again would be again the coefficients involved in expanding in terms of $\phi 2T-N$.

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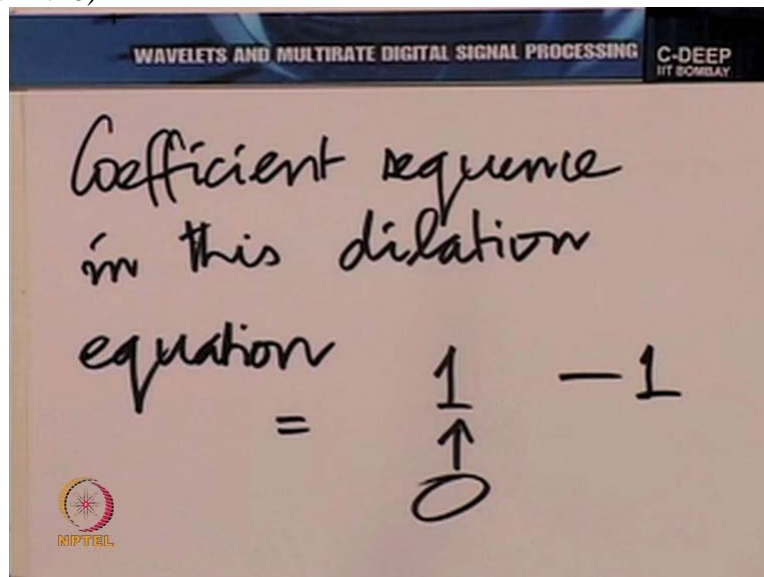
Dilation equation
for $\psi(t)$:

$$\psi(t) = \phi(2t) - \phi(2t-1)$$

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As I can see from this dilation equation, the coefficients are 1 and -1 respectively at 0 and 1.

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WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING C-DEEP IIT BOMBAY

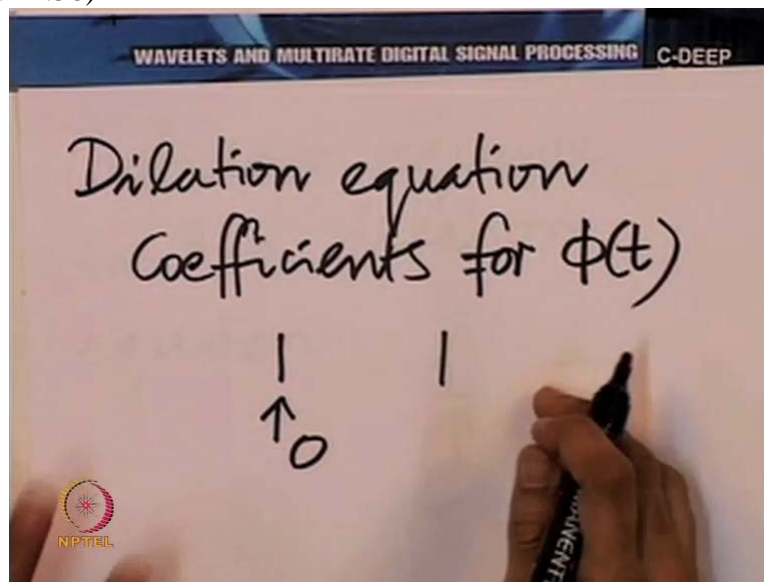
Coefficient sequence
in this dilation
equation

$$= \begin{matrix} 1 & -1 \\ \uparrow \\ 0 \end{matrix}$$

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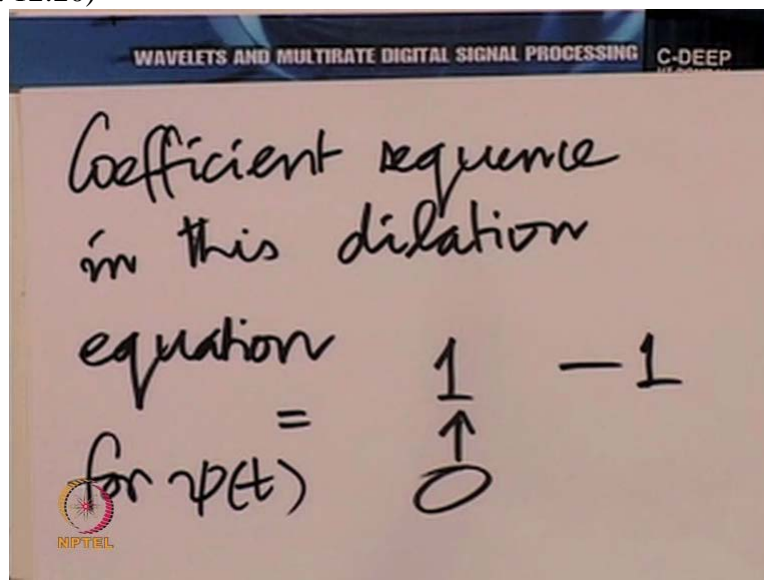
So I put that down. Now things are beginning to make sense and perhaps even ring a bell. Let me put both these coefficient sequences before you once again and then I'm sure it will ring a bell.

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Look at the dilation equation coefficients for the $\phi(t)$ itself. In fact, let me write it down. The dilation equation coefficient for $\phi(t)$.

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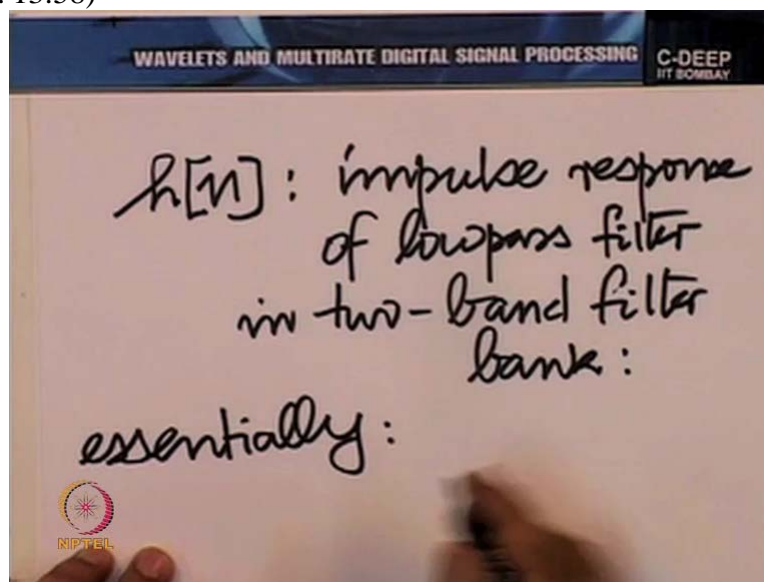


And let me then put before you the dilation equation coefficients for $\psi(t)$. So just write it down, for $\psi(t)$. Does this ring a bell? Yes indeed. If you look at the impulse responses, either on the analysis side or the synthesis side of the low pass filter and the high pass filter these coefficients are functionally those impulse responses. A minor difference. You see on the analysis side, you

have a factor of half. For the moment, keep aside that factor of half. Otherwise, these dilation equation coefficients are just those very impulse responses.

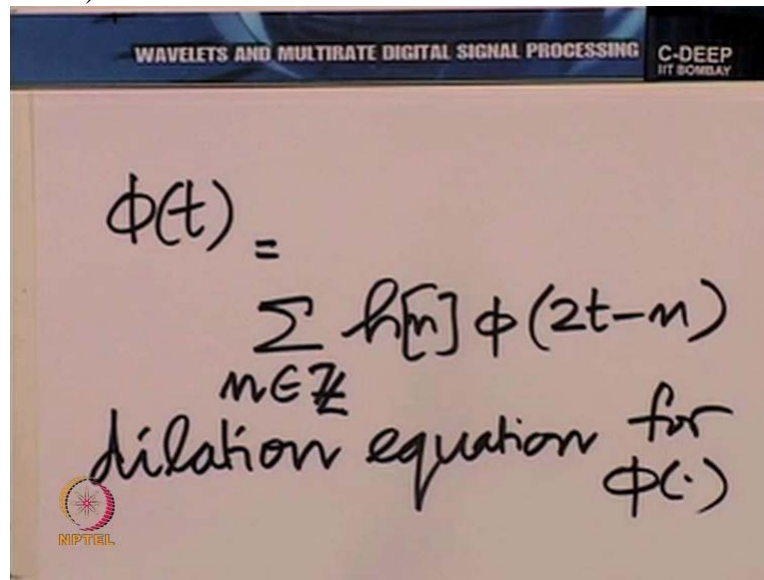
Synthesis side, similarly. Perhaps with a plus minus ambiguity but otherwise the same. So we have a very intimate relation which we have seen here. The coefficient sequences in these dilation equations that no one ϕT and ψT are actually the impulse responses of the filters. Now in fact I go step further. We shall progress to show that if I know these impulse responses, I can go the other way too. So here by serendipity so to speak by surprise or chance discovery, with this relationship. Now we will take that serendipity, that discovery further.

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So indeed, let us know at the moment that within a scaling factor, if H of N so to speak is the impulse response of the low pass filter in question in the Two band filter bank is essentially what we have is the following dilation equation. So we will continue the dilation equation.

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$$\phi(t) = \sum_{n \in \mathbb{Z}} h[n] \phi(2t - n)$$

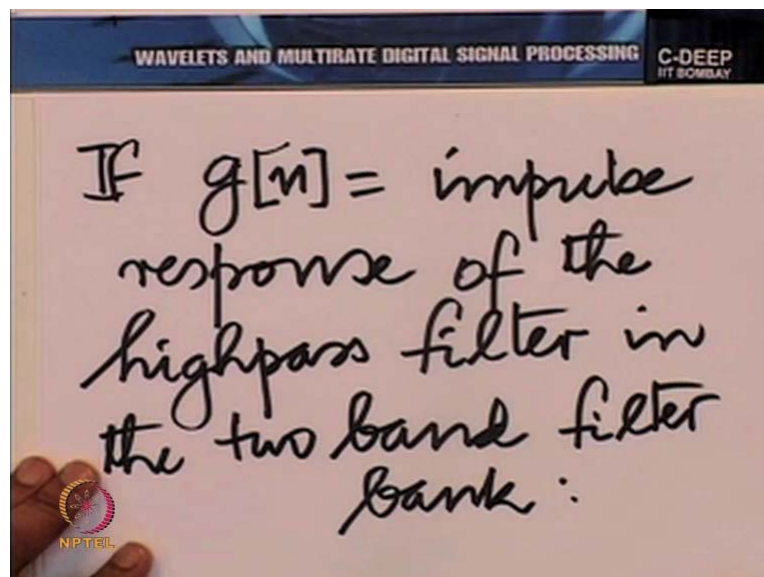
dilation equation for $\phi(\cdot)$

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$\phi(t)$ is a summation on n . n over the set of integers, $h[n] \phi(2t - n)$. This is the essential dilation equation for ϕ .

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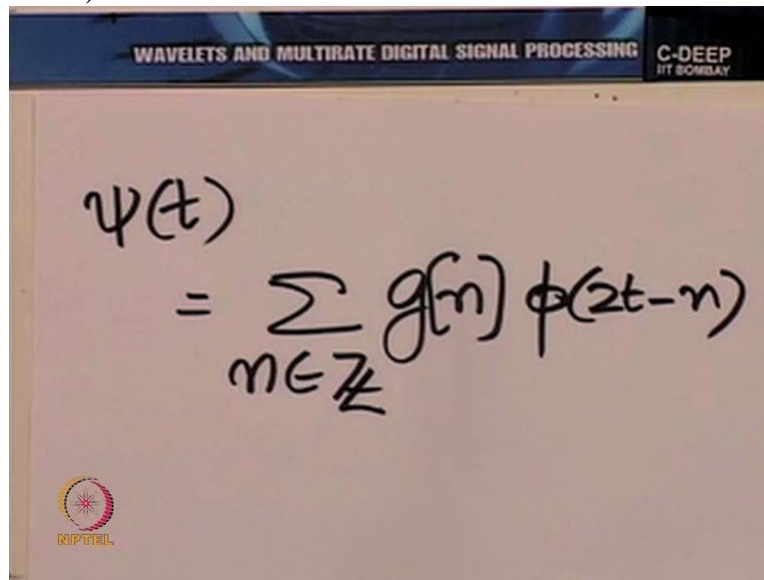


WAVELETS AND MULTIRATE DIGITAL SIGNAL PROCESSING C-DEEP IIT BOMBAY

If $g[n]$ = impulse response of the highpass filter in the two band filter bank:

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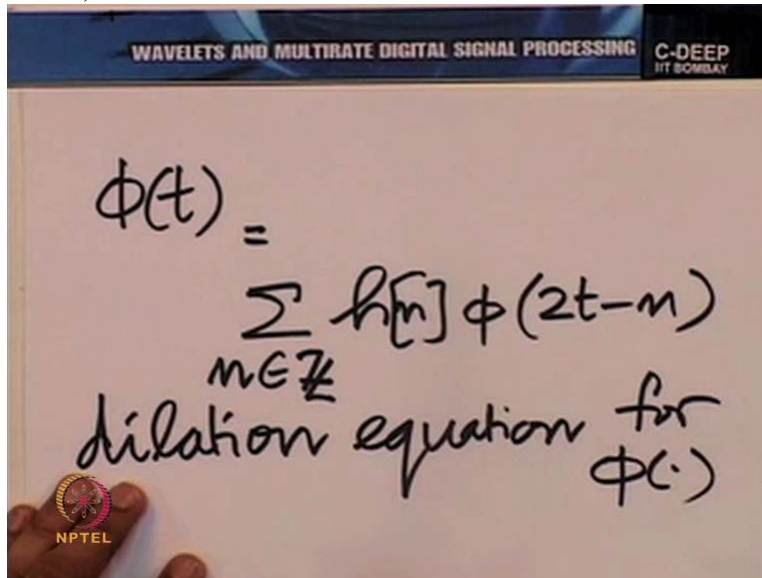
$$\psi(t) = \sum_{n \in \mathbb{Z}} g[n] \phi(2t - n)$$

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And conversely, if G_N is the impulse response of the high pass filter in the Two band filter bank then we have ψ_T the summation N over the set of integers $G_N \phi_{2T} - N$. So in the time domain, we have made an intimate relationship. The low pass filter impulse response allows us to expand essentially ϕ_T in terms of its own dilates and translates. The scaling function in terms of its own dilates and translates. The high pass filter helps us expand the wavelet in terms of the dilates and translates of the scaling function. Once again, low pass filter impulse response, a recursive expansion of the scaling function in terms of its own dilates and translates, the high pass filter response and expansion of the wavelets in terms of the dilation and translates of the scaling concept.

Now, we want to go a step further. We want to show that once you have this dilation equation, we can actually completely characterise ϕ_T and ψ_T knowing that Two band filter bank. I shall in the next couple of minutes only give the strategy for doing so at which will actually do this in the succeeding lectures, the lectures to follow. Let me put before you the strategy that we are going to follow. And for that purpose, let me put down the equations before you once again. So let's recapitulate the 2 equations we have written.

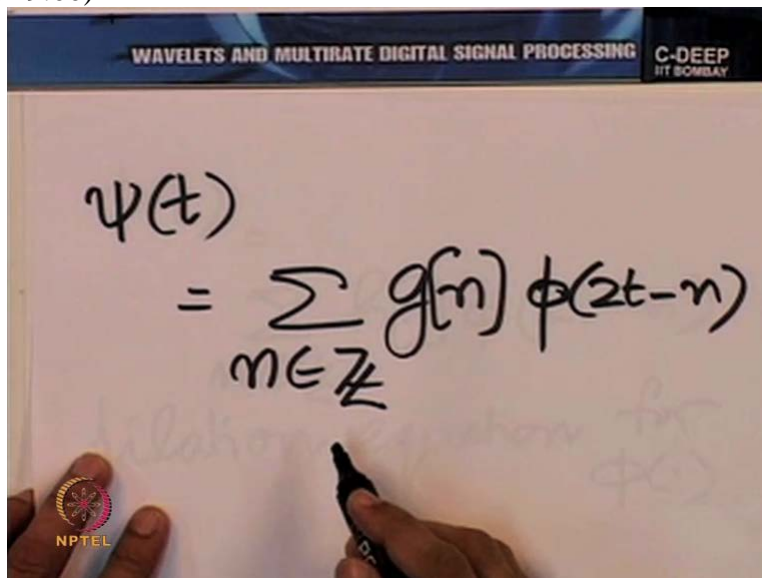
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$$\phi(t) = \sum_{n \in \mathbb{Z}} h[n] \phi(2t - n)$$
 Below the equation, the text "dilation equation for $\phi(\cdot)$ " is written. In the bottom left corner, there is a small logo for NPTEL.

We have this dilation equation relating ϕ to its own dilates and translates. What we shall do with the next lecture is to take Fourier transforms on both sides and noting that we can express the Fourier transforms of ϕ or rather $\phi(2t - n)$ in terms of that of ϕ we shall have a recursive equation in the Fourier domain on the Fourier transform of ϕ . From this we shall be able to completely characterise the Fourier transform of ϕ in terms of the discrete time Fourier transform of the sequence H .

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$$\psi(t) = \sum_{n \in \mathbb{Z}} g[n] \phi(2t - n)$$
 Below the equation, the text "dilation equation for $\psi(\cdot)$ " is written. In the bottom left corner, there is a small logo for NPTEL.

Having done so, we shall then progress to this dilation equation here and we will relate the Fourier transform of the wavelet to the Fourier transform of the scaling function effectively. And

then, noting that the discrete time Fourier transform of the sequence G_N can be used to make this relationship. We can obtain the wavelet from the scaling function. So, it is with these 2 steps that we shall begin the next lecture. For the time being, let us keep our curiosity alive to see how beautifully we can enmesh the design of the Two band filter bank and the design of scaling function and a wavelet for building a whole multiresolution analysis. With that note of curiosity and anticipation, let us conclude this lecture. Thank you.

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