

Foundations of Wavelets and Multi-rate Digital Signal Processing.

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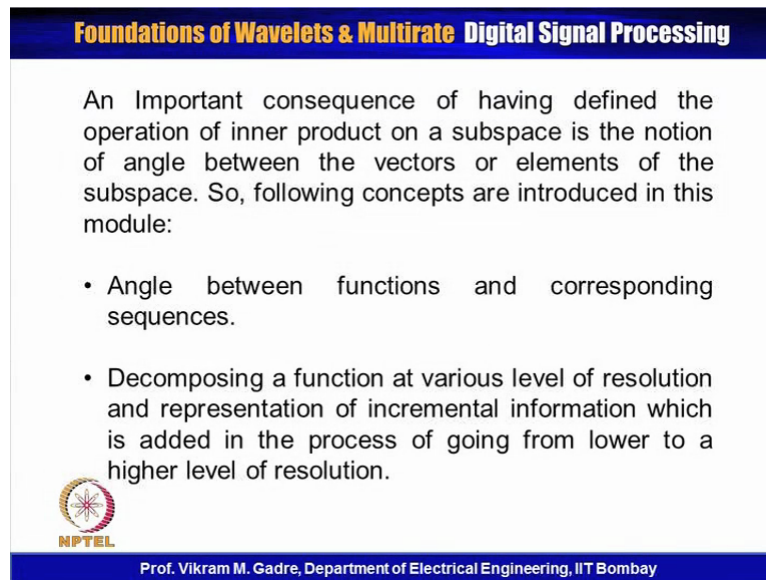
Indian Institute Of Technology Bombay.

Lecture -5.

Module-2.

Angle between Functions and their Decomposition.

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Foundations of Wavelets & Multirate Digital Signal Processing

An Important consequence of having defined the operation of inner product on a subspace is the notion of angle between the vectors or elements of the subspace. So, following concepts are introduced in this module:

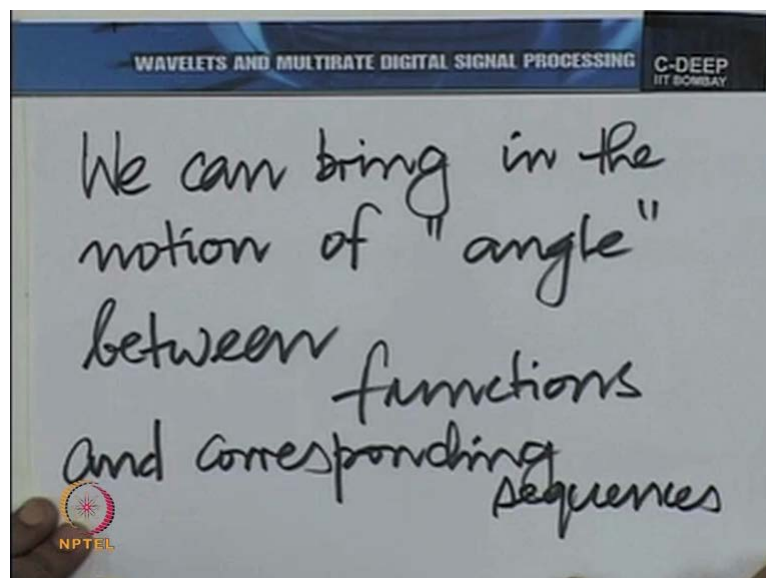
- Angle between functions and corresponding sequences.
- Decomposing a function at various level of resolution and representation of incremental information which is added in the process of going from lower to a higher level of resolution.

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So, we can talk about the angle.

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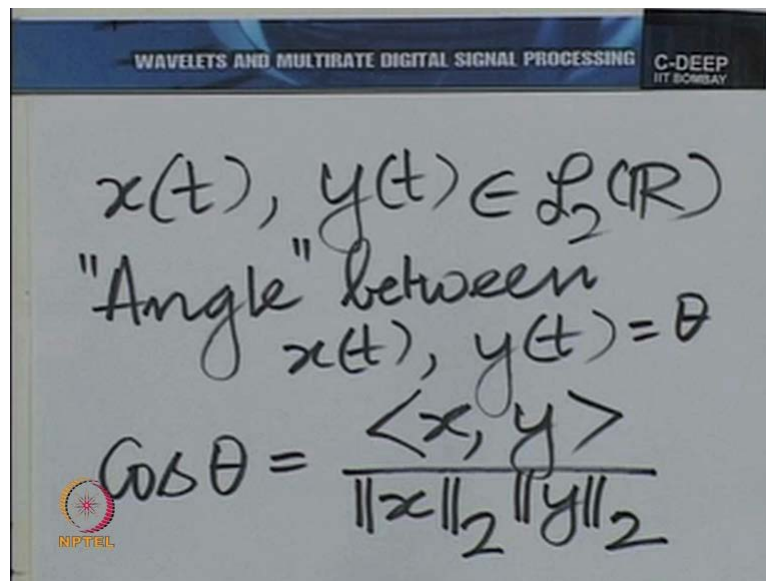
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We can bring in the notion of "angle" between functions and corresponding sequences

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In fact once we bring in the notion of angle between functions, we can also bring in the notion of angle between the corresponding sequences.

(Refer Slide Time: 01:09)



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$$x(t), y(t) \in L_2(\mathbb{R})$$

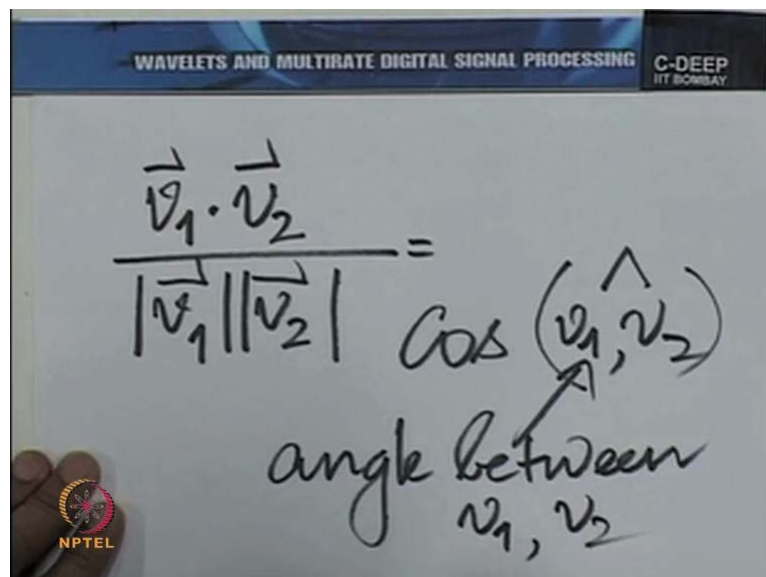
"Angle" between $x(t), y(t) = \theta$

$$\cos \theta = \frac{\langle x, y \rangle}{\|x\|_2 \|y\|_2}$$

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So, if you have 2 functions, X_T and Y_T , of course belonging to $L_2\mathbb{R}$, we are going to confine ourselves to that space. Then the angle between X_T and Y_T is essentially defined by the following. Let it be θ , we say $\cos \theta$ is essentially the inner product of X_T with Y_T divided by the norm in X_2 of in L_2 of X . And the norm of Y in L_2 multiplied together.

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$$\frac{\vec{v}_1 \cdot \vec{v}_2}{\|\vec{v}_1\| \|\vec{v}_2\|} = \cos(\hat{v}_1, \hat{v}_2)$$

angle between v_1, v_2

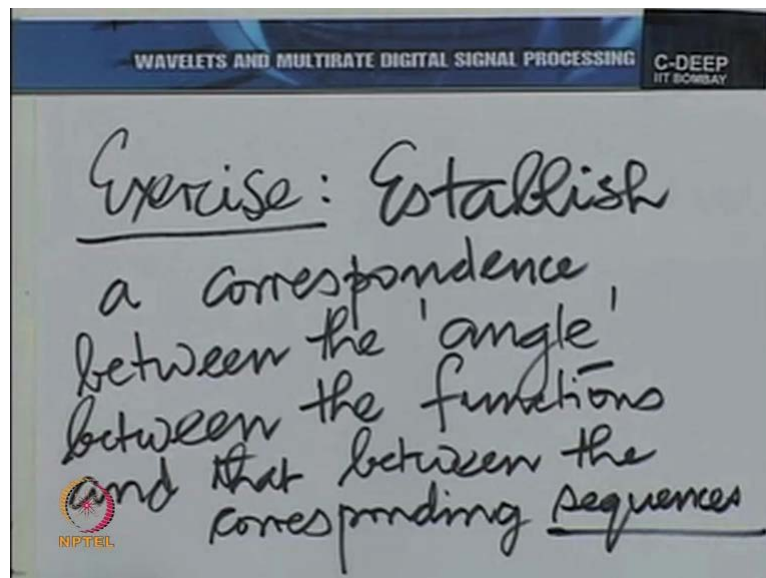
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Now this is very similar to the idea of a dot product between 2 vectors. So, if you recall, if you have 2 vectors, let us say V_1 and V_2 , then $V_1 \cdot V_2$, divided by the magnitude of V_1 and the magnitude of V_2 gives us the cosine of the angle between V_1 and V_2 . So, in a restricted sense, you do have the notion of angle between functions and whatever you did to construct the angle for the functions can also be done for the corresponding sequences

associated with the functions and therefore you have the notion of angle even between the corresponding sequences.

And I ask questions and I leave it to you to ponder over the answers, are those 2 angle the same, do they actually match? I think they should, should not they?

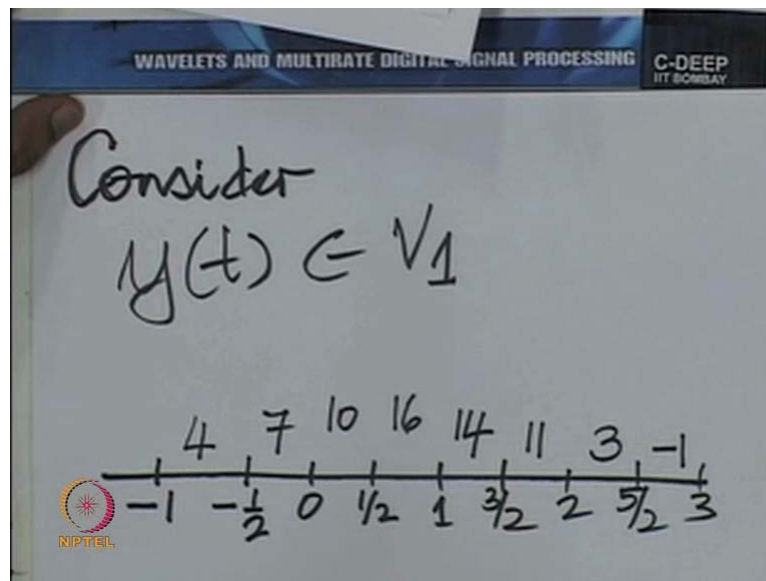
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And I leave it to you as an exercise to actually show that they do. So, exercise. Establish a correspondence between the angles, between the functions and the corresponding sequences. Anyway, our correspondence has become deeper and deeper and whatever we had been doing with the functions, we discovered can now be done with the sequences.

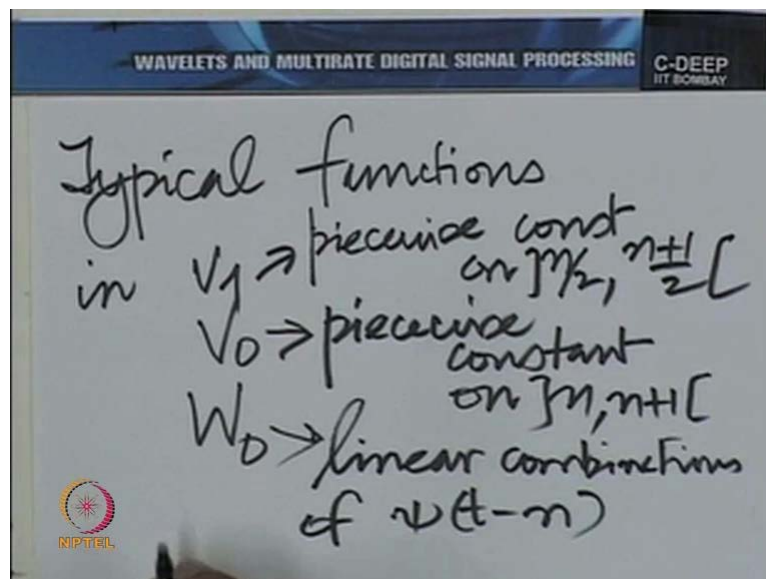
Now the next step is to ask can we also think of decomposition in terms of decomposition of the sequences. So, for example, let us go back to that function in V_1 that we had a few minutes ago.

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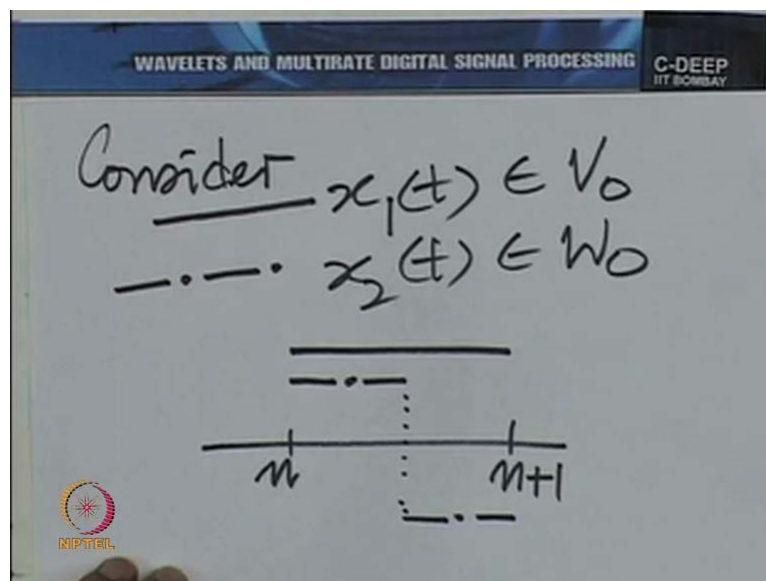
We had this function in V_1 and we could then decompose V_1 into V_0 the orthogonal sum of V_0 and W_0 . Now, just for a minute, let us keep aside, the discussion of this particular function and let us look at a typical function in V_1 , a typical function in V_0 and a typical function names W_0 . So, let us look at typical functions in V_1 , V_0 and W_0 .

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Now, these are piecewise constants on the standard unit intervals. These are linear combinations of $\psi(t-n)$ for integer N . And these are piecewise constants on $[n, n+1]$ by $2n+1$ for integer N . Suppose we take a given function in V_0 and a given function in W_0 , now what would the dot product be?

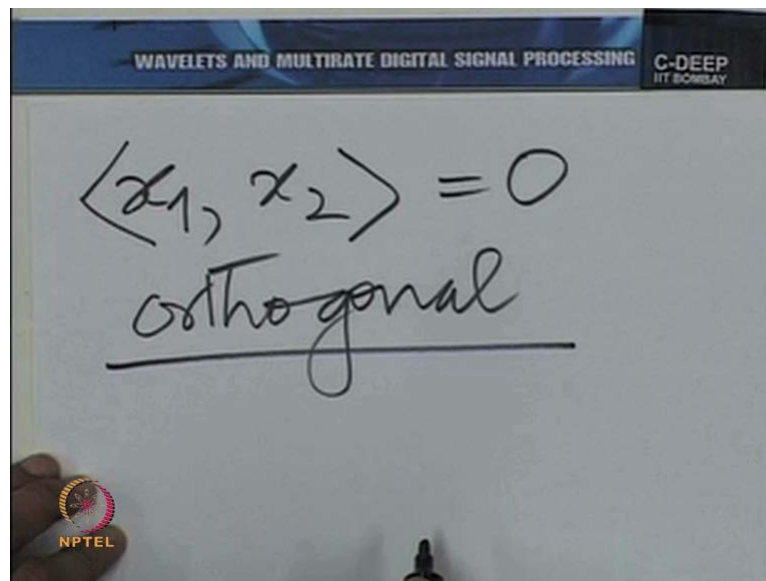
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So, what I am saying is, suppose we consider, say $x_1(t)$ belonging to V_0 and $x_2(t)$ belonging to W_0 , let us focus our attention on a particular interval. Let us say N to $N+1$, let us draw the function $x_1(t)$ by a solid line and this one by a dot dash line.

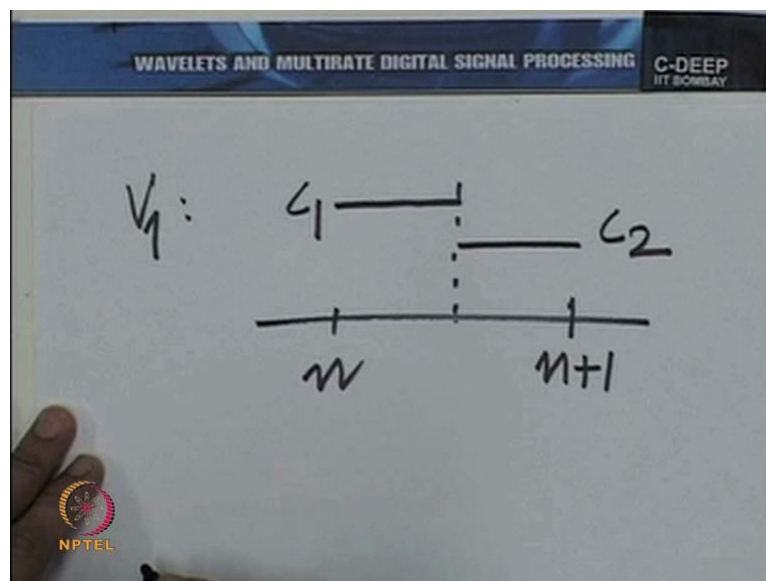
Now for a typical function look like, $x_1(t)$ function would look like this and $x_2(t)$ function might look something like this. If I multiply these functions together and integrated, you can visualize the integral piece by piece on each of these intervals N to $N+1$. The integral of any one of these pieces is obviously 0 because the positive and negative areas are equal. And this can be seen to be true of all the intervals and therefore obviously these 2 functions are perpendicular or orthogonal because their dot product is 0.

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So, the dot product of x_1 and x_2 is 0, they are perpendicular or orthogonal. You know, we do not use the word perpendicular anymore, when we talk about functions, we should talk about orthogonal. What is more, click any particular function in V_1 .

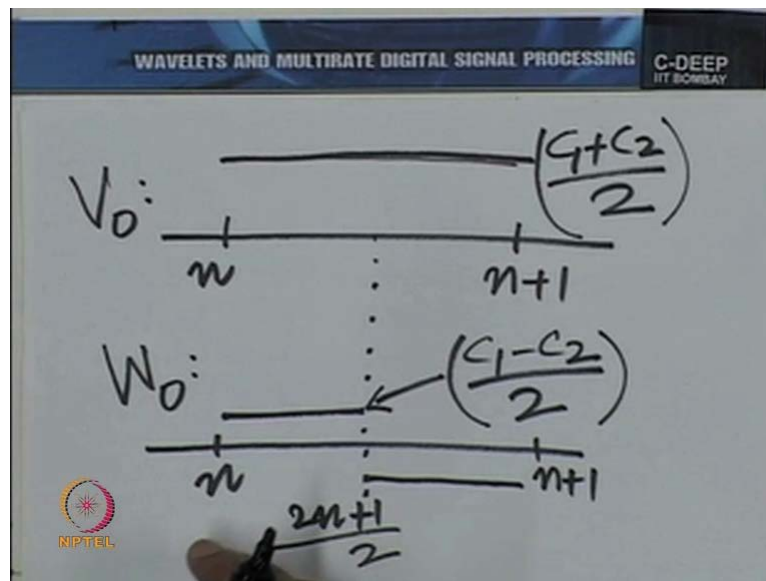
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So, let us take the function, let us say, again focus on any one particular interval of function V_1 , let us take the interval N to $N+1$ and let the function have the value, let us say C_1 for the 1st half interval and C_2 for the 2nd half.

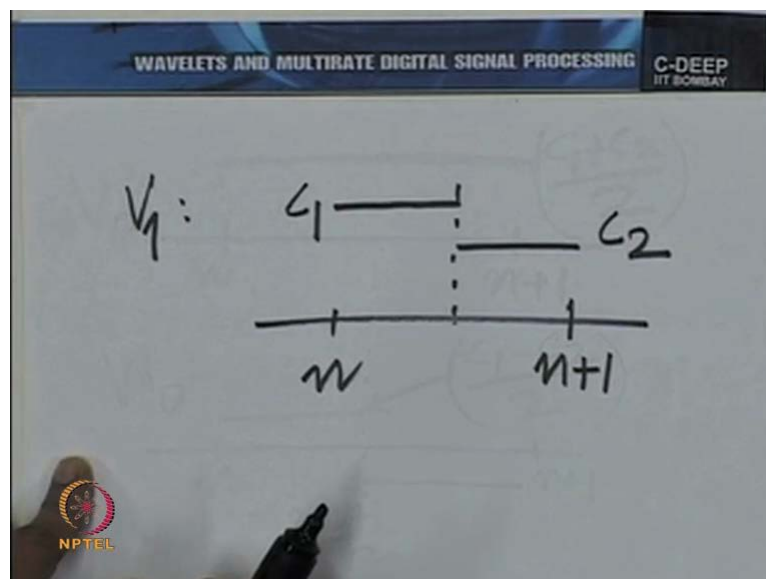
It is very easy to see that this can be treated as a function belonging to V_0 + a function belonging to V_1 where the corresponding function belonging to V_0 looks like this.

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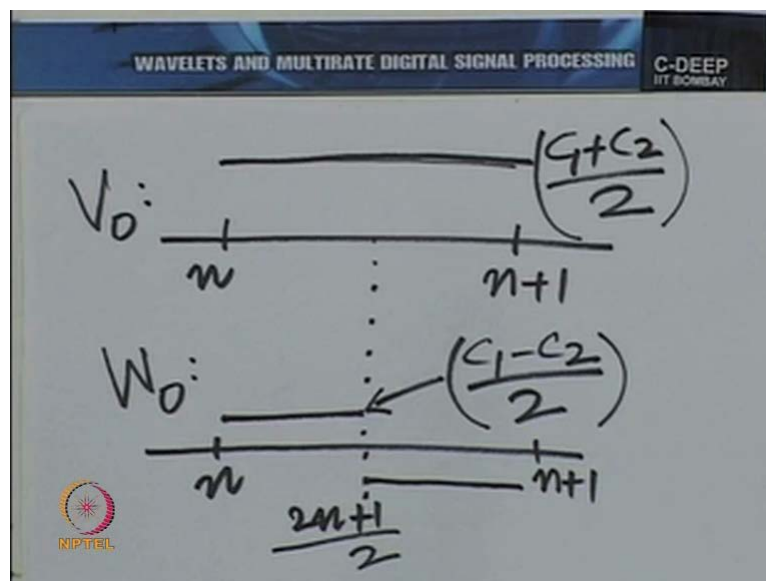
So, this function is equal to this function $C_1 + C_2$ on the interval $C_1 + C_2$ by 2 over the interval N to $N + 1$ + the function $C_1 - C_2$ over the 1st half interval and the negative of the same thing over the 2nd half interval.

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So, if you take this point to be N to $N + 1$ by 2, the middle of the interval, and this height here is $C_1 - C_2$ by 2. So, this is you see... What I am saying is the function coming from V_0 and the function coming from W_0 , I am just showing one segment of each of these functions, the same thing can be done for each of these intervals from N to $N + 1$.

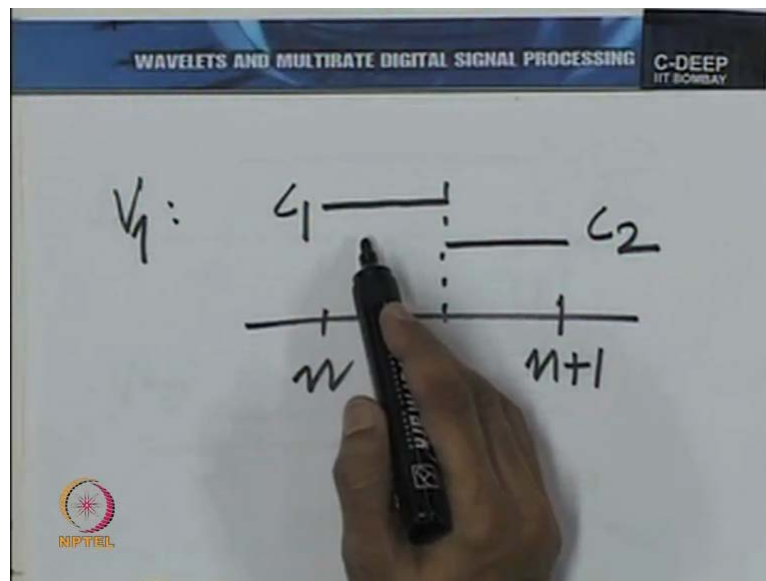
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What I am saying is this function whose segment over N to $N+1$ I have shown here being C_1 on the 1st half interval and C_2 on the 2nd half interval of course it belongs to V_1 is equal to the sum of this function $C_1 + C_2$ by 2 on the entire interval belonging to V_0 . + this function belonging to W_0 which is $C_1 - C_2$ by 2 of the 1st of interval and the negative of the same thing on the 2nd half interval. So, it is very easy to see that we can in general decompose a function in V_1 into a function in V_0 + a function in W_0 in a unique way.

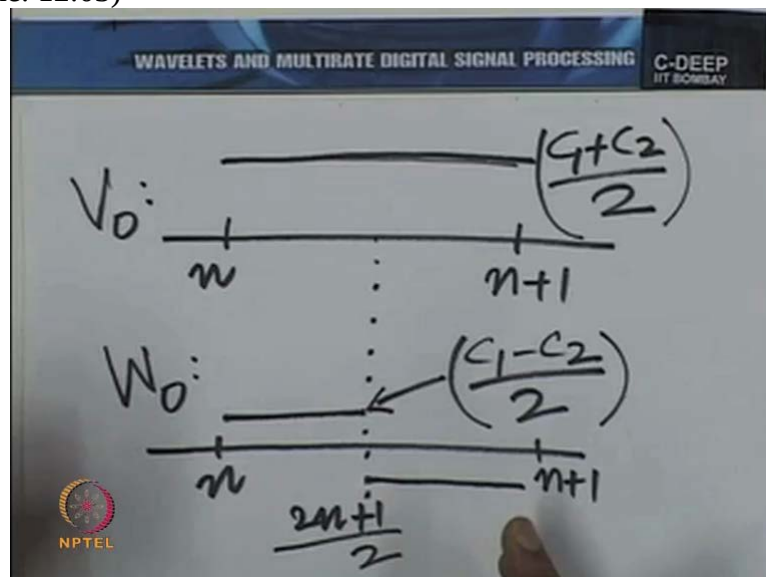
And therefore, the orthogonal decomposition of V_1 and V_0 and W_0 is easy to construct. Now, can we also make corresponding construction on the sequences and in fact to some extent, we have already answered the question.

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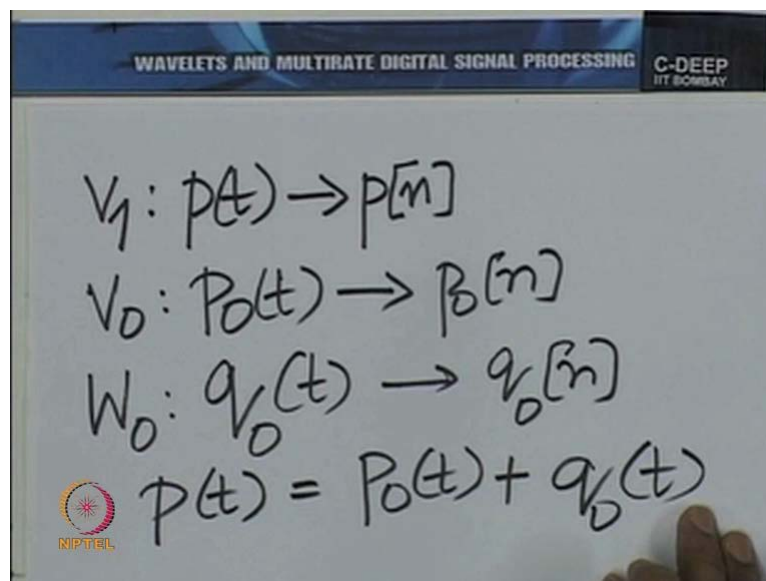
If you look at it, here C_1 would be the value of the sequence at $2N$, the sequence corresponding with the function in V_1 and C_2 will be the value of the sequence at $2N+1$. Interestingly, the value of the sequence corresponding to the function in V_1 at the point $2N$ and $2N + 1$ relate to the values of the sequences corresponding to the functions in V_0 and W_0 but at the points N and not $2N$.

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So, you have the value $C_1 + C_2$ by 2 for a sequence corresponding to this function, at the point N and not $2N$ and you have the value $C_1 - C_2$ by 2 corresponding to the function in W_0 but at the point N and not $2N$.

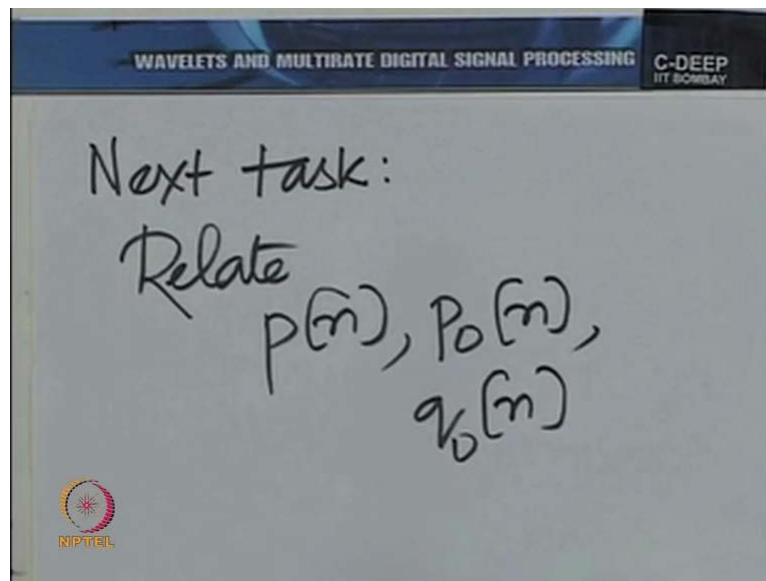
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$$\begin{aligned} V_1: p(t) &\rightarrow p[n] \\ V_0: P_0(t) &\rightarrow P_0[n] \\ W_0: q_0(t) &\rightarrow q_0[n] \\ p(t) &= P_0(t) + q_0(t) \end{aligned}$$

What I am saying is, now if you think in terms of sequences, let us do that, so you had this function in V_1 , so let us say this function is, now you know for variety, let us use P_T belonging to V_1 and the corresponding sequence P_N , you have the corresponding function P_0T let us call it and the corresponding sequence P_0N , where P_0 is the component in V_0 so to speak. And you have Q_0 let us say as a component in W_0 . And the corresponding sequence Q_0N .

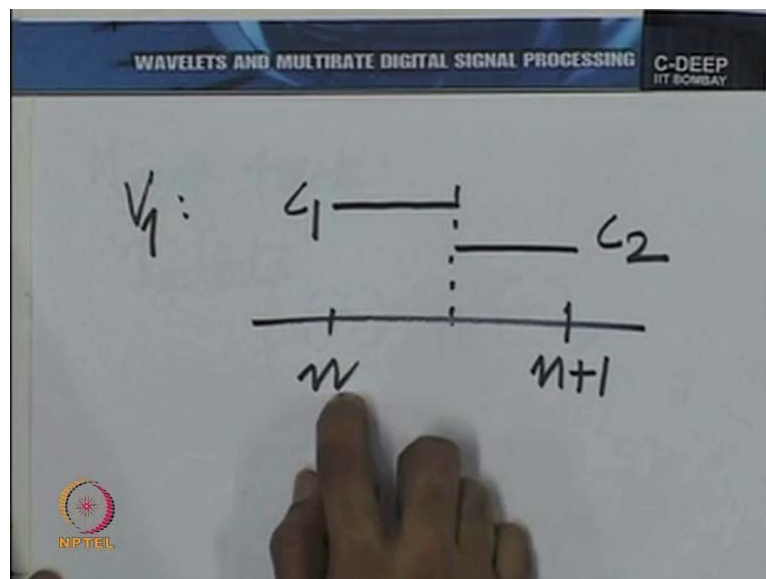
So, what I am saying is P_T is of course equal to $P_0T + Q_0T$. But P_N is not equal to $P_0N + Q_0N$, that is not correct, that is because the orthonormal bases are different. So, now we need to establish a relation between P_0N , Q_0N and P_N , that is the next task that we would like to undertake.

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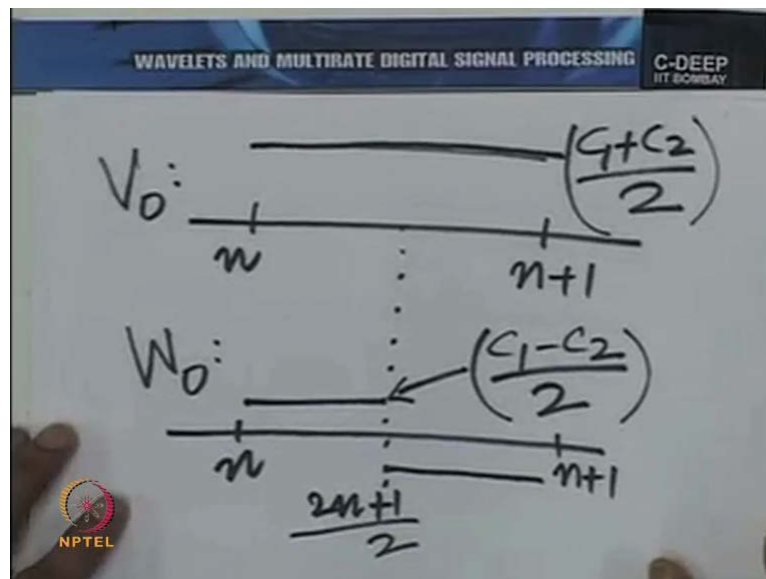
So, our next job is to relate P_N , P_{0N} and Q_{0N} . And in fact, we already had the answer to that question, let me just go back one step here.

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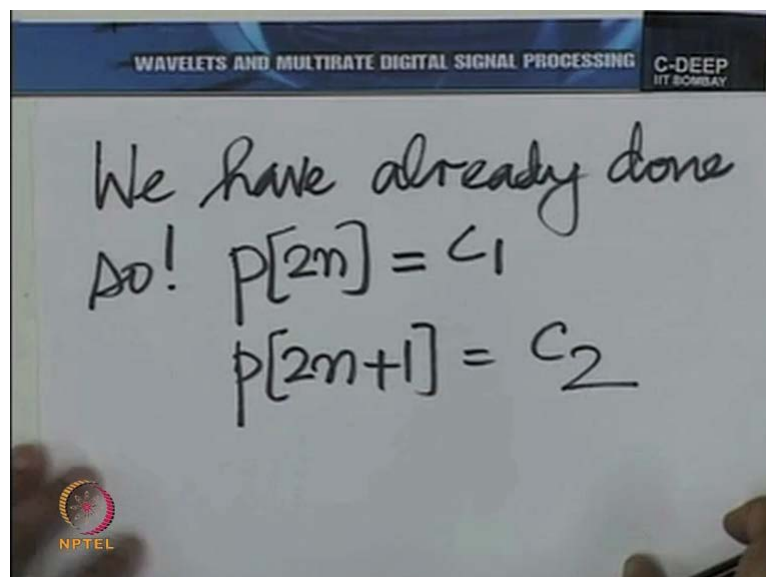
So, we have the answer here.

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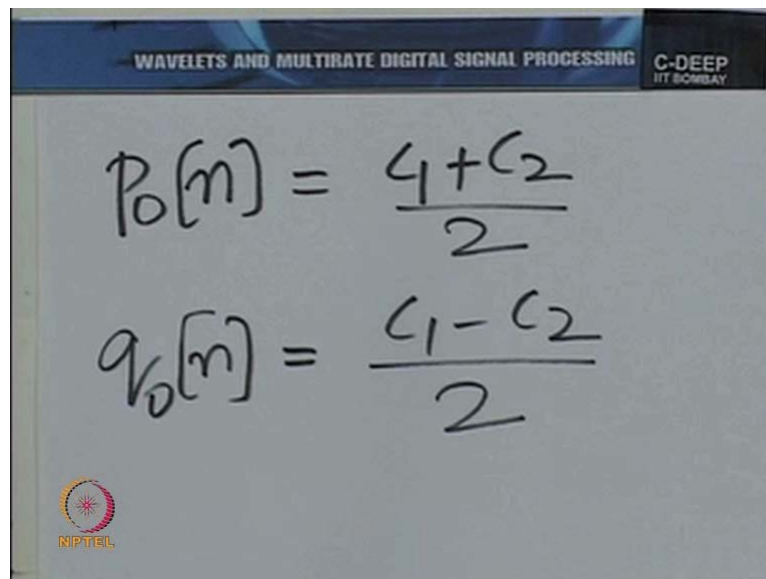
You see, P at the value $2N$ is C_1 , P at the value $2N + 1$ is C_2 , P_0 at the point N is $C_1 + C_2$ by 2, Q_0 at the point N is $C_1 - C_2$ by 2. Let me write down all this formally. So, we have the relationship there, I have almost done my job.

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So, what have we said, we said P at $2N$ is C_1 , P at $2N + 1$ is C_2 , P_0 at N is $C_1 + C_2$ by 2 and Q_0 at N is $C_1 - C_2$ by 2.

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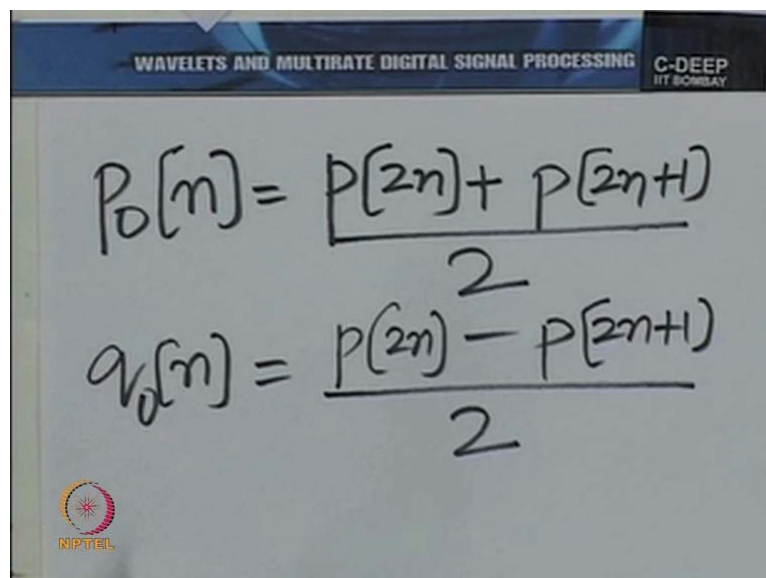
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$$p_0[n] = \frac{c_1 + c_2}{2}$$
$$q_0[n] = \frac{c_1 - c_2}{2}$$

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Now let us combine these equations. So, we have P_0N is $P_2N + P_2N + 1$ by 2.

(Refer Slide Time: 15:43)



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$$p_0[n] = \frac{p[2n] + p[2n+1]}{2}$$
$$q_0[n] = \frac{p[2n] - p[2n+1]}{2}$$

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And Q_0N is $P_2N - P_2N + 1$ by 2. And now this brings before us a very beautiful perspective. When we talk about sequences, we can also extend that context to talk about discrete time filters acting on sequences. Can we visualize what we have done here as discrete time filters acting on the sequences? So, suppose you have the following discrete time filter.

(Refer Slide Time: 16:43)

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Consider the following discrete time filter:

$$x[n] \rightarrow \boxed{\text{filter}} \rightarrow y[n]$$
$$y[n] = \frac{1}{2} \{x[n] + x[n+1]\}$$

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Let the input be X of N and output be Y of N and Y of N is half $X_N + X$ of $N+1$. You know, it is a non-causal filter, let us not worry too much about it for a moment, let us accept it even if it is non-causal. What have we done here, in this relationship?

(Refer Slide Time: 17:39)

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$$p_0[n] = \frac{p[2n] + p[2n+1]}{2}$$
$$q_0[n] = \frac{p[2n] - p[2n+1]}{2}$$

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Let us reflect on the connection between for example this relationship here and the filter we just constructed. If you think of $2N$ as a variable, let us call it, let us say it is L , so, this is P of $L + P$ of $L+1$ by 2, then in fact this is essentially the filter acting on the sequence P .

(Refer Slide Time: 18:12)

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Consider the following discrete time filter:

$$x[n] \rightarrow \boxed{\text{filter}} \rightarrow y[n]$$
$$y[n] = \frac{1}{2} \{x[n] + x[n+1]\}$$

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So, what we are saying is use this filter and put PN here but then a little bit of work needs to be done at this point because if you are putting PN, let us do that, let us put in PN.

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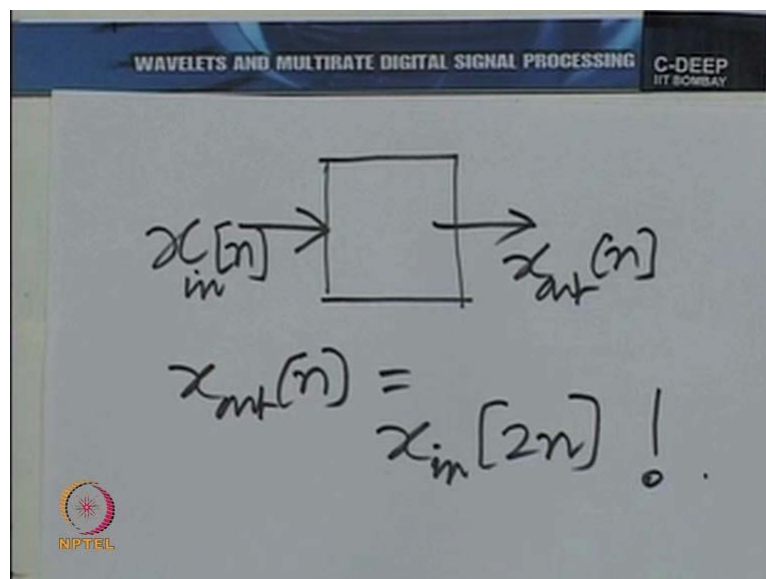
$$p[n] \rightarrow \boxed{\text{filter}} \rightarrow \frac{1}{2} \{p[n] + p[n+1]\}$$
$$y[n] = \frac{1}{2} \{x[n] + x[n+1]\}$$

we want $2n$ not n !

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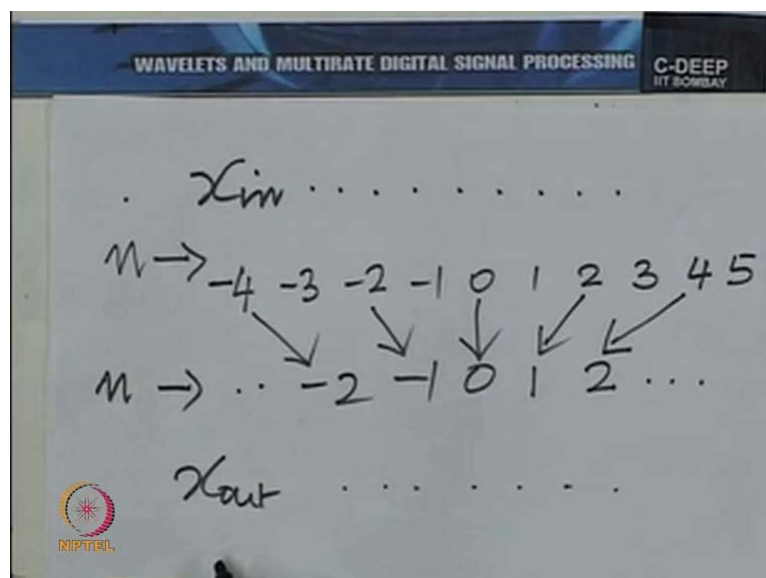
So, if you put in PN there, if you put into this filter, we just wrote down here, Y of N is half $XN + X$ of $N + 1$, what we would get here is half of $PN + PN + 1$. But we do not want PN and $PN + 1$, we want to replace here.

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So, what should we do, we should have another system now following this where putting x_{in} and put out x_{out} where x_{out} of N is x_{in} of $2N$, we want a system like this.

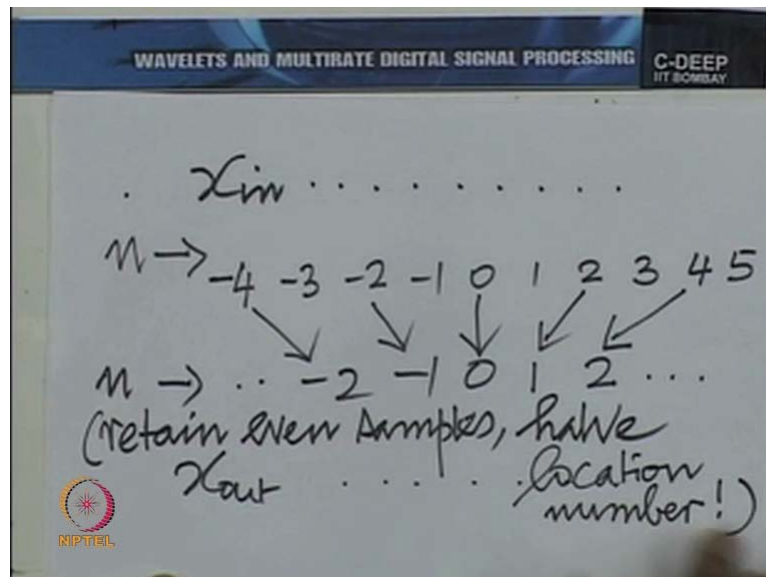
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Let us interpret this system. What we are seeing in this system is x_{in} at N , let us write down N $-4, -3, -2, -1, 0, 1, 2, 3, 4$ and so on and the sequence x_{in} here, put at these points. So as far as x_{out} goes, you have the index for x_{out} , as far as x_{out} goes, 0 comes from 0 , 1 here comes from 2 , 2 here comes from 4 , -1 comes from -2 , -2 comes from -4 and so on. So, in other words, what you are doing, you are retaining the samples at the even locations and throwing away the samples at the odd locations.

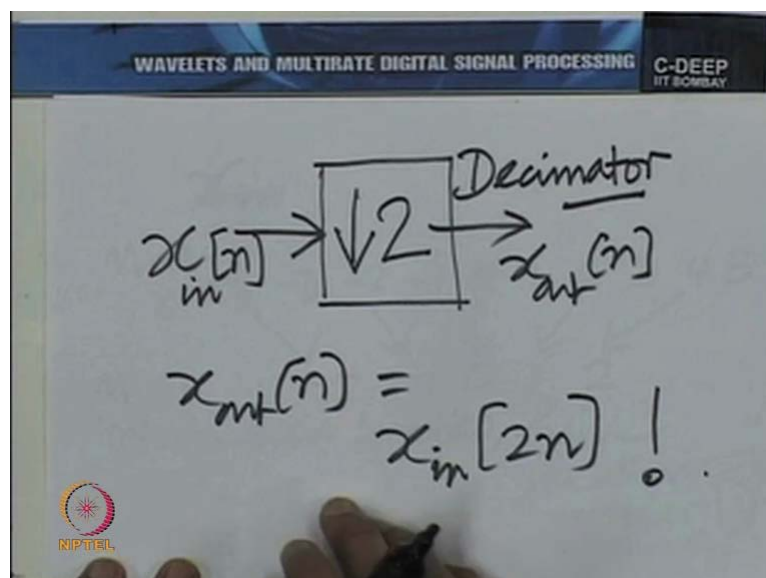
Not only that, after retaining the samples at the even locations, you are rating those samples at half the location number. So, let us summarise this.

(Refer Slide Time: 21:22)



Retain even samples and halve the location number. Now, this system is a new system as far as a basic course on discrete time signal processing is concerned, we need to christen it, we need to give it a name, in fact let us go back to that system.

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And let us give it both a symbol and a name. The symbol that we shall give it is a down arrow followed by 2. And we shall call it a decimator.

You know the word decimate actually has a very cruel meaning. I am told that in the days of war, during the Roman Empire, a very cruel thing that warriors used to do was to kill 1 out of 10 or maybe 9 out of 10 and that was what was called decimation. Take 10 of them and eliminate from each group of 10, that was a cruel way to deal with people. But the word decimate has also percolated down to the literature on digital signal processing. Here decimation means retaining one out of so many samples.

So, in this case, decimation by 2 means retaining one out of 2 samples. In fact the 1st of each pair of 2 samples. Out of 0 and 1, you retain 0. Out of 2 and 3, you retain 2. Not only that, after you retain only one out of two, compress so that the Sample number is halved or if you retain one out of three samples, then compress, so the Sample number is multiplied by one third. So, if you are decimating by a factor of 3 for example, the 0 Sample will go to 0, the 3 Sample will come to 1, the 6 Sample will go to 2, the -3 Sample will come to -1 and so on. If you are decimating by a factor of 2, then 0 Sample will come to 0, the 2 sample will go to 1, the 4 Sample will go to 2, the -2 Sample to -1 and so on as we have just shown.

So, what do we have here, we have a filter followed by a decimator and that together helps us to construct the sequence P_0N from the sequence PN . Now, we shall see us in the next lecture that they can similarly construct the sequence Q_0N from the sequence PN by using another filter and decimator and we shall build up further from there to do something to reconstruct PN from P_0N and Q_0N and all this shall together lead us to a totally different structure in discrete time signal processing which we shall call a 2 band filter bank. With this little trailer for the next lecture, let us conclude the present lecture, thank you.