Foundation of Wavelets and Multirate
Digital Signal Processing
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Additional Information on Direct Sum

Hello everyone, my name is Shivam Bharadwaj, I am a Ph.D. student in the Department of Electrical engineering IIT Bombay. So today I am here with you all to discuss a very important concept from linear algebra which is directly applicable to wavelets. And this concept is popularly known as the concept of direct sum of vector spaces.

Now, I will be giving you the definition of as to what this operation is and I will be giving you 2 examples. One example would be from finite dimension vector spaces and another example will be from in finite dimensional vector spaces. Then let us proceed to the definition of this operation.

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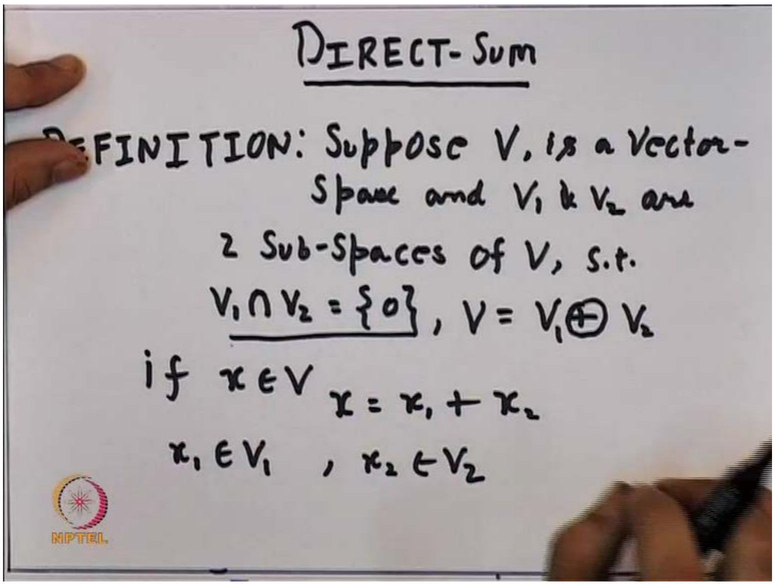

So definition V is a vector space and V 1 and v 2 are 2 sub spaces of V, such that by s dot t dot I mean such that.

V 1 intersection V 2 is just the 0 vector. Then this is true, this is a very important thing in this definition, if this thing is true then we say that V is a direct sum of V 1 and V 2 if any vector x in V can be represented as x equal to x 1 + x 2 where x 1 belongs to the set V 1 and x 2 belongs to the set of vector subspace V 2. Okay, so there are a few important things which I would like to dwell upon for a second.

That for defining the direct sum, we need to have this thing very precise that 2 sub spaces that are involved in the direct sum should have 0 intersection or they should have only origin of that vector subspace as their common point. Now let us look at 2 examples from finite dimensional vector spaces and one example from infinite dimensional vector spaces.

So in finite dimensional with space, let us look at our Euclidean two-dimensional plane which is also popularly represented as R 2. So it is a normal Euclidean two-dimensional plane.

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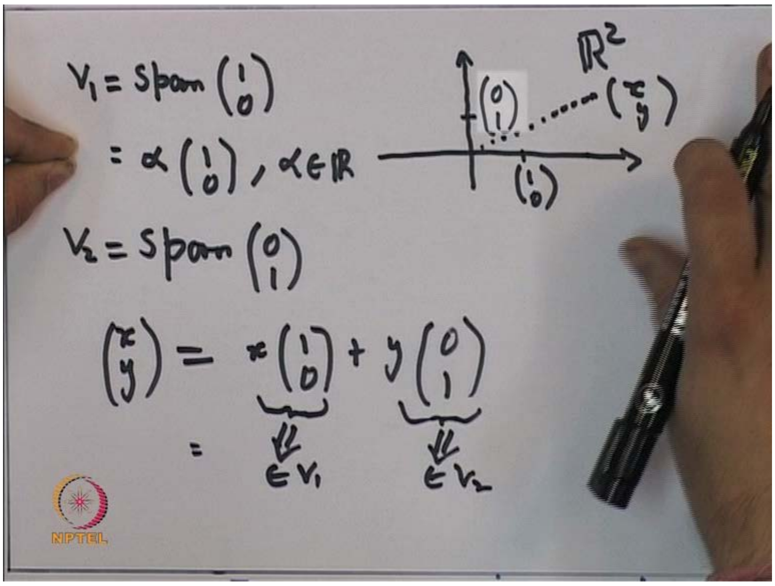

Now let us consider 2 sub spaces of this Euclidean space; one subspace let me call it as span of the vector 1 0. Now in this two-dimensional vector subspace by saying that V 1 is a span of 1 0 what do I mean is that, 1, 0 is this point.

So I am representing it as a column vector and by span means that I am allowing any scalar multiplication okay by span means any vector which is of the form alpha times 1 0 where alpha is any real number. So if you think on this for a second, you will realise that I am talking of this whole real axis which is from - infinity to + infinity. And let us consider another vector subspace, which is given by span of 0, 1.

So similarly, span of 0, 1 will be the vector which lie on this y axis or the vertical axis. Now that you can see that if I consider any point x y in this R 2 or in a two-dimensional Euclidean space, then I can write this point x y as x times 1, 0 + y times 0 1. So for any real number x and y, I have shown that any point in R 2 can be written as a scalar times 1 0 + scalar times 0 1.

So this first part, as we can see that this part belongs to V 1 and this part belongs to V 2. Now all that remains to show is that the only point which is common to both V 1 and V 2 is nothing but the origin of the 2-dimensional Euclidean space, so we will show it now. Let us suppose that there is a point x 1, Y 1 which lies in intersection of V 1 and V 2.

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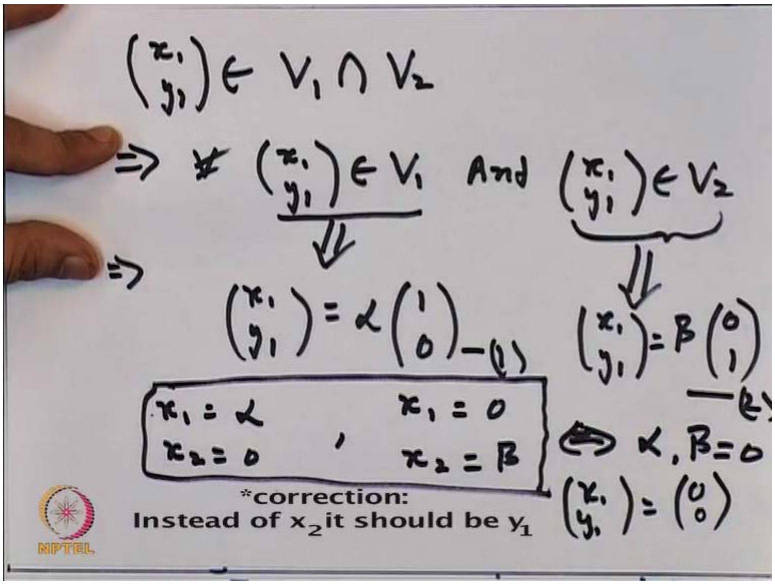

This implies that if x 1 belongs to V 1 intersection V 2, then this point x 1, Y 1 should belong to V 1 and x 1, Y 1 should belong to V 2. But if x 1, Y 1 belongs to V 1, then from here we have another implication that x 1, Y 1 is of the form from scalar times 1 0. And from here, we have the implication that x 1, Y 1 is of the form beta times sum scalar beta times 0 1.

So from here I have actually this point has to set 4 simultaneous linear equations, and what are those 4 simultaneous linear equations, this is true and this is also true. So 4 simultaneous linear equations that my point x 1 Y 1 has to satisfy our x 1 is equal to alpha, x 2 is equal to 0 and x 1 is equal to 0 and x 2 is equal to beta of.

So simultaneous word is very important here, you know this point x 1 Y 1 has to simultaneously satisfy all these 4 equations. Now when this can be true, if and only if alpha, beta both the reals are 0 that is, my point x 1 Y 1 is nothing but the origin in R 2.

So what we have essentially shown here that my R 2, what we have shown here that R 2 is the direct sum of span 1, 0 and span of 0, 1, very simple.

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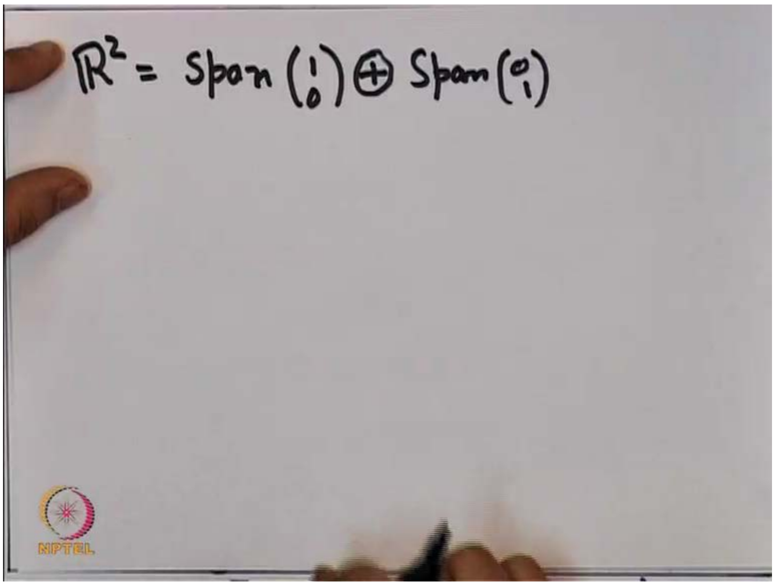

Now, I would like to point of very important subtlety in the case of finite dimensional vector spaces.

When we talk of finite dimensional vector spaces, it is important that the 2 sub spaces which are involved in the operation of direct sum should be such that dimension of those 2 sub spaces should add up to the dimension of the whole space for example, when we were talking about R 2, we saw that R 2 is the direct combination of this subspace and this subspace.

And these 2 are one dimensional sub spaces and the dimensional of these 2 sub spaces add up to 2 which is the dimension of the whole R 2. So this is an important subtlety in the case of finite dimensional vector spaces. Now let us take an example from in finite dimensional vector spaces, okay, this is very interesting.

Let us consider a subspace V which is set of all the functions f from R to R okay.

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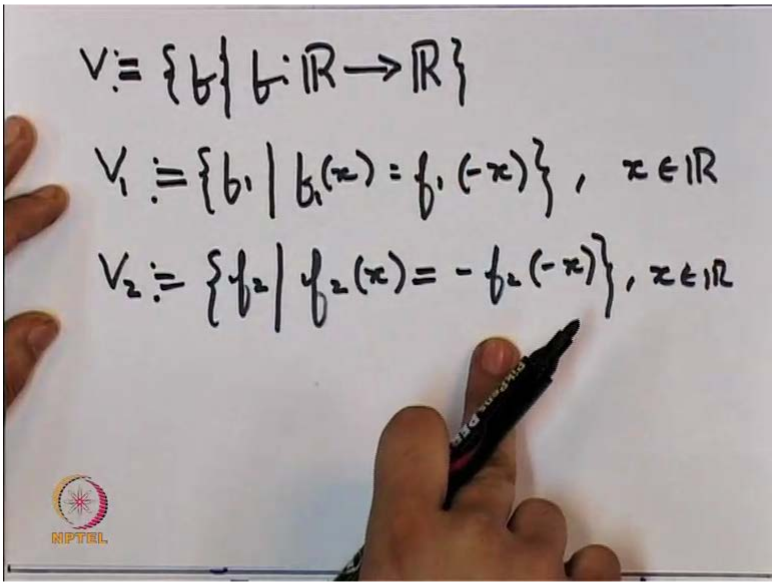

Very simple, I am considering a subspace V which is a set of all the functions from R to R. Now, let us define to other subspaces. This symbol means I am defining V 2 with this set; now let us define two more subspaces. One subspace is V 1 which is the set of all the functions f 1 such that f 1 x is equal to f 1 of - x, okay.

So this is the subspace of all the even functions. I am defining under subspace V 2 of all the functions f 2 such that f 2 x is equal to - f 2 of - x. So this is the set of all the add functions. Note here that in both these cases, x I am talking about x in R, okay, real number.

So what I will be showing that any function which is of mapping from real number to real number is actually a direct sum of functions from set of even functions and a function from the set of odd functions, so let us show that. Now consider a function g of x, okay. Now here g I am taking as in element of V.

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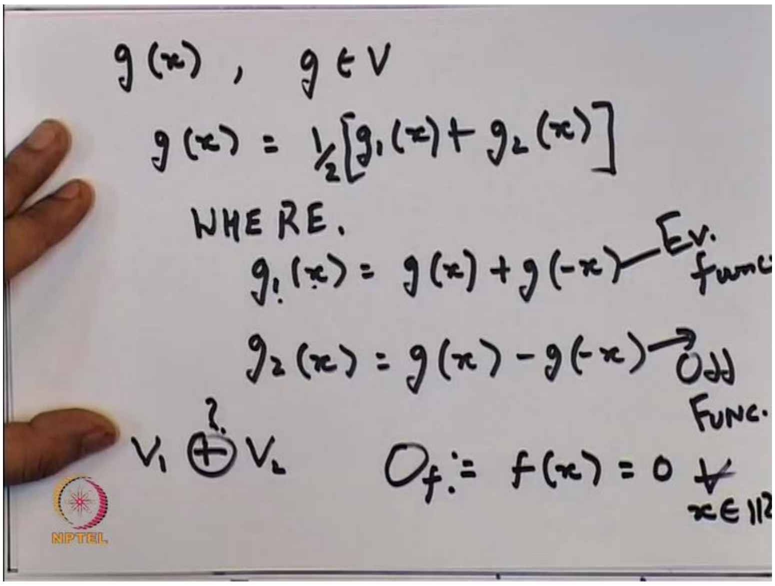

Now we know that any g can be written as half times g 1 x + g 2 x, okay where g 1 x is equal to g of x + g of - x and g 2 x is equal to g of x - g of - x, okay.

This is a very elemental thing that we all must have studied in a basic calculus course, so note here that g 1 is an even function because if in the place of x we go ahead and substitute - x, then what we will get is g 1 of - x is g of - x + g of - of - x which is g of x which is again equal to g 1 of X. So this part is even function, by E v I mean even. And this part can therefore be shown as an odd function.

Now, what we are shown here that any element of V of the vector subspace V can be represented in the terms of element from the vector subspace V 1 which is g 1 x and element from the vector subspace V 2 which is g 2 X. Now for showing that this is actually a direct some okay, what remains to show is that the only function which is common to both V 1 and V 2 is nothing but 0 function okay, so what is 0 function?

0 function is a function which for all the argument x gives you the value 0 okay, this is a 0 function. So let us take a function f which is in the intersection of V 1 and v 2. So if f is in that section of V 1 and V 2, this implies that f belongs to V 1. This is true then this implies that f of x is equal to f of - X.

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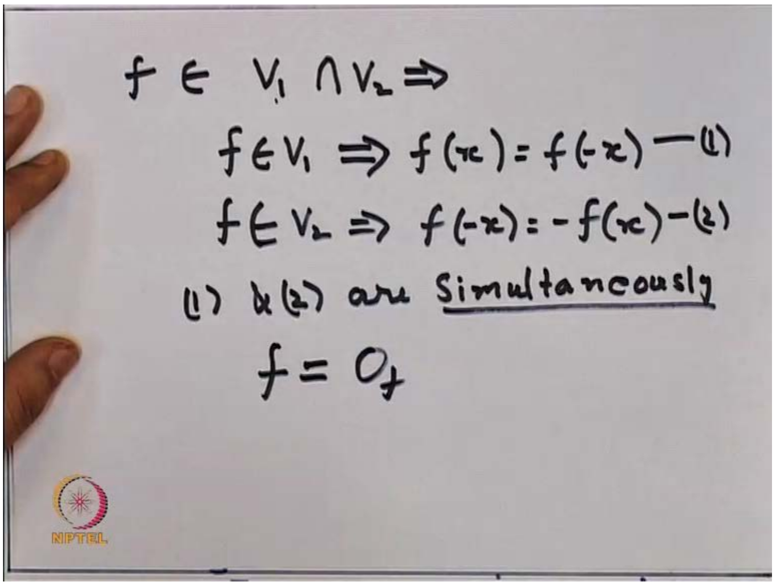

And if f belongs to interaction of these 2 subspaces, then this also implies that f is in V 2 which implies that f of x or f of - x is nothing but - f of x okay. So what do we have here? We have here basically these 2 equations, now these 2 equations if we look at closely, then these 2 equations are true if and only if 1 and 2 are simultaneously true, this simultaneously word is very important.

These 2 equations are simultaneously true if and only if f is your 0 function which we have earlier defined. So what we have shown here is that any function which has its domain as a whole real line and its range as a whole real line can be written as the addition of 2 functions which are even and one of them is odd okay.

So now this direct some operation I think, if you all want to read about it more then you can, there are various books on linear algebra actually which you can consult. One of the very good books which I have found very interesting for beginners is “Linear Algebra and its Applications”. This is the book by Gilbert Stang. It is available in low-price edition in PHI publications; this is available in PHI publication.

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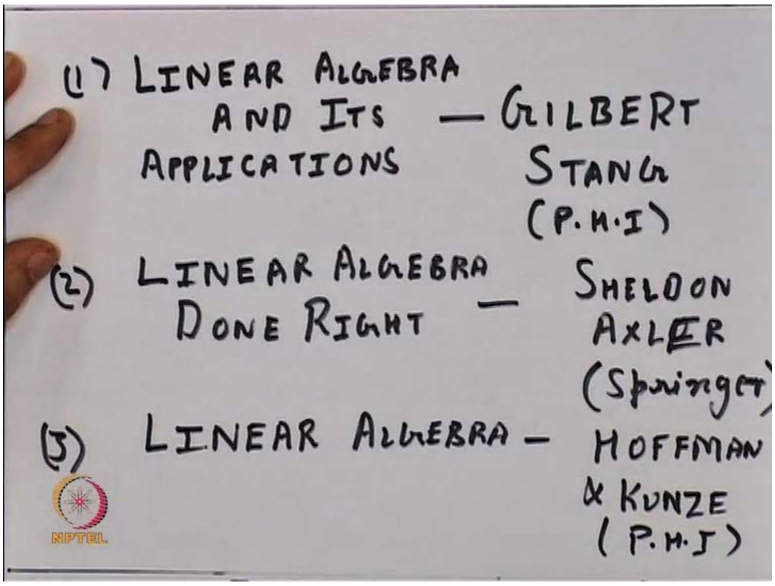

So we can also buy it if you want, another very good book which I find very interesting is Sheldon Axler "Linear Algebra Done Right" Sheldon Axler, I think it is available with Spriger. And the third and very important book which I have recommended you all to study very thoroughly is Hoffman and Kunze, its name is just "Linear Algebra". It is also available in low price edition with PHI. Thank you very much.