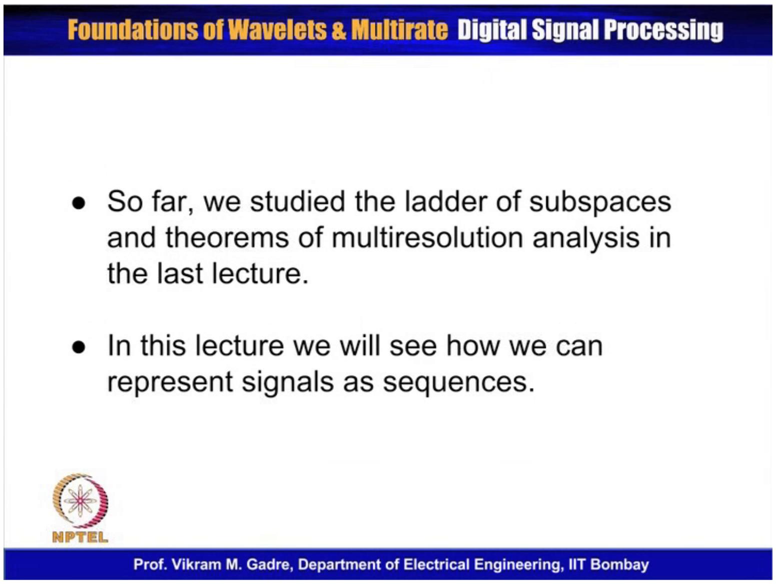
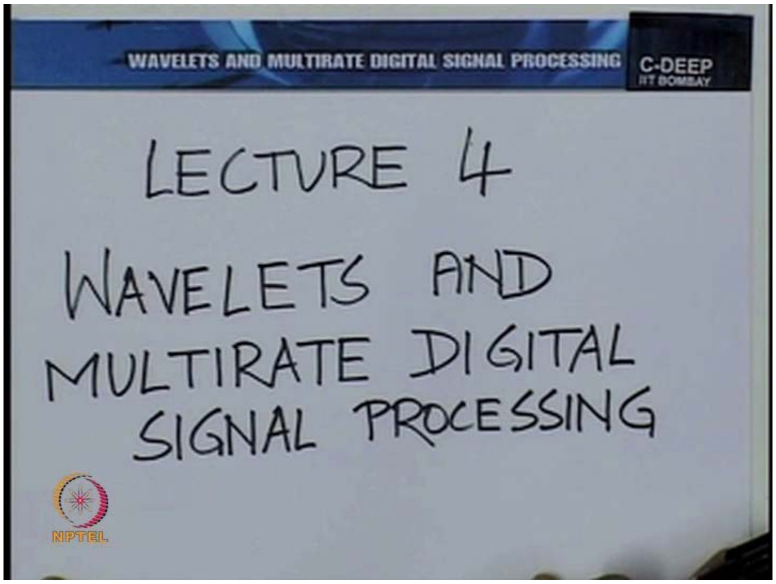
Foundations of Wavelets and Multirate Digital Signal Processing
Prof. Vikram M. Gadre
Department of Electrical Engineering
Indian Institute of Technology, Bombay
Lecture - 4
Module - 1
Vector Representation of Sequences

(Refer Slide Time: 00:17)



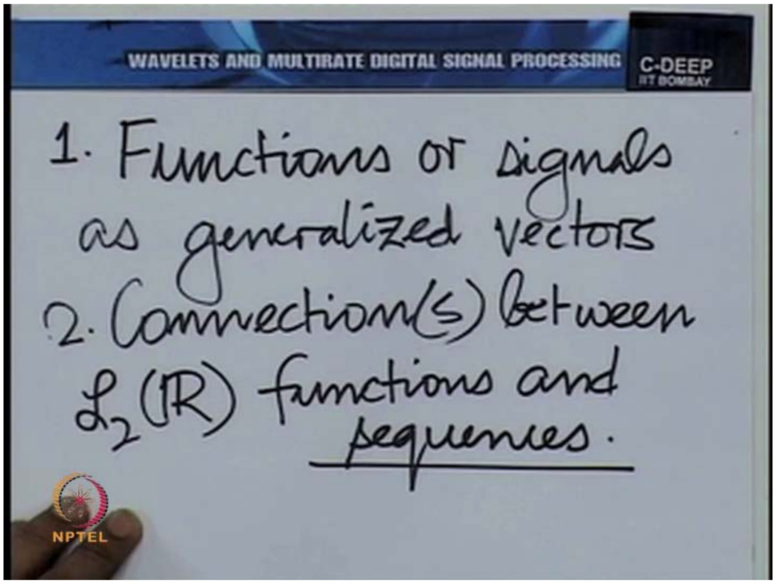
A warm welcome to the 4th lecture on the subject of wavelets and multirate digital signal processing in which we intend to build further the connection between signals or functions in L2r and vectors and therefore we wish to build further the idea of thinking of functions as belonging to linear spaces and characterizing them in a manner slightly different from what we were doing in the previous lecture.

(Refer Slide Time: 01:14)



So just to put our discussion in perspective this is the fourth lecture on the subject of wavelets and multi rate digital signal processing and what we intend to discuss in this lecture is the following let me put down the point one by one.

(Refer Slide Time: 01:40)

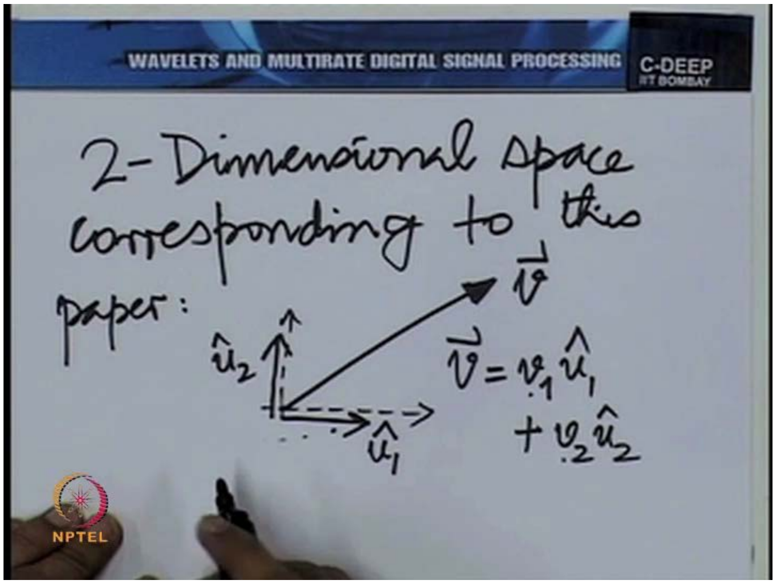


The first thing that I wish to talk about today is to think of functions as generalized vectors. This idea is going to be useful to us in many different contexts in this course. So we need to understand this connection between functions or signals and vectors in depth we shall spend some time on it today. Secondly the connection between L2r function, connection or connections between L2R functions and sequences. We wish to understand this in greater depth. So what we are going to show in the later part of this lecture is that one can intimately relate processing of a function to processing of an equivalent sequence.

And whatever we are doing to try and gain information from or modify a function can be done by equivalently processing or modifying that sequence corresponding to the function. Let us then embark on the first of these 2 objectives now. You see let's begin by asking what characterizes a vector after all let's take a minute and reflect. What characterizes a 2 dimensional vector for example?

A 2 dimensional vector is essentially characterized by 2 coordinates which are independent. We call them perpendicular coordinates; actually the idea of perpendicularity there is also intimately related to the idea of independence.

(Refer Slide Time: 04:20)



So for example let me treat the plain of the paper as a 2 dimensional space, 2 dimensional spaces corresponding to this paper. Well let's take any vector on this 2 dimensional space. Let this vector be V, I am marking this vector as v.

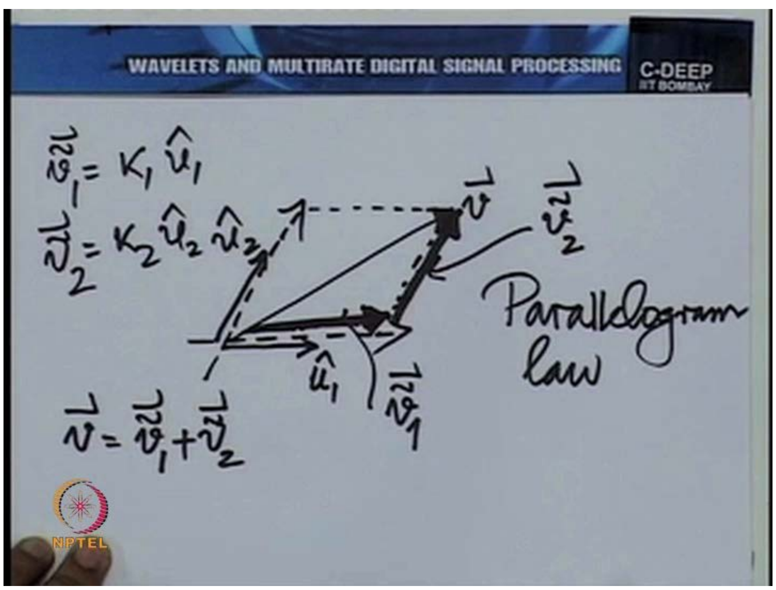
There are many different ways to characterize this vector. In fact notionally an infinite number of ways and one of those ways is to choose the following two so called perpendicular axes. So we choose one axis like this and another axis like this and choose a unit vector along each of them.

So I have say a unit vector let me call it u1 cap along this axis and other unit vector u2 cap along this axis.

And then I could write v1 or I could write sorry just this vector v uniquely as say v1 times u1 cap plus v2 times u2 cap. Whereby v1 and v2 characterizes vector v uniquely in this 2 dimensional space with respect to the coordinate system generated by u1 and u2. And there is an infinity or such coordinate systems. In fact 1 infinity of such coordinate systems can be generated simply by rotating this coordinate system of u1 and u2. It is very easy to see that if I take this structure u1 and u2 and rotate it by any angle in this 2 dimensional plane it would give me a new coordinate system.

So there is an infinity of orthogonal coordinate systems in 2 dimensional space and in fact there is also relation between all these infinite orthogonal coordinate systems. Simple enough and orthogonal coordinate systems are not the only kinds of coordinate systems for a 2 dimensional vector. So for example the same 2 dimensional space can be described by the following different coordinate systems.

(Refer Slide Time: 07:05)

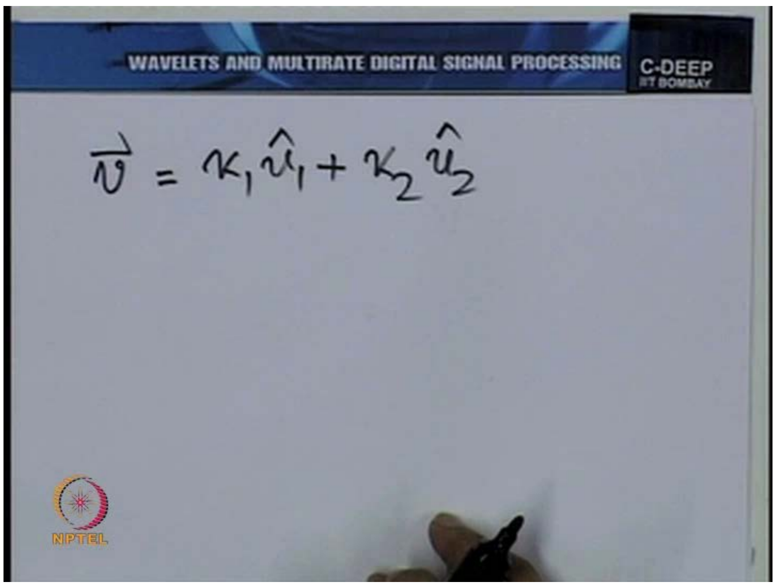


I will draw the same vector v and it is perfectly alright to choose a coordinate system something like this. I could choose 1 coordinate like this and another coordinate like this. And of course I could again have the unit vectors in these 2 directions. U1 cap so to speak, u2 cap and I could express v in terms of u1 cap and u2 cap. Indeed I could complete a parallelogram here. So using the parallelogram law I could draw a line parallel from the tip of this vector to this u2.

And another one parallel to u1 from the tip of the vector and it is very easy to see that this dot dash vector here plus this dot dash vector here gives me v. So let me highlight that dot dash vector this vector here plus this vector here gives me v. Let me call this v1 tilde and it's a vector and let me call this v2 tilde that’s again a vector. Of course we have v is v1 tilde plus v2 tilde and it is very easy to see that v1 tilde as a vector is some multiple of u1.

And similarly v2 tilde as a vector is some multiple of u2.

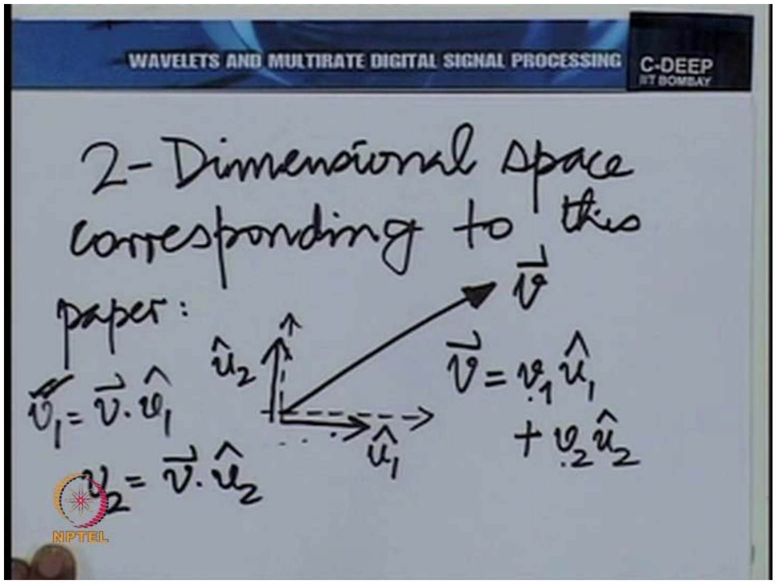
(Refer Slide Time: 09:35)



There upon I have v is some multiple of u1 plus some other multiple of u2. K1u1 plus k2u2.

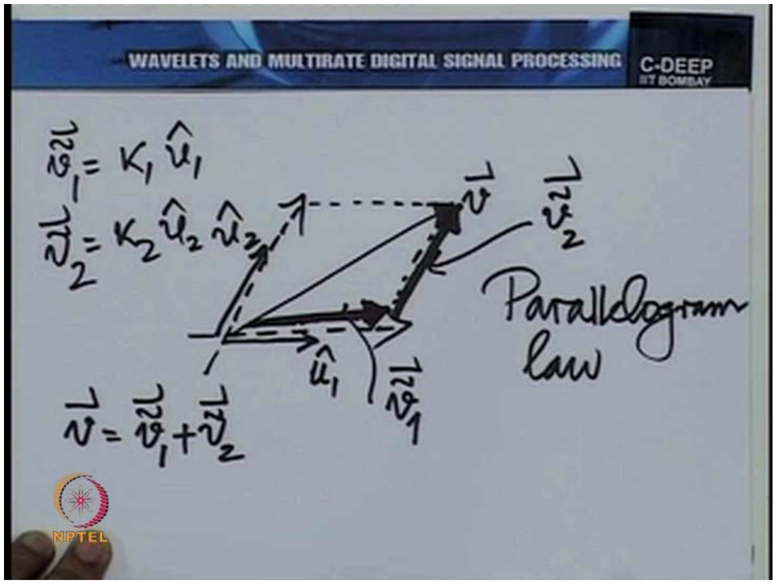
The only catch is determining k1 and k2 is a little more difficult then determining the constants in the previous representation.

(Refer Slide Time: 10:01)



In fact let me go back to that previous representation. I had this representation previously where v is v1u1 cap plus u2v2 cap and remember v1 and v2 of course here are constants. And very easy to obtain, because I can simply obtain them by taking a dot product of v with u on cap and v with u2 cap. So in fact in the sense of dot products v1 is indeed v dot u1 cap and v2 v1 is a coordinate, not as a vector, v2 is a coordinate is the dot product of v with u2 cap simple enough. Such a simple relationship does not exist in this context. While we are not hard put to describe the process by which we obtain k1 and k2.

(Refer Slide Time: 11:00)



It simply says construct a parallelogram. Expressing this analytically is a bit of work. So it is definitely very clear from this example that an orthogonal or a perpendicular coordinate system has its advantages. It's always nice to have a perpendicular coordinate system in 2 dimensional space to represent any 2 dimensional vector. The same idea can of course be extended to 3 dimension to and then one could also conceive of more than 3 dimensions, 4 dimensions, n dimensions.

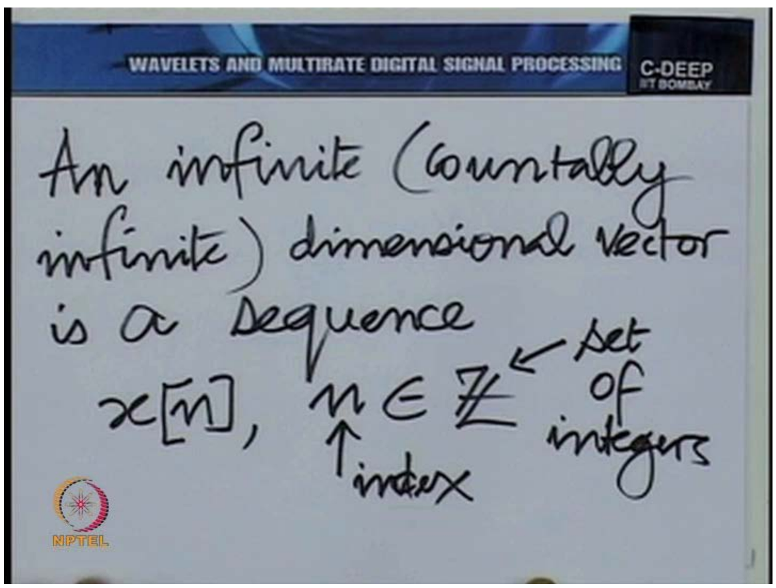
And then in principle an infinite number of dimensions too. Now there again when we talk about infinite dimensional situations we have countably infinite and uncountably infinite finer points. But for the moment infinite is difficult enough. So infinite dimensional vectors in fact lead us to the idea of functions. Now it is a little difficult to understand infinite dimensional vectors all at once.

So to progress towards infinite dimensional vectors it is easier first to start from finite dimensional vectors of a larger and larger dimension. And all that we need to do is to understand that what characterizes the dimension of a vector is really the number of independent coordinates that is has. For example a 3 dimensional vector has 3 independent coordinates a four dimensional vector would have 4 and an n dimensional vector n.

And a countably infinite dimensional vector would have a countably infinite number of dimensions or a countably infinite number of coordinates. By countable we mean we can put the coordinates or dimensions into one to one correspondence with the set of integers. So we can talk about the zeroth coordinate, we can talk about the oneth coordinate we can talk about the minus 1 coordinate, minus 2th coordinate and so on so forth.

What are we talking about here there if we talk about an infinite dimensional vector we are in fact talking about sequences. So we will develop the idea from there.

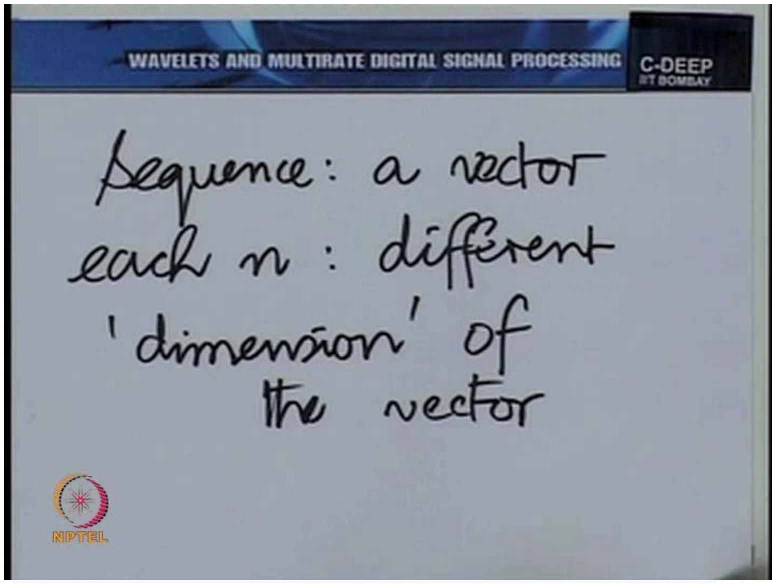
(Refer Slide Time: 13:40)



So here we are let's make a note of this. An infinite dimensional vector or rather an infinite countable infinite dimension. Vector is essentially a sequence. So for example we have a sequence x of n where n belongs to set of integers all the integers.

We call that this script z is a representation of the set of integers and this is called the index variable. So now we have a different interpretation of sequence. Sequence is like a vector and each n is a different dimension of that vector ok, I think that is important enough for us to write it down explicitly.

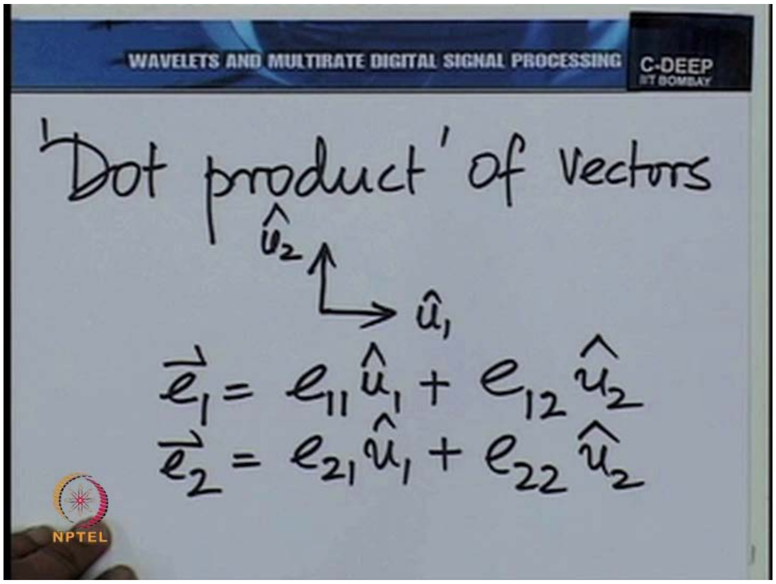
(Refer Slide Time: 15:00)



So a sequence is a vector. Each n is a different dimension of the vector. And once we have this analogy then extending other ideas of vectors to this context is not difficult at all.

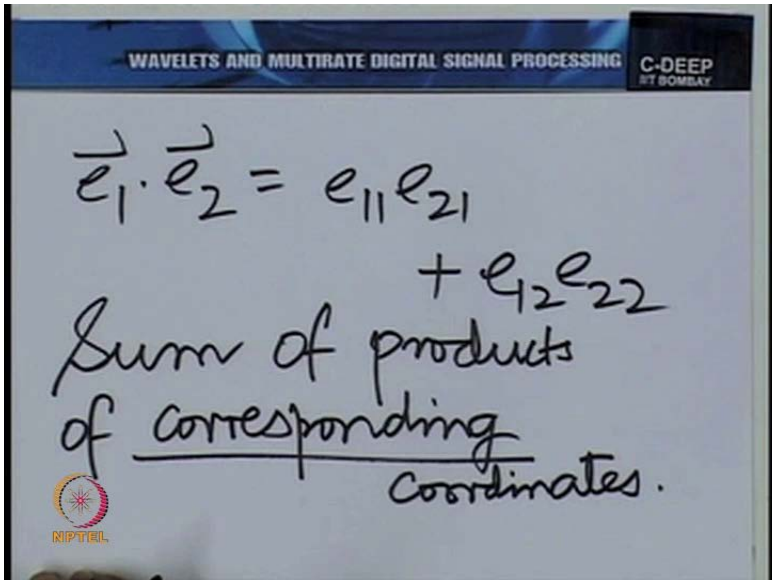
For example 2 vectors simple add the sequences point by point, multiplying a vector by a constant very simple. Multiplying each point of that sequence by that constant. What we would like to do now is to extend some of the other ideas of vectors that we have some of the geometrical ideas to this context of infinite dimensional vectors. And one of the very useful ideas that we have in the context of vectors is the idea of a dot product.

(Refer Slide Time: 16:25)



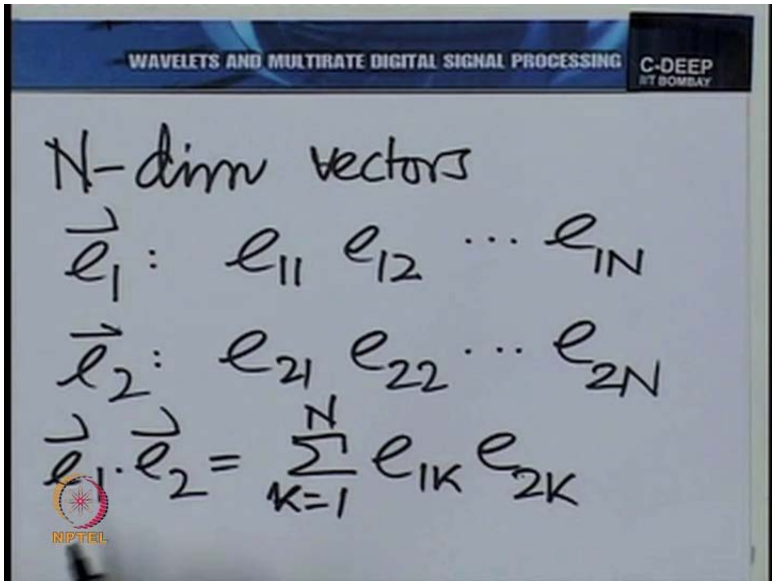
How do we take dot the product of 2 vectors in 2 dimensional space so let us recall. So suppose for example we choose a pair of orthogonal coordinates. So we have u1 cap and u2 cap as we did some time ago. Orthogonal to one another, perpendicular to one another. And we have 2 vectors let us call them e1 which has the coordinates e11 and e12, so e1 is e1,1u1cap plus e12u2 cap and similarly e2 has a vector has the coordinates e21 u1 cap plus e22u2 cap.

(Refer Slide Time: 17:20)



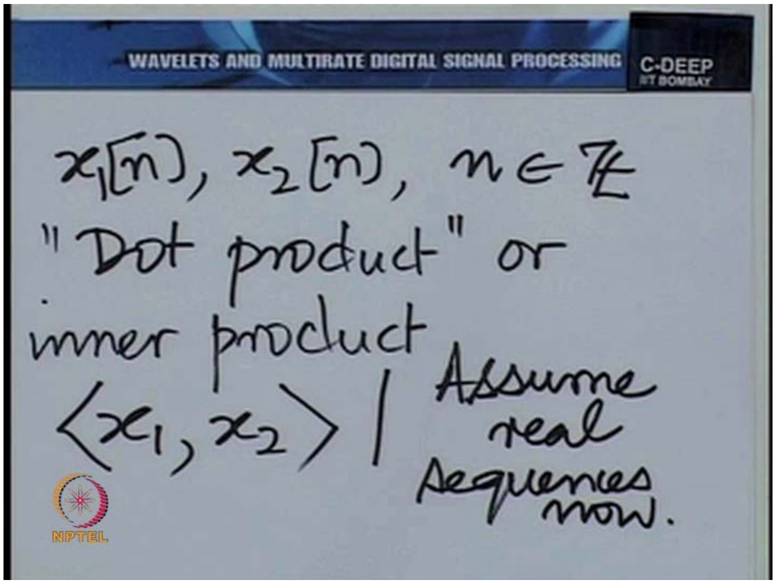
Then the dot product of e1 and e2, e1 dot e2 as we write is essentially e11e21 plus e12e22 so it is the sum of products of corresponding coordinates. 2 dimensions easy enough to understand 3 dimensions easy to extent in fact n dimensions equally easy to extent.

(Refer Slide Time: 18:10)



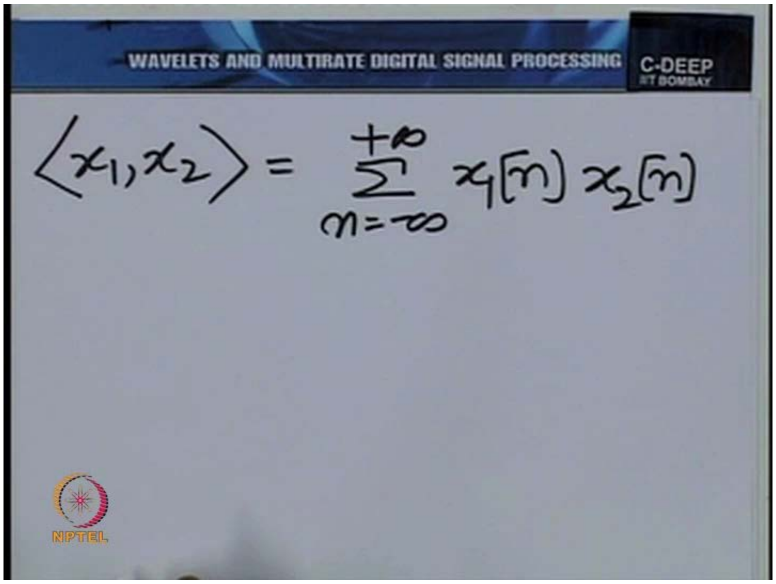
Suppose we had 2 n dimensional vectors characterized by coordinates say e11 to e1n. so you have 2 n dimensional vectors. E1 characterized by coordinates e11, e12 up to e1n and similarly e2 characterized by the coordinates e21, e22 up to e2n. Then of course e1.e2 is easy to express if we generalize this. Essentially summation k from 1 to n, e1k times e2k so dot product generalized to n dimensions. Of course we assume these are orthogonal coordinates. Now we can even take this to infinite dimensions, so we can think of the dot product of the 2 sequences.

(Refer Slide Time: 19:25)



Let's say x1 and x2. So we have here for example 2 sequences x1n and x2n defined over the set of integers and over all the integers. They are so called dot product or inner product has the formal name is, so you see instead of dot product now you would like to use the term inner product to generalize. And we denote the inner product this way. So the moment let's assume these are real sequences, for the moment.

(Refer Slide Time: 20:20)

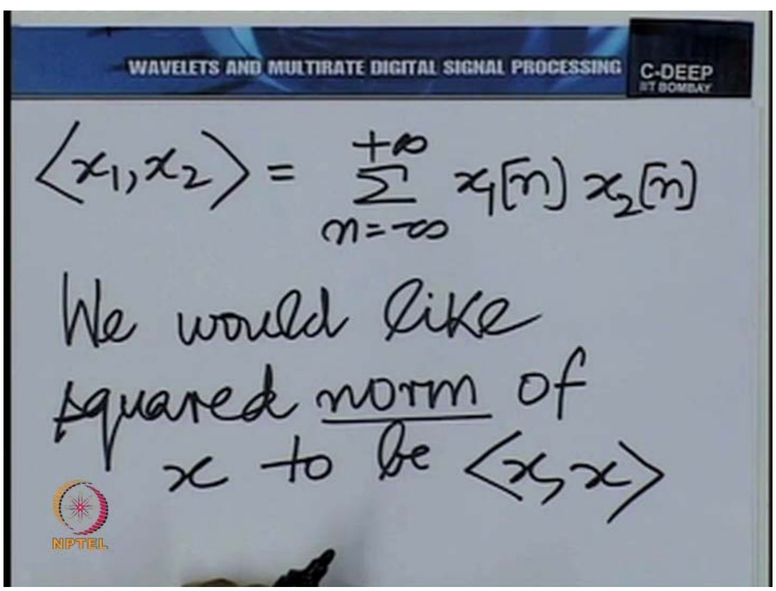


In that case if we generalize it is easy to see that the inner product, inner product of x1 and x2 is simply summation on n. n running from all the way minus to plus infinity x1n times x2n and of course it is clear that the dot product of the inner product as we are going to call it in this generalized situation is commutative. That means if I interchange the roles of x1 and x2 the result does not change.

However we would like this inner product or dot product notion to give us some of the powers and some of the conveniences that the dot product offers in the context of vectors. One so called convenience or one so called interpretation or meaning that we derive from the dot product is the notion of magnitude. So in fact one could think of the notion of magnitude as induced from a dot product if one desires.

Or in other words one could calculate the magnitude of a vector by using the notion of a dot product as one path towards the calculation of magnitude. Incidentally the word magnitude of vectors is used for small dimensional vectors like 2 and 3 dimensional, but when we go to these generalized situations of n dimensional vectors or countably infinite dimensional vectors we replace the word magnitude by the word norm.

(Refer Slide Time: 21:55)



So we say that we would like the square norm of x to be the dot product of x with x as is the case with vectors. So if you recall a dot a where a is a vector for 2 or 3 dimensions for that matter is the magnitude square of a. The same should hold good here. When we take a dot product of a sequence with itself it should give us the square norm of that sequence where norm is the more general word of the magnitude.

In fact in L2R the norm is representative of the energy but at this moment we are not talking about this L2R because we have not yet come to that situation where we are dealing with functions of continuous variables. So we'll postpone that interpretation for a minute not very far away from now and once again come back to sequences. Even for sequences when we take a dot product of real with itself we indeed get something that will likel to an energy of the sequence.

So it is not uncommon to refer to the dot product of the real sequence or for that matter sequence with itself as the energy in that sequence. Anyway I kept emphasizing real for a good reason. When we talk about the magnitude of a vector or for that matter there is more generalized word norm. What is it that we expect of a magnitude? We want the magnitude or the norm to be a non-negative number and in fact strictly positive if that vector is non 0.