Foundations of Wavelets and Multi-rate Digital Signal Processing.
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Lecture -5.
Module-1.
Equivalence of Functions and Sequences.

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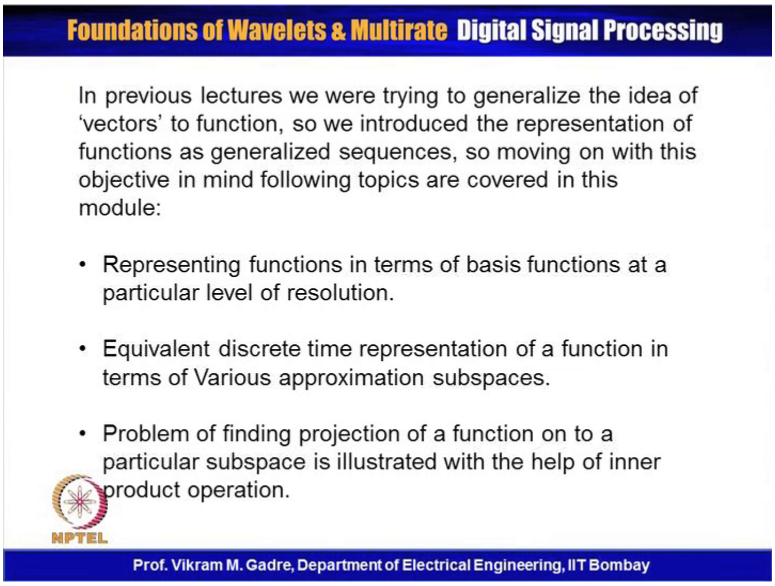

A warm welcome to the 5th lecture on the subject of wavelets and multi-wave digital signal processing. To put the discussion of the current lecture in perspective, let us recapitulate what we did in the previous lecture. In the previous lecture, we had looked at the equivalence between functions and vectors in depths. In fact, in that equivalence we saw that by bringing in the notions of the inner product between functions and of course the inner product between sequences and then noticing that Parseval's theorem is a version of a similar statement on the inner product. We understood that this analogy between functions and vectors leaves us much to gain because it helps us picturize very well, it helps us visualize very well what we were talking about when we talked about ladder of subspaces.

In general, in function analysis this is a very serious analogy. So, it is not just a simile so to speak, it is a metaphor. We can actually think of functions as generalised vectors and gain a lot from that kind of analogy rather that kind of an equivalence or generalisation. And now we need to bring in another dimension to this discussion which we had briefly begun in the previous lecture. The dimension of replacing work with functions with work with sequences.

Sequences are easier for us to deal with. In fact sequences can be dealt with using a computer. You could store the samples of a sequence in memory point after point. You could process them in discrete time step-by-step and produce an output which is again a sequence and lo and behold, is whatever you are doing with sequence maps exactly with what you wish to do with the original continuous time functions, then this is an added advantage. In fact this is true for the spaces V0 Containing V1 containing V 2 and V-1 containing V0 and so on.

As we saw briefly in the previous lecture but which we shall now go into greater depths today. So, let us come back to the discussion. How could we think of a function in V0 has a sequence, has an equivalence sequence? Well, very simple, essentially what we are seeing is look at the coefficients in the expansion of that function with respect to the basis the space V0. So, there you are.

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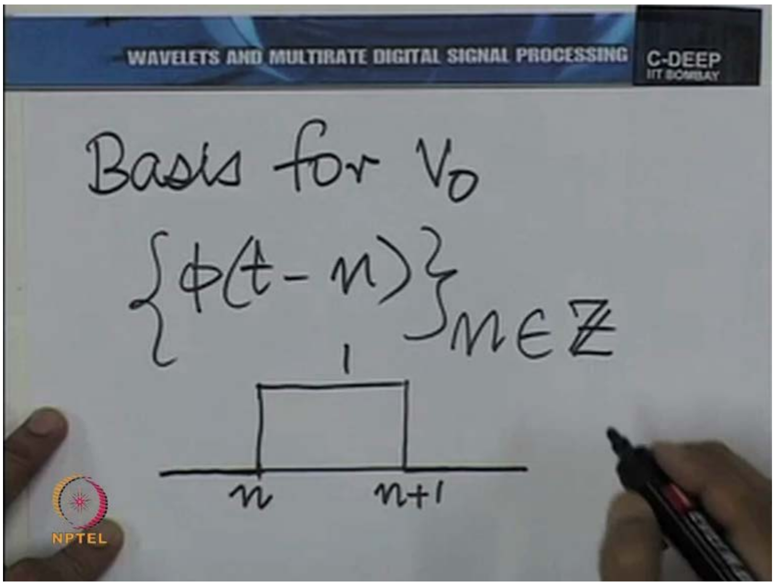

We have the following standard basis for V0 given by Phi of T - N, N over all the integers and let us sketch it typical Phi of T - N, it is 1 between N and N +1 and 0 elsewhere, 0 everywhere else. This is how Phi of T - N looks.

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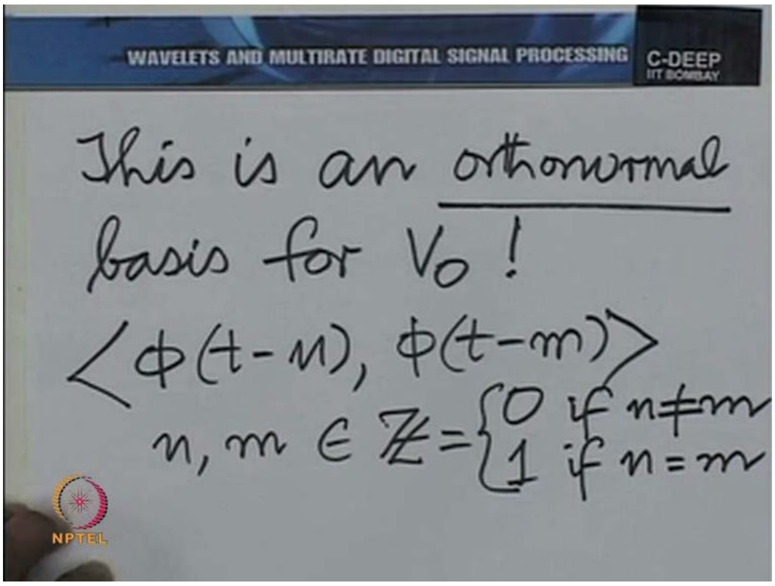

Now in fact, this is also an orthonormal basis, I will introduce that word, orthonormal basis. What they mean by that? If I take the dot product of 2 of them, Phi of T - N and Phi of T - M for any 2 integers N and M, N and M belong to the set of integers. Then this dot product is equal to 0 if N is not equal to M and 1 if N is equal to M. Very easy to check, it does not require too much of working to prove this and I leave it as an exercise for you to show.

All that you need to do is to calculate the integral in the product after all, Phi of T - N and Phi of T - M are in fact nonoverlapping when N is not equal to M. And if N is equal to M, the overlap completely and then of course the integral is just the integral of a rectangle of unit height over unit length, so it is one. Whatever it be, let us consider the function in V0 again to fix our ideas of the connection between functions and sequences. So let us take this example again little bit of a repetition from the previous lecture but let us fix our ideas with it. So, we have this function say X of T given by the following graphical representation.

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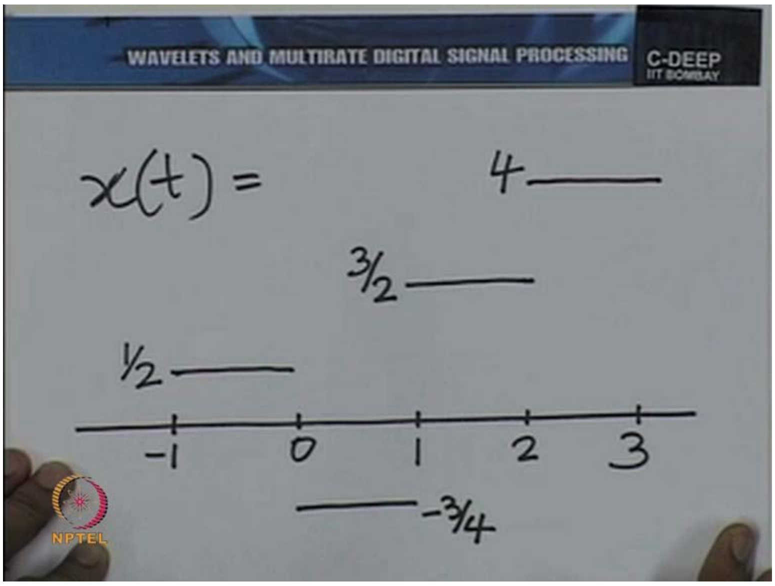

So, if I take the value, let us say half or some variety between -1 and 0, -3 by 4 in the region from 0 to 1, 3by 2 in the region 1 to 2 and let say 4 in the region 2 to 3.

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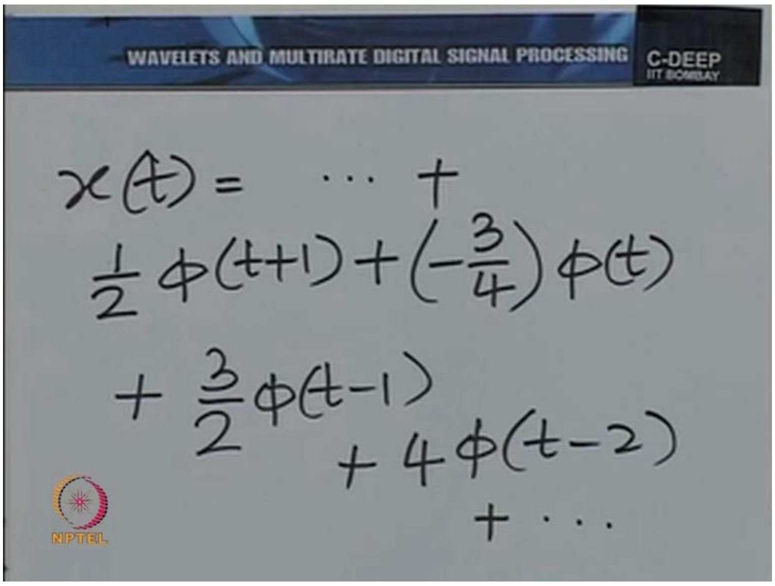

So, it is very easy to see that X of T can written as half Phi of T +1 + -3 by 4 times Phi of T + 3by 2 times Phi of T -1 + 4 Phi of T -2 and so on. So, you could continue this beyond 3 and you can continue it before -1. And what we said in the previous lecture was that equivalent to this continuous function is a sequence constructed by dot dot dot and then 1by2 at -1, -3 by 4 at 0, 3by2 at 1, 4 at 2 and so on.

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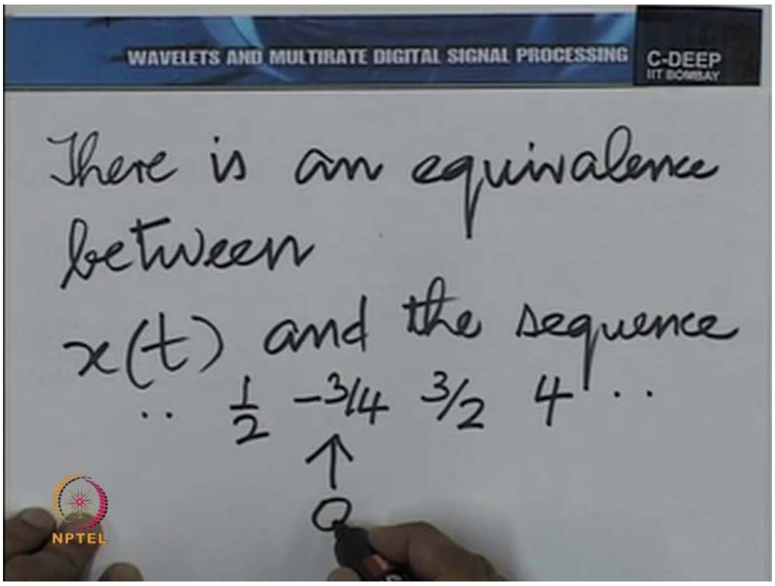

So, in other words, there is an equivalence between XT and the sequence. Now again, just to recapitulate the notation for a sequence, we write the 0 point and Mark the RO 0 here to show that this is the sample at N equal to 0. And then of course you have 3 by 2 at 1, letter 4 at 2 and 1by2 at -1 and so on. So, when we write the sequence in this notation, what we mean is that this is a sample at 0, N equal to 0. And then the other samples are arranged in the right order. So, for example, this is a sample at N equal to1, the sample at N equal to 2, sample at N equal to -1 and so on in the correct order around the samples that we have marked.

Now as an alternative, we could have as well written the same sequence in the following way. This is just to introduce the notation properly. So, we could also have written…

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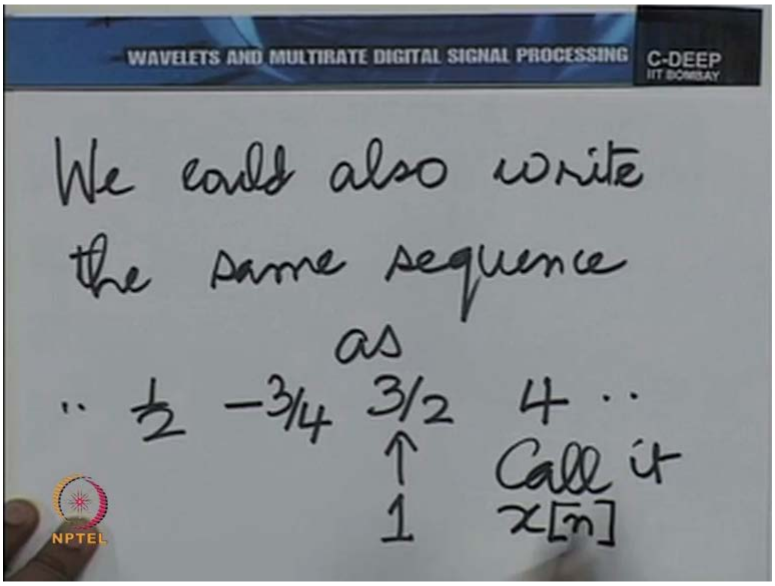

So, you could mark 1 for example, the sample at point number 1. And then the sample at point number 2 is 4, the sample at 0 is -3by4, the sample at -1 is 1by2 and so on. So, it is the same sequence written differently, just for some variety notations. Sometimes we might prefer to do something like this. Anyway coming to the point and stressing it once again, there is an equivalence, there is an equivalence between functions and the sequence. This function belongs to V0 and the sequence belongs to what is called the set of square integrable sequences.

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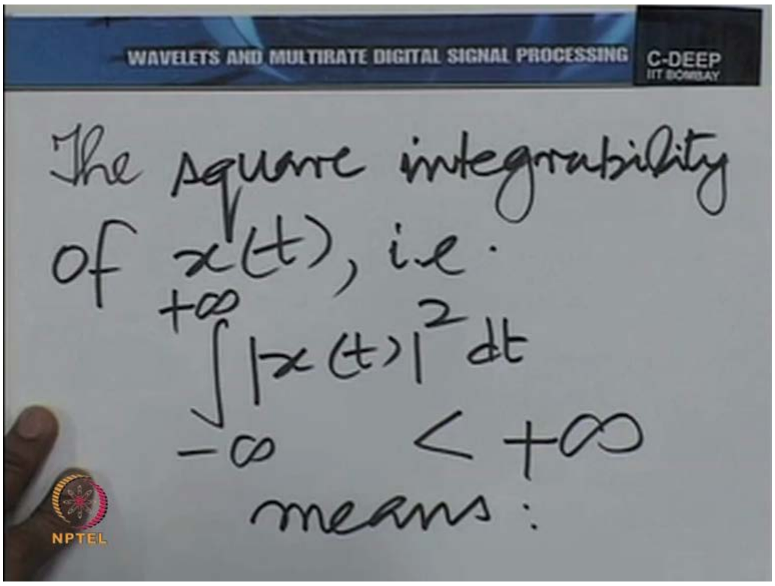

So, let us call this sequence X of N, in that case, the square integrability of XT, in other words,

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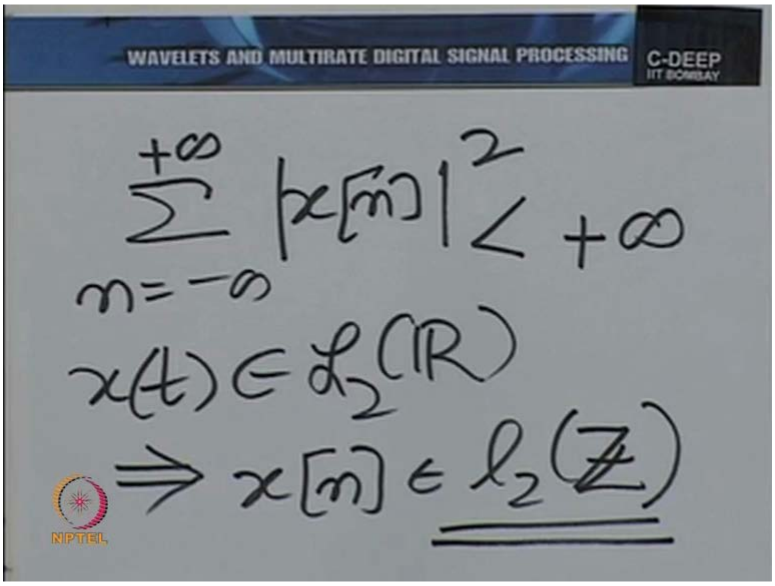

the fact that integral from – to + infinity mod XT square dt is finite means summation from N going – to + infinity is also finite with the arguments as model XN square, this is also finite. And therefore, we use a term for this, we say that the XT belong to L2R implies X of N belongs to small L2 Z, so introduce this notation. This is a new term we went to do, small L2 Z. Just like we have L to denote spaces with continuous arguments, you have small L to denote spaces with discrete arguments. And again, we would like to define LP Z in general.

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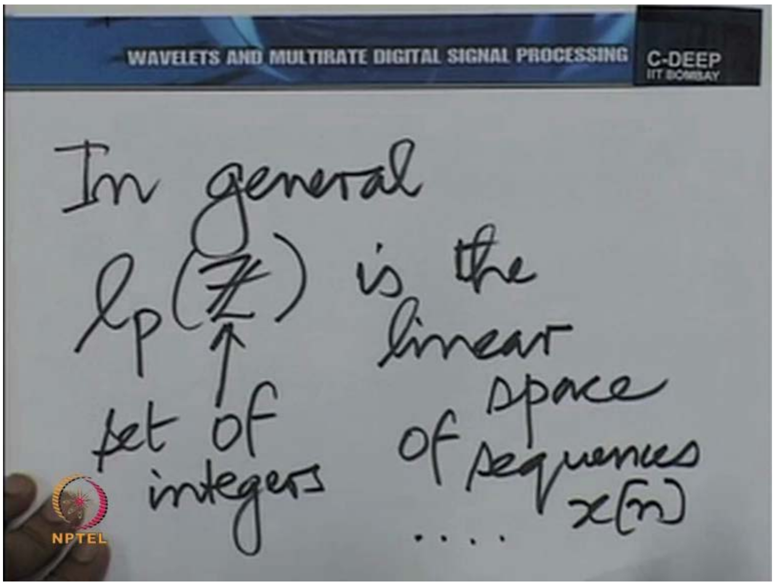

So, in general, L PZ, this Z of course refers to set of integers. So, L PZ is a set of sequences, in fact it is the linear space of sequences. Let the sequences be written as XN, such that,

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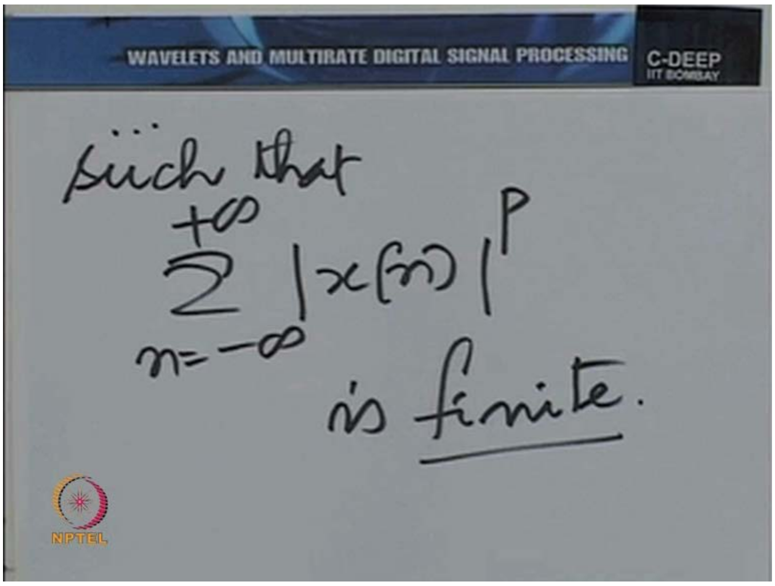

such that this is continued, such as that summation M going from - to + infinity, mod XN to the power P is finite. So, in particular, if you could P equal to 2, you get small L2 Z.

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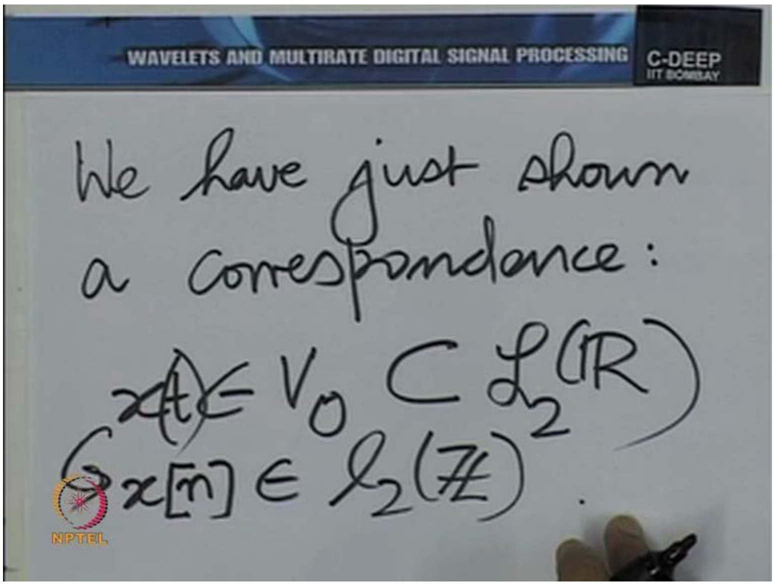

So, what we have just shown is that there is a correspondence. So, we have just shown a correspondence. If X belong to V0, which in turn is a subspace of L2R, of course here were talking about a continuous time function X of T, then we have the corresponding X of N belonging to small L2 Z.

Of course we can make a similar inference for other values of P but that is not of consequence at the moment, so we shall not go into it. What is important is when we talk about inner products, now if you have an orthonormal basis, so here for example we have an orthonormal basis.

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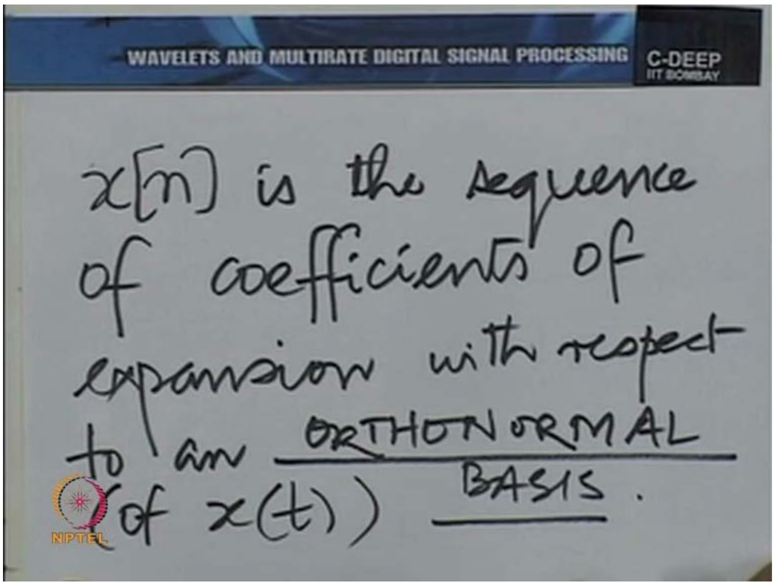

You see, so X of, in fact here XN is the set of coefficients is a sequence of coefficients of expansion with respect to an orthonormal basis. So, you know that orthonormal is important here. If the basis is not orthonormal, what we are now going to say very soon is not going to be true. So, the orthonormal basis is important here.

If XN is a sequence of coefficients of expansion with respect to an orthonormal basis, of course of XT I mean, then there is also mapping between the inner products, that is what is interesting. So, not only is there just an equivalence, there is also a mapping of the other operations.

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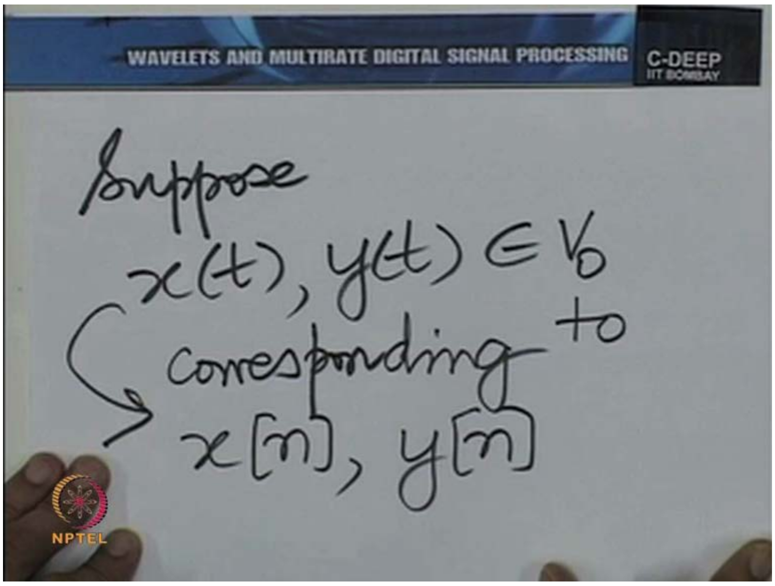

So if we have 2 such functions in V0…, So, suppose you have XT and YT belonging to V0, both of them belong to V0, which of course belong in turn to L2R. And that corresponds to the sequences X square N and Y Square N, square meaning square bracket.

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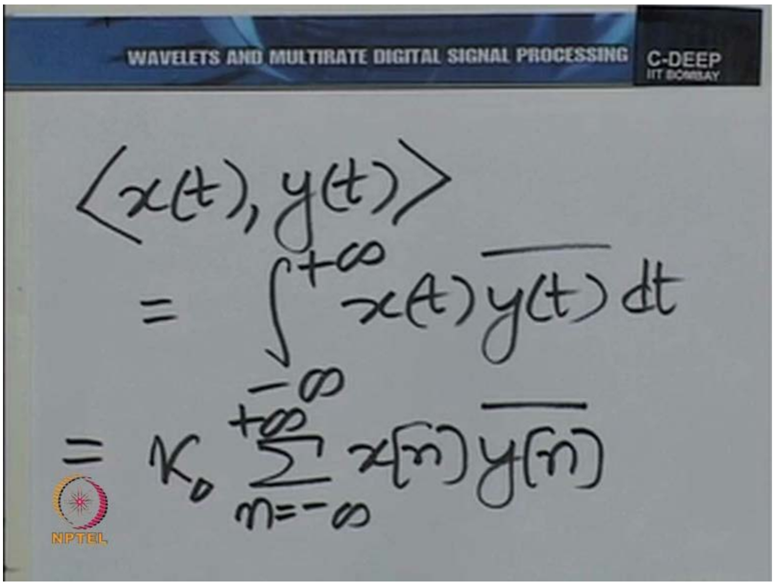

Then the dot product of XT with YT in continuous time, so inner product of XT and YT is understood on the continuous time axis which is of course given by summation - to + infinity integral XT Y bar T dt is a multiple, is some constant times Summation N going from - to + infinity XN YN bar. This is important. There is also a carry onward of the equivalence to the domain of inner products. So, whatever we are doing in the context of the continuous functions can be done, can be equivalently done or can be equivalently derived or dropped forward to the context of the sequences which are associated.

So, we can go to the extent of forgetting about underlying continuous functions and deal with the sequences directly. Now what is the physical or practical meaning of this? Well, let us go back to the 1st lecture where we motivated the very idea of piecewise constant representation. And we started with a two-dimensional situation. We said let us take images and we said let us divide those images into very small areas which we represent using picture elements or pixels. On each of the pixels, we put down one number corresponding to the average intensity of the picture in that pixel area.

And we said that is reasonably representative of the image or the picture if those areas are small. Now, of course we are representing the original image by piecewise constant function and equivalently we could now think of the image as being represented by a two-dimensional sequence. So, sequence index with 2 integer variables, let us say N1 and N2. N1 going from 0 to 511 and N2 also going from 0 to 511, in case we have 512 **cos** 512 picture representation or resolution on the computer screen. Good.

So, what we are saying now is easy to understand. Whatever we wanted to do with the picture, we can equivalently do with the two-dimensional sequences. If you have 2 pictures and if you want to mix-and-match and do whatever you want to do, you could do it equivalent with this two-dimensional sequence. If you want to gain some inferences, you could gain the same inferences by looking at these sequences. Good. But where does this take us?

Now what is going to be useful to us is to see how we can move from one resolution to the next, that is what is of interest. You see, ultimately our movement in all this discussion is to extract incremental information and incremental information is extracted by going from one subspace of L2R to the next in the ladder.

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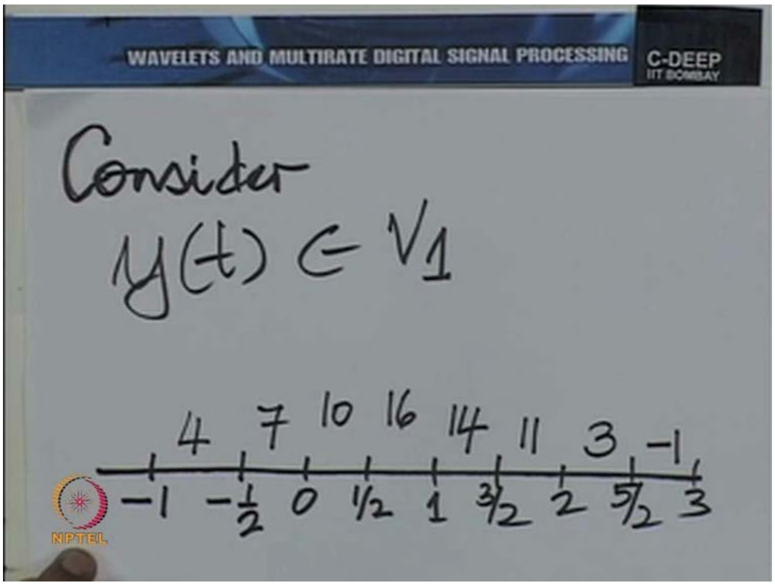

So, let it T the function belong to V1. Consider a function, let us say YT belonging to let V1. Now V1 you will recapitulate is the space of functions piecewise constant on intervals of linked half.

So, let us take one such, let us take a function from -1 to say 3. So, -1, -0.5, 0, 0.5, 1, 3 by 2, 2, 5 by 2 and 3. So, I will just write down, I will not sketch, I will just write down values, the piecewise constant values in each of these intervals. So, in this interval the value can be let us say 4. In this interval, the value is 7, in this interval, the value is 10, in this interval, the value is 16, let us say in this interval, the value is 14, 11, let us say 3 and -1 some variety. These are the piecewise constant values in these respective intervals.

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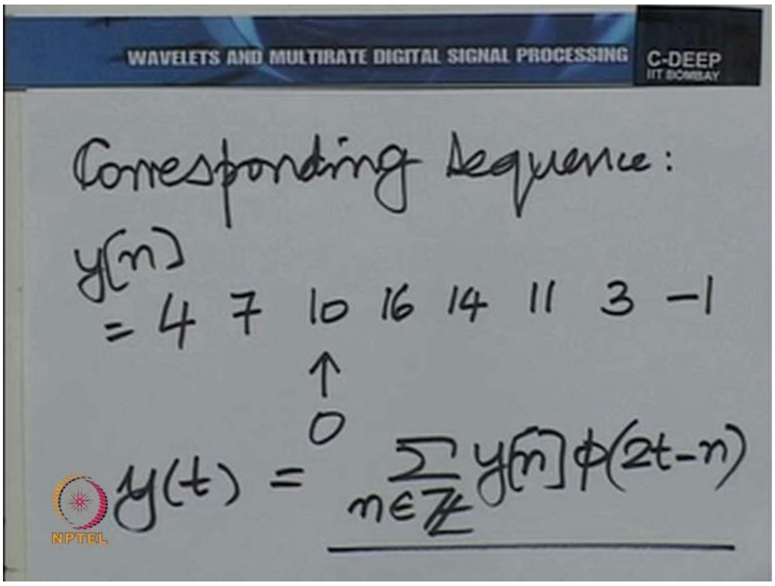

So, in fact we have the sequence here, the corresponding sequence, so I will use the notation with the standard marker of 0. So, at 0 I have 10, and then I have in order 16, 14, 11, 3, -1, 7 and 4, this is my corresponding sequence. So, in fact what we are saying in effect is YT, if I call this sequence Y of N, what we are saying in effect is YT is summation N over all the integers YN. Note, Phi of 2 T - N, that is important.

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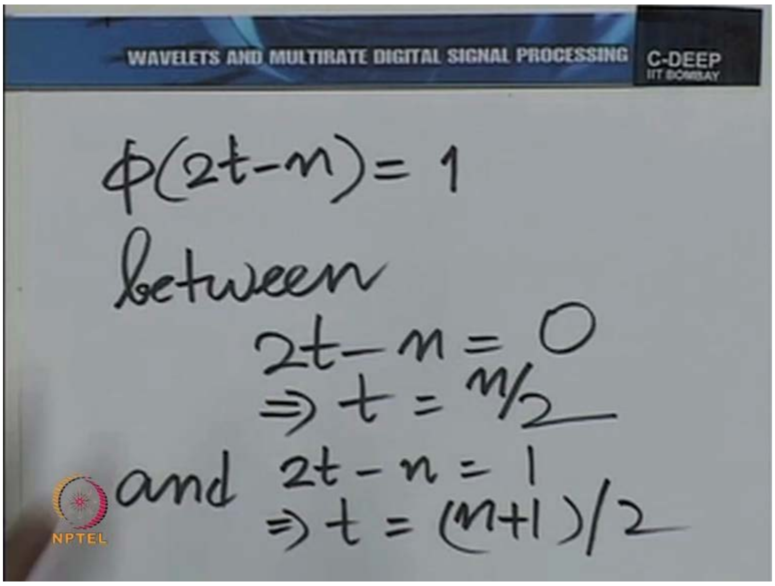

Notice that Phi of 2T - N is by definition going to be equal to 1 between 2T - N equal to 0, that is the equal to N by 2 and 2T - N is equal to 1.

Which means T is N +1 by 2. So, for example, Phi 2T -1 is going to be 1 between 1by 2 and 2 by 2. 0.5 and 1, and that agrees with what we have just wrote down. Good.

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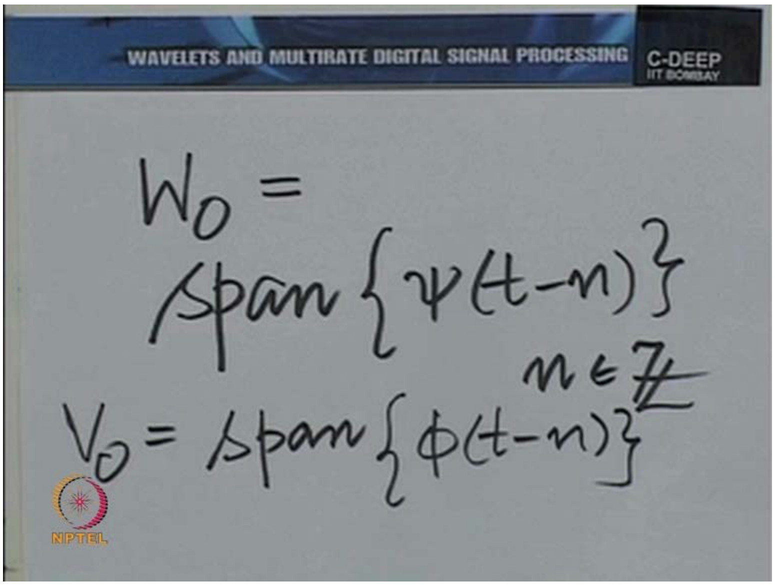

Now suppose that we decompose, we know that we have this decomposition of V1 into W1 and V0 or rather W0 and V0, I am sorry. So, we said that we have this space W0 given by the span of Psi T - N. N over all the integers. And V0 of course is the span of Phi T - N for all integers N. And now we are going to introduce the notion of what is called an orthogonal complement.

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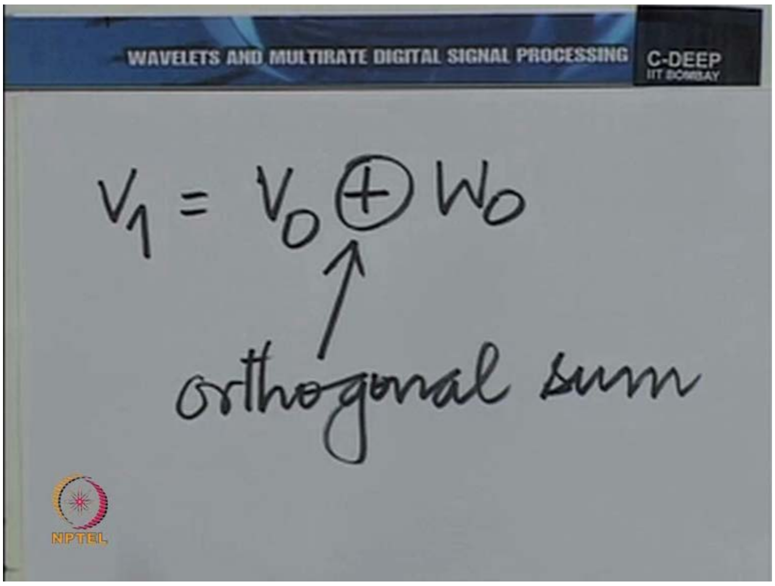

So, we are going to say, well V 1 is the orthogonal sum, this represents orthogonal sum and we are going to explain this in more detail. We say that the space V 1 is the orthogonal sum of the spaces V0 and W0, if there is a unique way to take a vector in V1 and decompose it as a sum of vector V0 and a vector in W0 with the vectors from the 0 and W0 in perpendicular, being orthogonal, in other words, inner product being 0.

So, what is this idea of an orthogonal sum, the idea of an orthogonal sum is to decompose a linear space into subspaces of course of smaller dimensions where we also have mutual perpendicularity or mutual orthogonality between the space, between the vectors in these 2 spaces of the composition. Now, I shall give you a simple example from a three-dimensional situation. Let us take that three-dimensional space of the room that we are in at the moment and let us take the two-dimensional space of the floor.

The three-dimensional space of the room is the orthogonal sum of the two-dimensional space of the floor and a one-dimensional space comprised of all multiples of a vector perpendicular to the floor. So, as expected, a three-dimensional space is an orthogonal sum of a two-dimensional space and a one-dimensional space. The one-dimensional space is formed by all multiples of a vector perpendicular to the floor. The two-dimensional space is formed by all vectors lying on the floor.

So if you take any vector lying on the floor and any vector in that one-dimensional space of multiples of a unit vector perpendicular to the floor, these 2 vectors are orthogonal, perpendicular. In 3 dimensions, it is very easy to visualize. Now you can always generalise to N dimensions, so for example, if you have a 10 dimensional space, it could be an orthogonal sum of 4 dimensional space and another 6 dimensional space where if you take a vector from that 4 dimensional space and a vector from the 6 dimensional space, they are perpendicular to one another.

Perpendicular as understood by taking the inner product between these 2 vectors in that 10 dimensional space. Now, I must at this point make a little remark, the inner product allows us to bring in the notion of an angle between functions. A more general version of orthogonality, so we say 2 vectors are perpendicular if their inner product is 0, in general we can also define the angle between 2 vectors using the inner products and we shall do exactly that in a minute.