Foundations of Wavelets and Multirate Digital Signal Processing
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Lecture - 4
Module - 2
Properties of Norm

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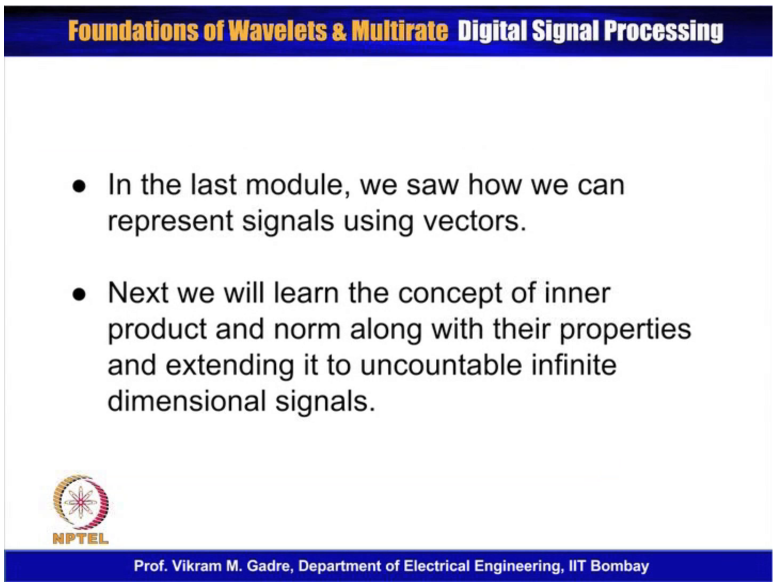

The following things that we demand of this concept of norm or magnitude.

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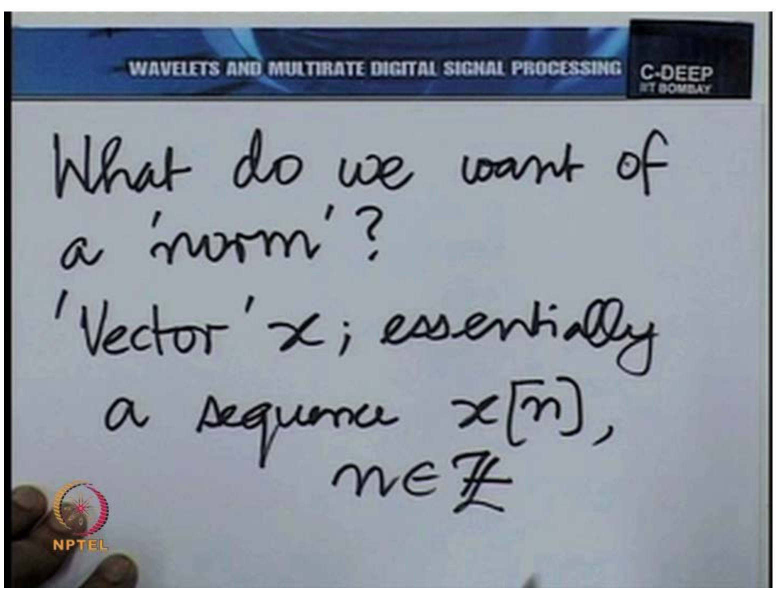

Let us write them down. This is a useful and a powerful idea to have around us. So what to want of a norm? So if I have a vector x essentially a sequence xn. N over the set of integers then it's a norm which we shall denote in a following way.

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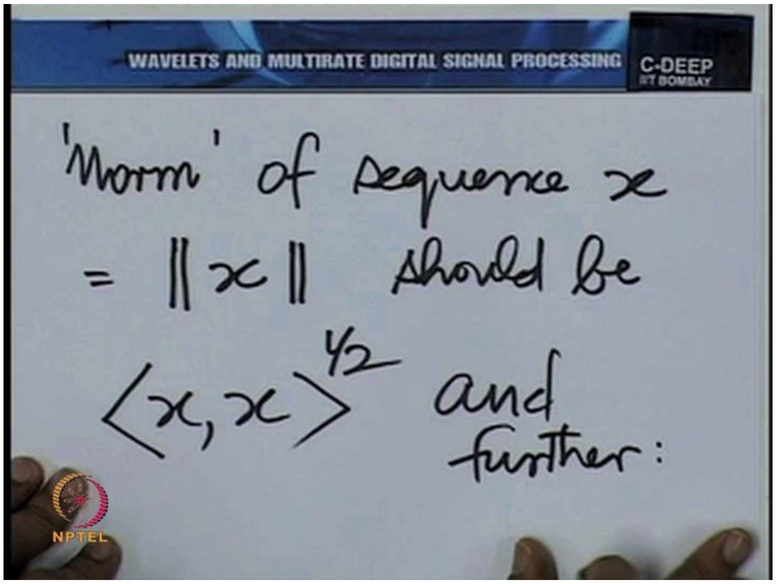

We will denote it like this. Should be essentially the doc product of x with x square root.

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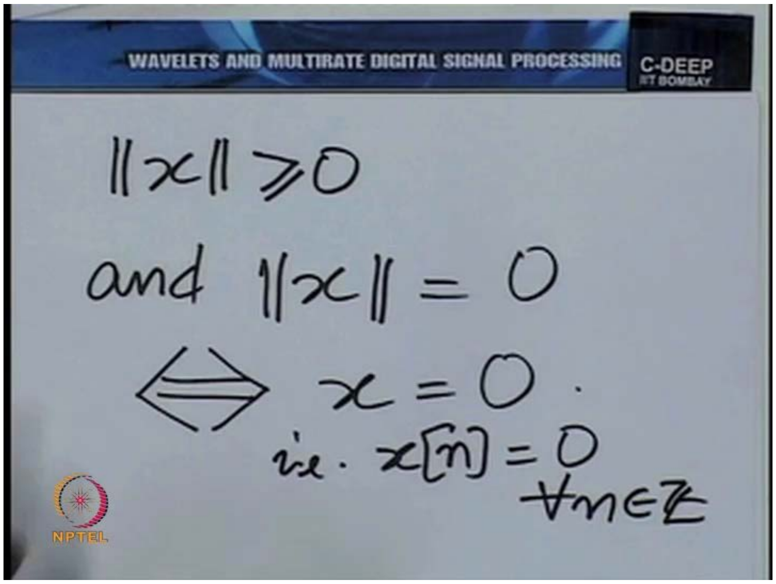

And further we would want norm of x to be non-negative and if at all the norm of x is 0 that implies and implied by the sequence itself being 0 everywhere that is x of n is equal to 0 for all n belonging to set of integers. This is important. So we don't want that norm to be 0 unless the sequence itself is a 0 sequence. A non 0 sequence even if it is non 0 at one point have a non 0 norm and a 0 sequence must have a 0 norm.

Does our dot product satisfy this? Well for real sequence it does.

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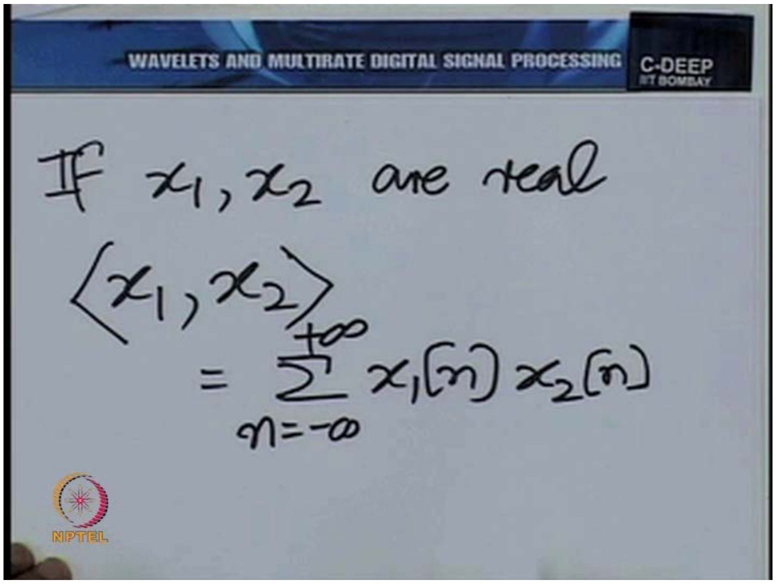

If xn is real rather if x1, x2 are real and we take the following definition the dot product of x1 and x2 is essentially the summation on n going from minus to plus infinity x1n, x2n.

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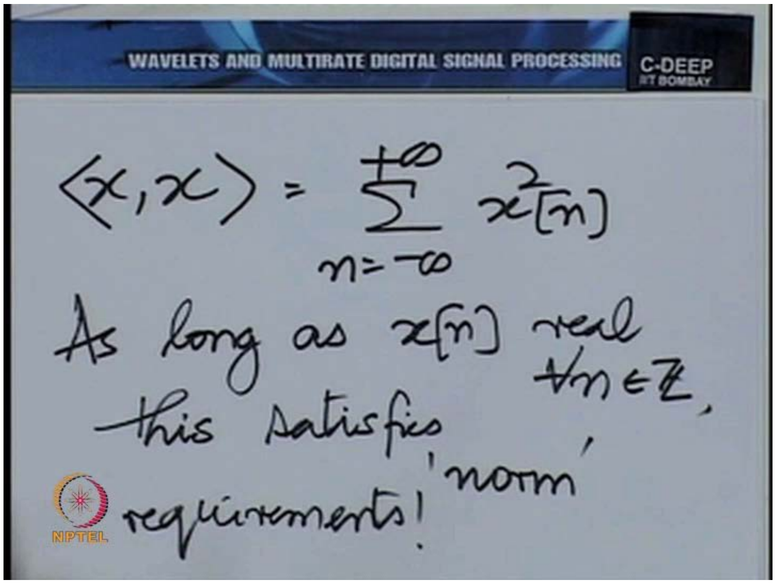

Then the dot product of x with x is essentially the summation of n running from minus to plus infinity. X square n, and as long as xn is real for all n belonging to z. This is non this satisfies the requirements of norm. It is non - negative and it is 0 if and only if the sequence is identically 0. But what if this is complex? So we have to allow complex sequences too. One of the coordinates could be complex and in fact the situation could be such that square could be plus 1 for one of the coordinates and minus 1 for some other coordinate in that case because when you square a complex number nothing guarantees the output is going to be non-negative.

In fact nothing even guarantees that the output is going to be real. Where is the question of non-negative? So this definition is not going to work when x1 and x2 are complex sequences in general. And we need to tweak the definition a little. Well it is not that difficult after all what we want is that for every coordinate you must get a non-negative quantity when you take point by point products. So all that we need to do for that purpose is to complex conjugate the second argument in that summation.

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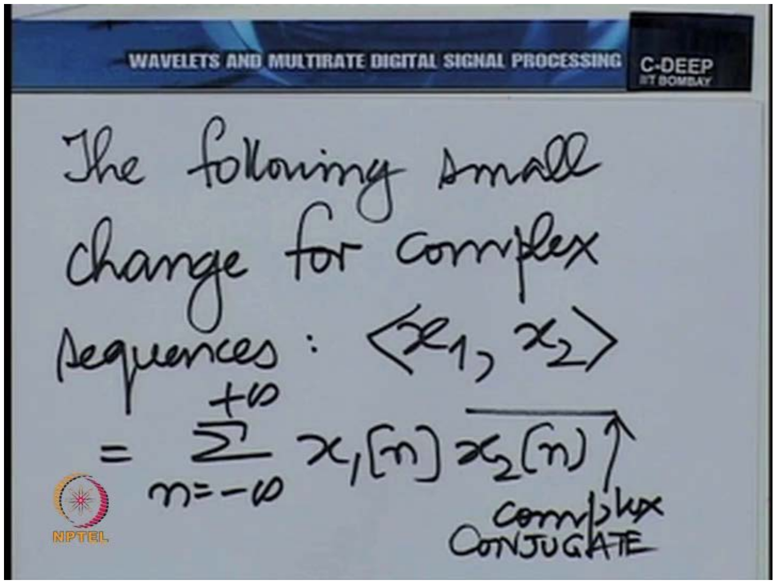

So the small change for complex sequences, we'll do our job. Dot product of x1 with x2 is summation over all n. x1n, x2 bar n where bar denotes the complex conjugate. Now one point to note here when we make this little change is that the commutativity property is lost.

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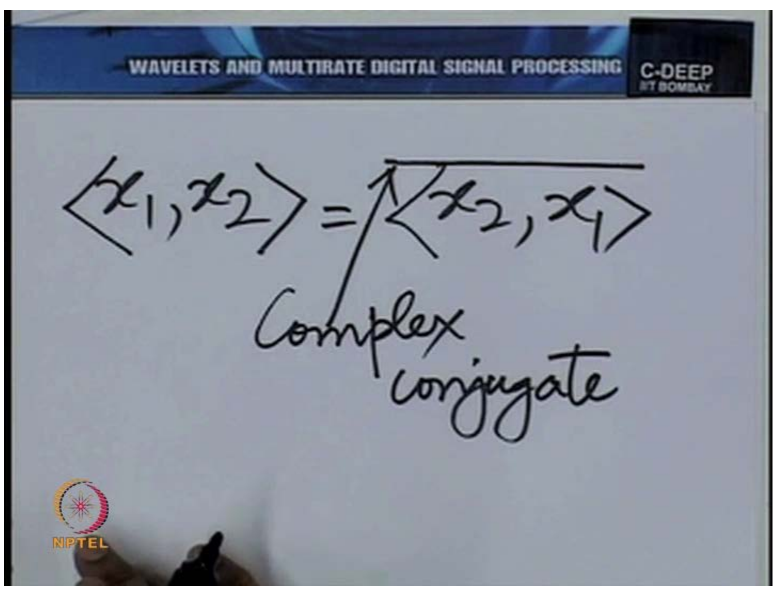

So if I take t inner product x1 with x2 and then I take inner product x2 by x1 there is a complex conjugate relationship. And this is a more general requirement of a dot product. In fact this is the simplest way in which one can define a dot product between sequences. There are many other ways again an infinite number of ways. But at this moment we shall not go into the other ways they will only confuse us. This is what is called the standard inner product. But one can have many other non standard inner products which obey the following conditions.

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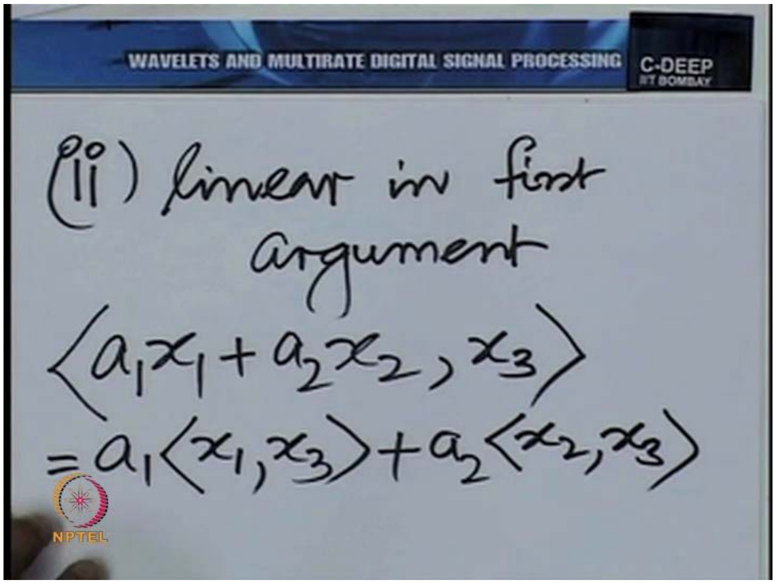

The first condition is this let me write down here. The inner product of the x1 with x2 is the complex conjugate of the inner product of x2 with x1. Secondly the inner product is linear in the first argument. In other words if I take a1x1 plus a2x2 where in general a1 and a2 would be complex and take the inner product x3, it is essentially a1 times inner product of x1 with x3 plus a2 times the inner product of x2 with x3. This is the second requirement of an inner product. Linearity in the first argument.

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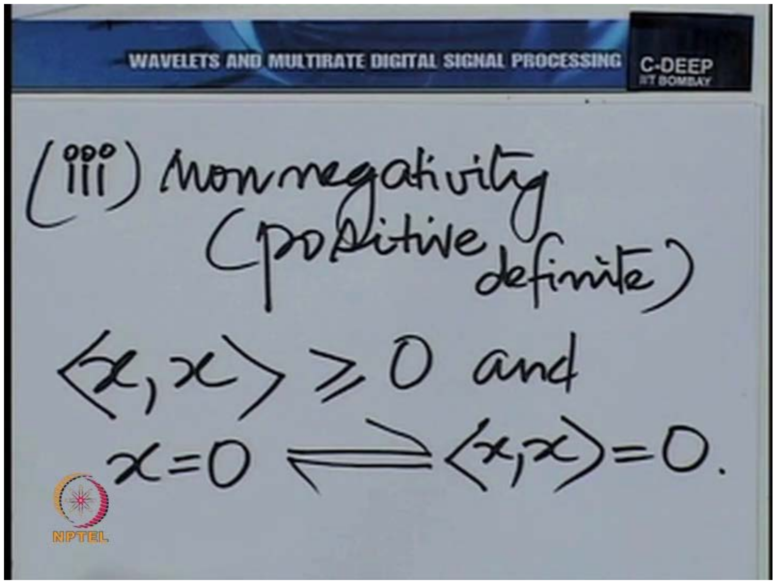

The 3rd requirement of the inner product is what we have been building towards all this while namely what is called the positivity or non-negativity. In fact positivity is more appropriate. Positive definiteness. Namely the inner product of x with x is always greater than equal to 0 and x equal to 0 implies and is implied by the inner product of x with x being 0.

In fact any operation between 2 sequences x1 and x2 which obeys these 3 conditions is called an inner product. And the standard inner product that we have just described is one such which we shall use very frequently. So in the discussions hence forth when we say inner product of sequences we mean the standard inner product unless otherwise specified. Alright so let us just verify this for completeness let us verify this for the standard inner product.

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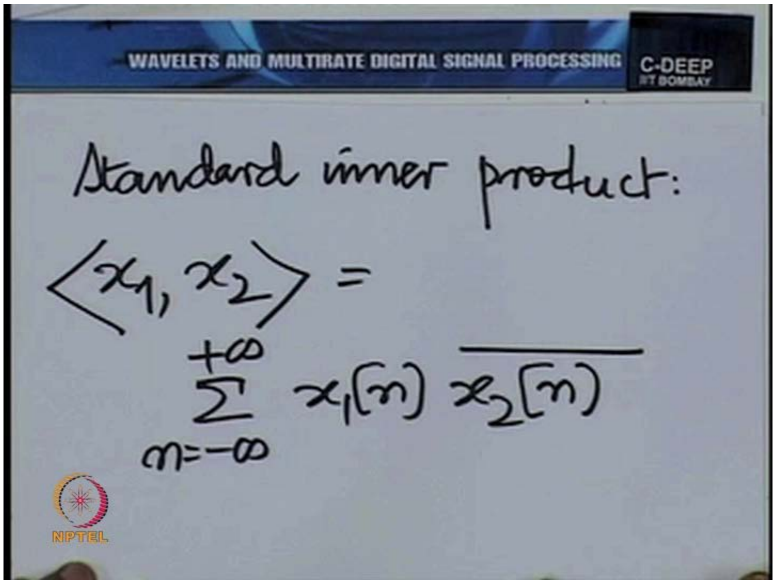

The inner product of 2 sequences x1, x2 is essentially the sum and going from minus to plus infinity x1n, x2 bar n definition.

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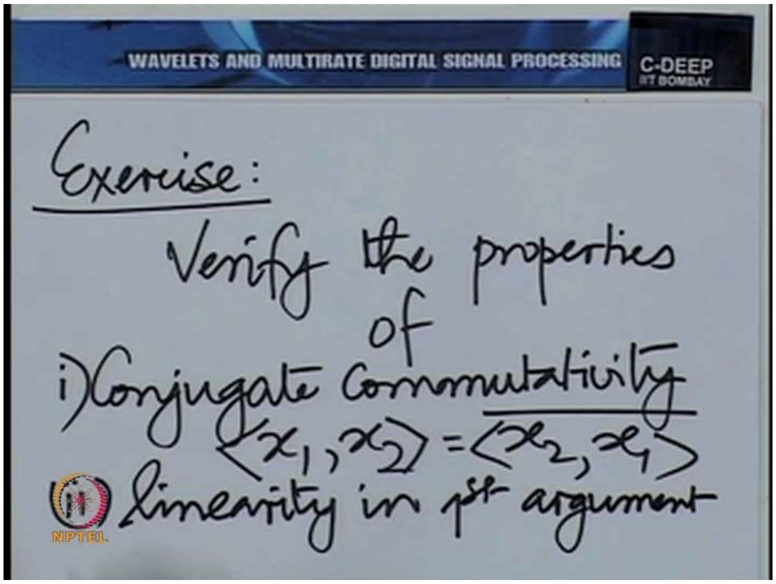

The first property as we said is complex conjugate easy to verify. So in fact I will leave it to you as an exercise. Verify the properties of what is called conjugate commutativity, the first property and linearity, linearity in the first argument. I will leave it as an exercise easy enough to do.

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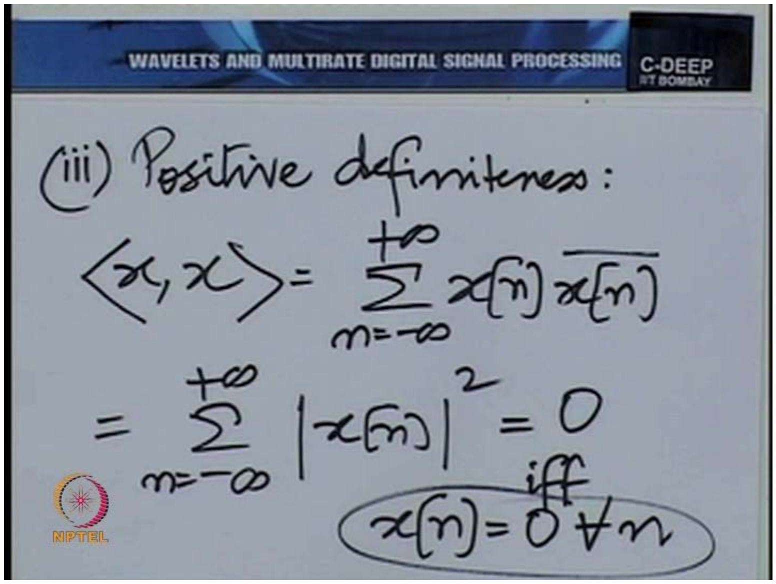

But we shall because it is so important verify the third property the positive definiteness. Indeed if we take the dot product of x with x it is summation n going from minus to plus infinity, xn, xn bar which is summation n going from minus to plus infinity mode xn square. And it is very easy to see that this is equal to 0 if and only if xn equal to 0 for all n. Even if one of the coordinates is non 0 that particular mode xn square is going to be non 0 and it is going to contribute a positive term.

And of course it is very easy to see that each term for every n I mean is strictly positive if xn is non 0. So far so good. So now we have built up the idea of inner product or dot product between two sequences which is going to be useful to us. So we have moved from 2 dimensional to 3 dimensional to n dimensional, n is infinite and to countably finite dimension. Now let us move to uncountably infinite dimension.

So suppose I take a function of the continuous variabile t how can I extend these notions.

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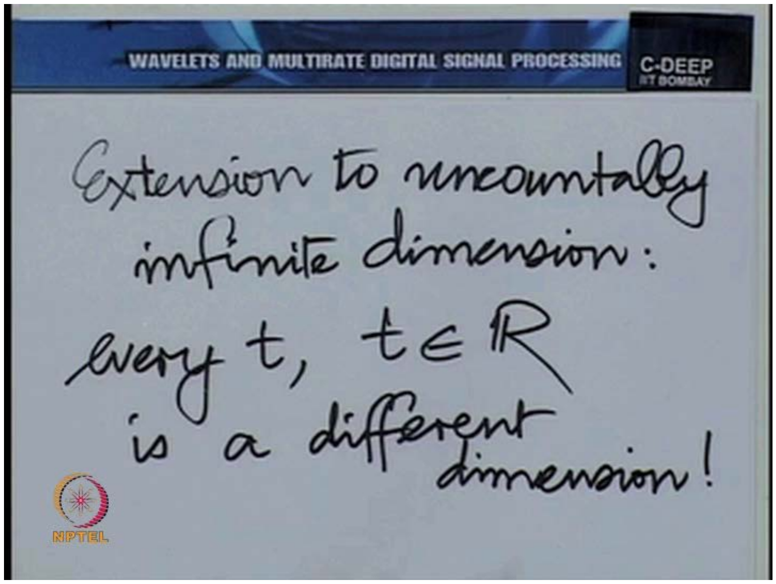

So extension to uncountably infinite dimension. Well this is going to be very difficult in general but very easy in particular if we simply accept that every t for real t is a different dimension.

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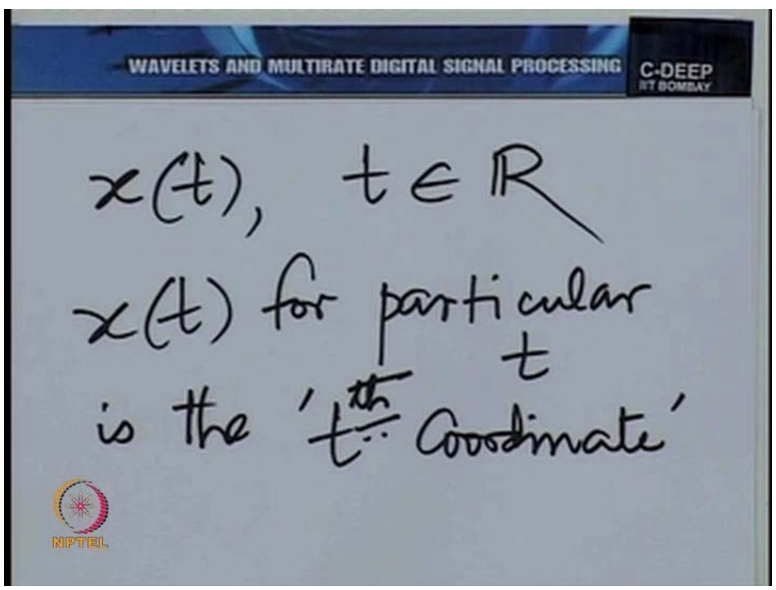

Simple so if you have a function x of t, t over the real numbers. X of t for a particular t is the tth coordinate so to speak and there is uncountably infinite number of such coordinates indexed by the real numbers.

So in principle in a given function you have complete liberty to put down the value of xt at every different point t. The only catch is we have agreed that we would like to make the functions square integrable. So that does put some restriction on xt but not a very serious one even so. Now you know dealing with infinite dimensional spaces if we wish to do it very rigorously and very very carefully and to satisfy the fastidious mathematician is a difficult job.

And we don’t really intend to do that all the way in this course. If some of us do wish to take that puritanical perspective then of course would benefit from it in some ways and one could look up a book on functional analysis but what we wish to do is rather to give intuitive understanding of some of the concepts at different places. The intuitive understanding will not be different from a more rigorous understanding for those specific situations.

But it may not quite be complete. Even so we would not suffer too much in our study of wavelets, in our applications of wavelengths if we take this intuitive path to some extend not all the time. I mean to some extend in the context of dealing with infinite dimensional spaces.