Transmission Lines and E. M. Waves
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Lecture – 58
Problems on Uniform Plane Wave in a Medium

In this session let us solve some problems related to uniform plane waves traveling in arbitrary direction in a medium and also the problems related to the polarization of an electromagnetic wave. We will also solve a problem related to the dielectric boundaries and so on. So let us consider a problem here that the complex vector electric and magnetic fields in a dielectric medium are given by this.

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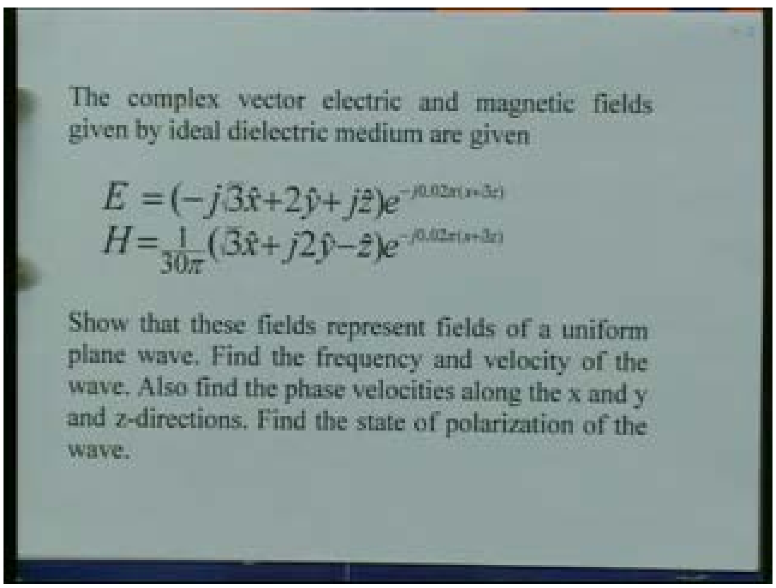

So e is equal to minus j root 3 unit vector x plus 2 unit vector y plus j unit vector z with a phase function which is e to the power minus j 0.02pi x plus root 3z and the magnetic field is given as 1 upon 30pi root 3 x plus j 2 y minus z cap with a phase function which is same as this which is e to the power minus j 0.02pi x plus root 3z. And what is asked is show that these fields represent fields of a uniform plane wave. Also you have to find the frequency and velocity of the wave. Also find the phase velocities along the x and y direction and z direction. Also find the state of polarization of this wave.

Then first we have to show that these fields represent the uniform plane wave. So we know from the property of uniform plane wave that the direction of wave propagation the electric field and the magnetic field they are perpendicular to each other and also at every instant of time the ratio of the magnitude of electric field and the magnitude of the magnetic field is equal to the intrinsic impedance of the medium. So essentially these are the properties we are going to use to establish the uniform plane wave nature for these fields. So essentially we have to show that E is perpendicular to H and is also perpendicular to the direction of the wave which we denote by the unit vector n.

Now we know that if we write down the fields in the standard form that any electric field is having some amplitude E 0 with the phase function which is e to the power minus j beta into n dot r and then we can compare this phase function with the phase function which is given for these fields, so this phase function (Refer Slide Time: 4:33). So essentially we can get from here that this quantity n dot r is n dot r is equal to as we know is cosine of phi x into x plus cosine of phi y into y plus cosine of phi z into z and in this case it is given this quantity here which is x plus root 3z. So first of all we can get this quantity for this and compare with this so that essentially it gives you this quantity as x plus root 3z and as we know this quantity should have... the n should have the magnitude which is unity.

So, if we can write down this quantity appropriately we can take this as 2 times x 1 upon 2 into x plus root 3 upon 2 into z so that is the quantity essentially which represents the n dot r so because this quantity here corresponds to cos of phi x so what that means is from here if I compare these two we get phi x is equal to 60 degrees phi x equal to 1/2 so that is equal to pi by 3 we get phi y, there is no terms for y here that means this cos of phi y should be zero so phi y is equal to pi by 2 and then phi z which is the angle which the wave vector makes with the z axis that is equal to cos of phi z is root 3 by 2 so that is equal to 30 degrees which is equal to pi by 6.

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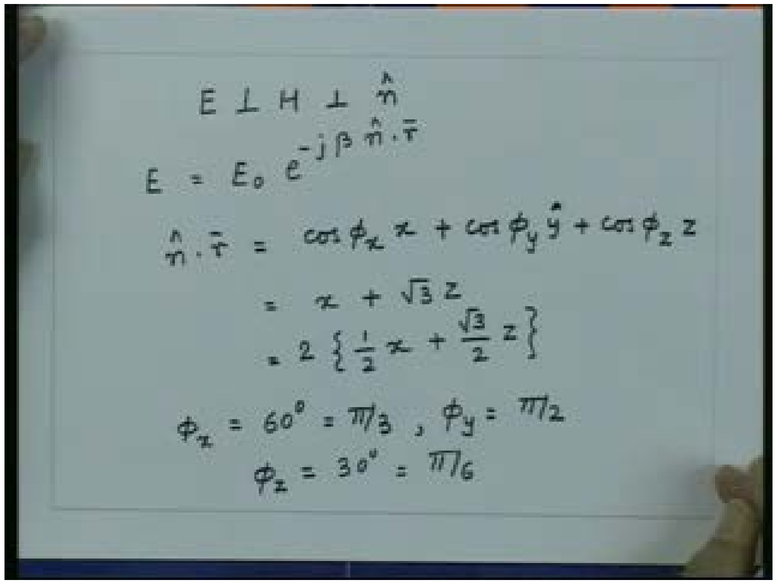

So one one thing we can note from here that since the phase function has only x and z terms in this essentially the wave is propagating in the xz plane and that is what essentially we see here that the wave normal makes an angle 60 degrees with the x axis, 30 degrees with the z axis and 90 degrees with the y axis that means the vector lies in the xz plane.

So essentially the wave is traveling... we write down the coordinate axis: this is x, this is y, this is z (Refer Slide Time: 7:39), the wave essentially is traveling in this xz plane so it is traveling since in that, this is your unit vector n where this angle is 60 degrees, this angle is 30 degrees and this angle with y that is 90 degrees. So from here essentially we get the unit vector n as we got that is 1 upon 2 x cap plus root 3 upon 2 z cap.

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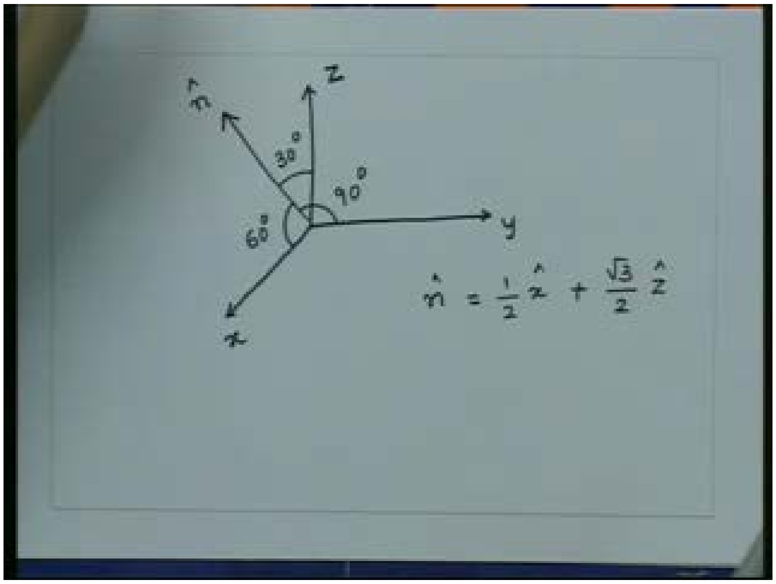

Now you have to show that the electric and magnetic fields are perpendicular to each other and they are also perpendicular to the direction of the wave vector which is n cap. So for this we can take the dot product of E H and e and the unit vector so we get E dot H that is equal to... if I take from here (Refer Slide Time: 9:00) we will get the dot product of these so this will become minus j into 3 so that that gives me minus j 3 plus 4 into j plus 4 into j minus j minus j so that is equal to 0. So that means the electric field of this is perpendicular to the magnetic field.

Same thing we can do for the unit vector and the electric field. we take electric field and the dot product of that to the unit vector that is equal to minus j root 3 x plus 2 y plus j z dot unit vector n which is 1 upon 2 x cap plus root 3 upon 2 z cap so that is equal to minus j root 3 upon 2 minus j root 3 upon 2 plus j root 3 upon 2 that is equal to 0. So now we have got E is perpendicular to H, E is also perpendicular to n that means E H and n are perpendicular to each other. That is one of the properties of uniform plane waves.

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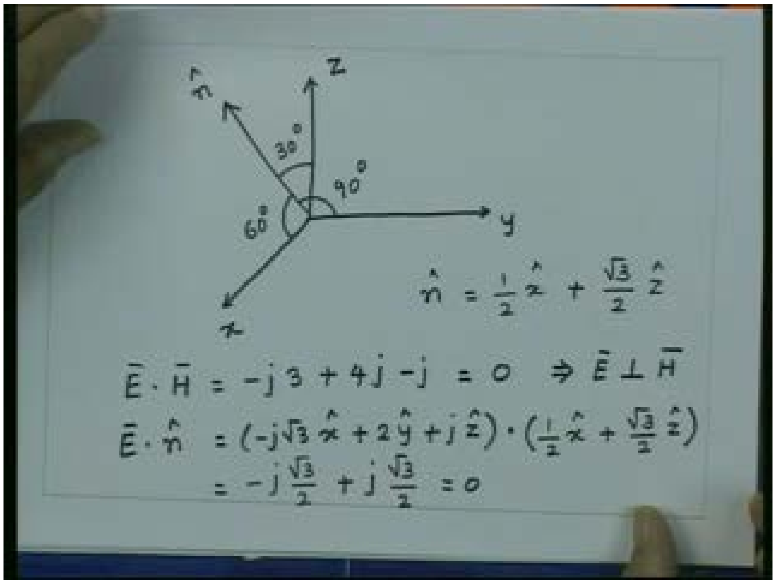

So it satisfies this property and therefore we say that the fields represent the uniform plane wave.

Now, to find out the frequency at the velocity of the wave essentially we have to find out the propagation constant beta and for that again we we take this term which is the phase term, this quantity this quantity here (Refer Slide Time: 11:29) and we do some rearrangement so that we can get this quantity propagation constant beta.

So, the phase function which is given there which is e to the power minus j beta into n dot r if I take this quantity now from here we will get beta into n cap that will be equal to 0.02pi this phase function here (Refer Slide Time: 12:00) into x cap plus root 3 into z cap. Writing this as unit unit vector this will be 0.02pi into 2 and this will be 1 upon 2 x cap plus root 3 upon 2 z cap so you multiply by 2 and divide it by 2 so this becomes the unit vector. So this quantity then essentially represent the the phase constant beta. So from here we get this phase constant beta which is equal to 0.04pi radians per meter.

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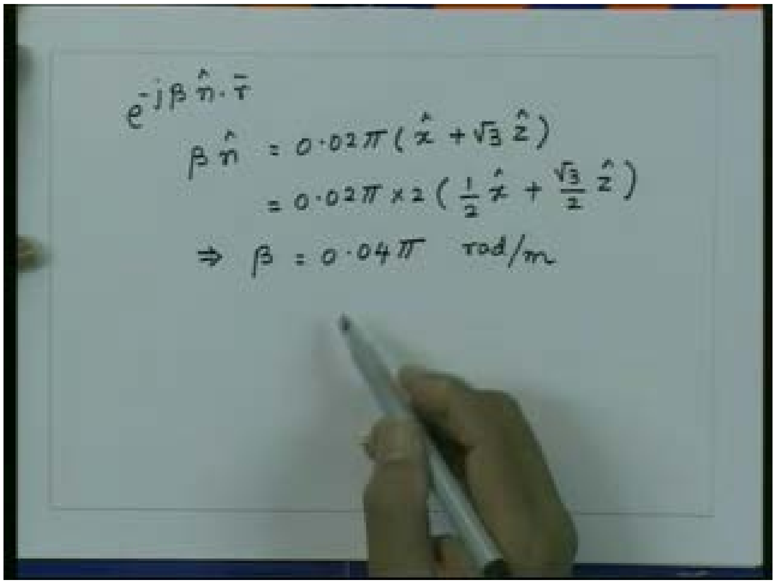

Now, to find out the dielectric constant... for finding out the velocity essentially we have to find out the medium properties, still we do not know because the frequency is not given; in fact we have to find the frequency of this wave so first we have to find out what is the medium property or what is the velocity of the wave and then we can find out essentially from the property of uniform plane wave that is the ratio of mod E to mod H that should be equal to the intrinsic impedance of the medium and since we are talking about a dielectric medium the permeability of a medium is same as the free space permeability, only the permittivity of the medium is different.

So we know that mod E upon mod H that should be equal to the intrinsic impedance of the medium which is square root of mu upon epsilon and since the medium is only dielectric it is equal to mu 0 upon epsilon 0 into epsilon r.

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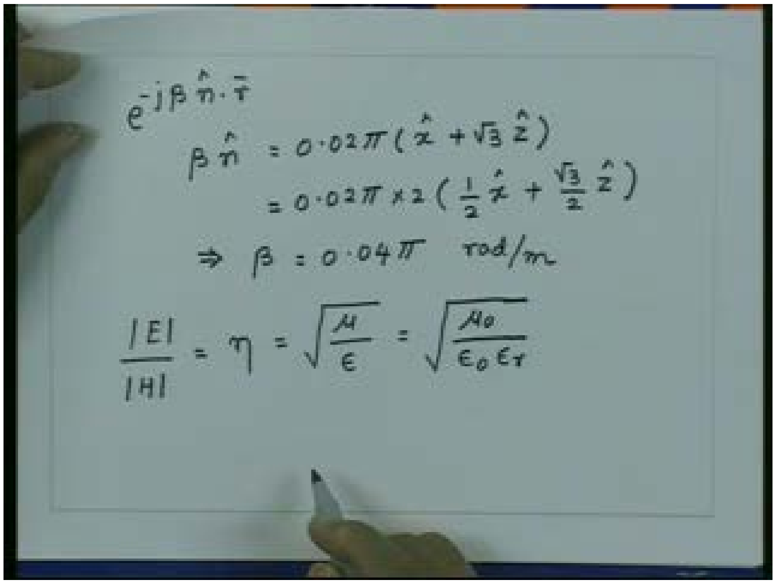

Now, to find out the ratio of this magnitude, since the electric and magnetic fields are having this component which are j so essentially it means that the fields are complex. So what we can do is we can find out the magnitude of electric and magnetic fields at some instant of time and since we know that the ratio of electric and magnetic field should be equal to the intrinsic impedance at every instant of time the ratio can be taken at some instant of time and the easier way to do that is take the time t equal to 0.

So what these components are showing you is that these two components are 90 degrees out of phase in time with respect to the y component so if I take t equal to 0 essentially these two components will be 0 at that instant of time because they are 90 degrees out of phase and at that same instant on time if I take the magnetic field this component will be 0 this will be 90 degrees out of phase, so at t equal to 0 if I take the electric and magnetic fields that will be corresponding to this term for electric field and (Refer Slide Time: 15:24) these two terms for the magnetic field.

So if I take at t equal to 0 sometime then the magnitude the electric field for this is given as 2 into y cap with of course a multiplier by the phase function and the magnetic field at that instant of time will be 1 upon 30pi into root 3 x minus z cap. So that instant of time now we can find out the magnitude of electric and magnetic fields, so this gives me that the magnitude of electric field mod of E that is equal to 2 it is oriented in y direction and mod of H will be mod of this so that is that will be equal to 1 upon 30pi into square root of 3 plus 1 so that will be mod of H from here mod H will be equal to 2 upon 30pi.

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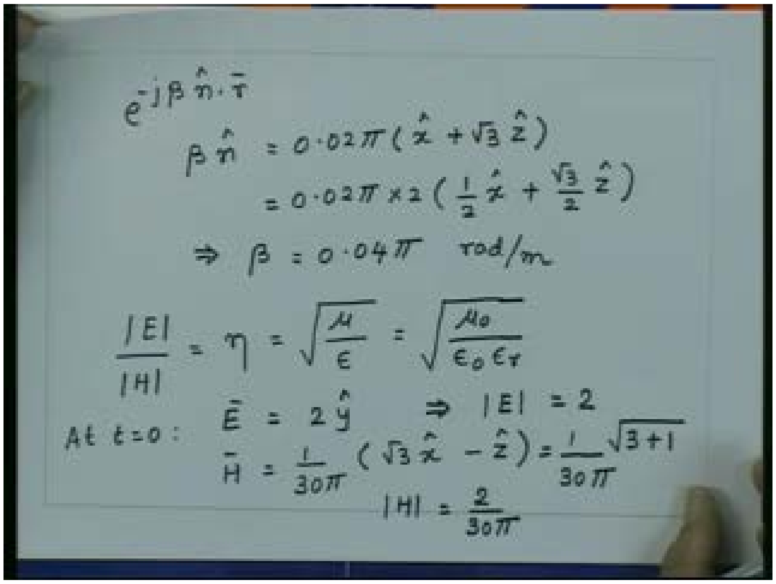

Then we can take the ratio of these two and from there we can find out the dielectric constant of the medium. So from here essentially we get mod E upon mod H that is equal to 2 upon 2 upon 30pi that is equal to 30pi that is equal to square root of mu 0 upon epsilon 0 into 1 upon square root epsilon r. we just take here the square root of epsilon r how? So we have this quantity which is square root of mu 0 upon epsilon 0 which is nothing but the intrinsic impedance of the free space and that number we know is equal to 120pi.

So this quantity square root of mu 0 upon of 1 epsilon 0 is 120pi from here we can get square root of epsilon r that is equal to 1 upon 30pi into this is which is 120pi which is 120pi that is equal to 4.

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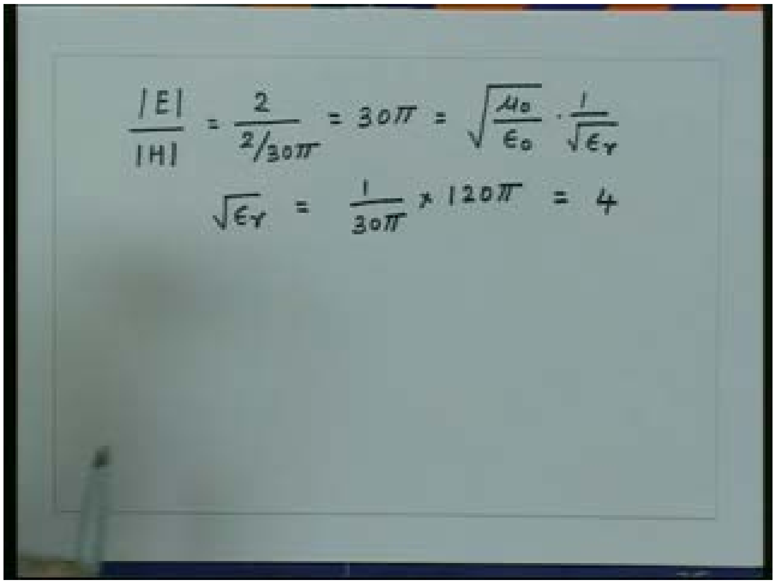

So the square root of the dielectric constant which we also call as a refractive index of the medium that is equal to 4 and therefore the dielectric constant epsilon r for this medium will be equal to 16.

Once we get the dielectric constant then we can find out the velocity of this wave in this medium and that is the velocity of the wave v will be equal to 1 upon square root mu 0 epsilon 0 epsilon r is equal to 1 upon square root mu 0 epsilon 0 square root epsilon r and 1 upon mu 0 square root mu 0 epsilon 0 is nothing but the velocity of light in the free space which is 3 into 10 to the power 8 meter per second so we can substitute here 3 into 10 to the power 8 meters per seconds for this and square root of epsilon r is 4 so the velocity will be 3 into 10 to the power 8 divided by 4 is equal to 0.75 into 10 to the power 8 meters per second.

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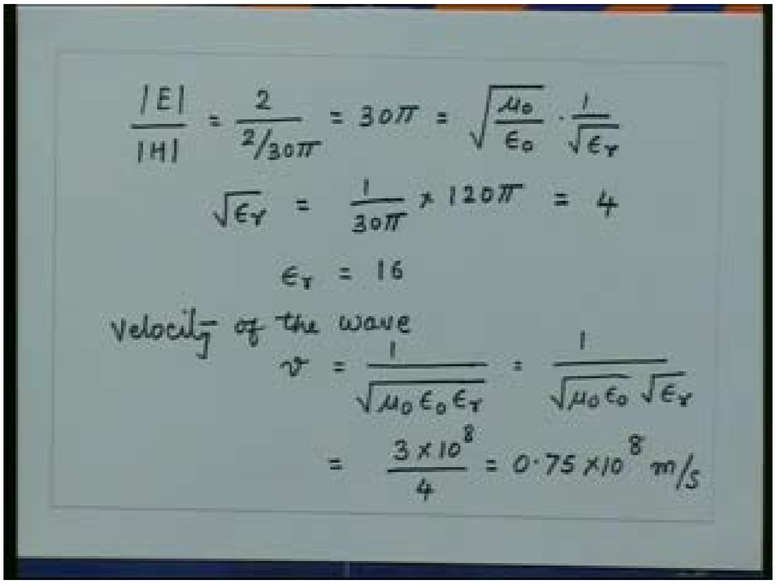

So, for this wave the velocity in whatever dielectric is given which is having a dielectric constant 16 the velocity will be 0.75 into 10 to the power 8 meters per second. Once we know the velocity then now we can find out the frequency of the wave because we know that the velocity for the wave is omega upon beta or from here since you want to find out the frequency which is equal to 2pi into frequency f divided by beta so the frequency from here will be equal to velocity into beta v into beta divided by 2pi.

So we have found out this velocity which is 0.75 into 10 to the power 8 meters per second, we also have found out the value of beta which is 0.04pi radians per meter, we can substitute this, this is equal to 0.75 into 10 to the power 8 multiplied by beta which is 0.04pi divided by 2pi that if we simplify that will turn out to be 0... 1.5 megahertz. So the frequency of this wave is 1.5 megahertz.

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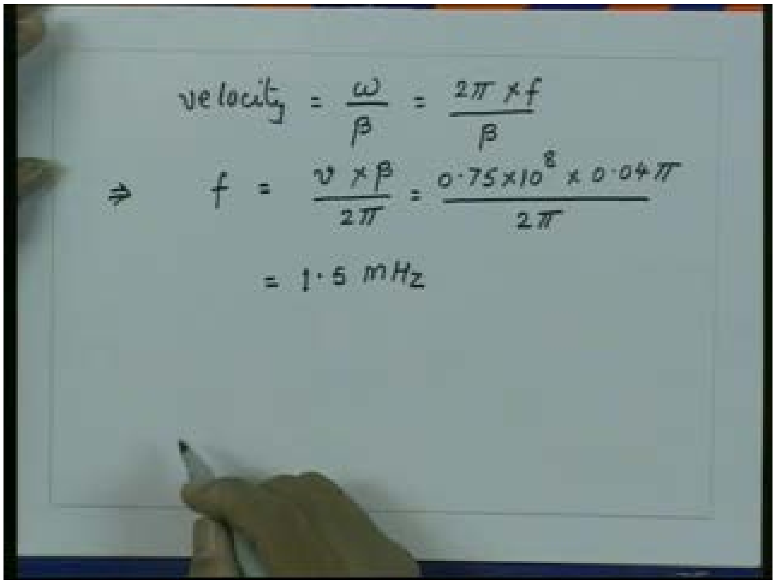

The next thing we have to find out is the velocity of the wave along different directions so we are supposed to find out the phase velocities along x, y and z directions. So as we know that the velocity v x the phase velocity along x direction is equal to v divided by cos of phi x, the velocity along y direction is v divided by cos of phi y and the velocity along z direction is v divided by cos of phi z and we have calculated this phi x, phi y and phi z so phi x is 60 degrees, phi y is phi by 2 so 90 degrees and phi z is 30 degrees so from here we get the velocity along the x direction that is 0.75 into 10 to the power 8 divided by cos of phi x which is 1 upon 2 you get 1 upon 2 so that will be equal to 1.5 into 10 to the power 8 meters per second.

The phi y is 90 degrees so cos of phi y is 0, the phase velocity along the y direction is infinity and phase velocity along z direction will be 0.75 into 10 to the power 8 divided by cos of phi z which is cos of 30 degrees which is root 3 by 2 so this will be root 3 by 2 so that is equal to 0.866 into 10 to the power 8 meters per second. So, since the wave is traveling in the xz plane the phase velocity in the y direction is infinity and these are phase velocities which are in the two directions x and z.

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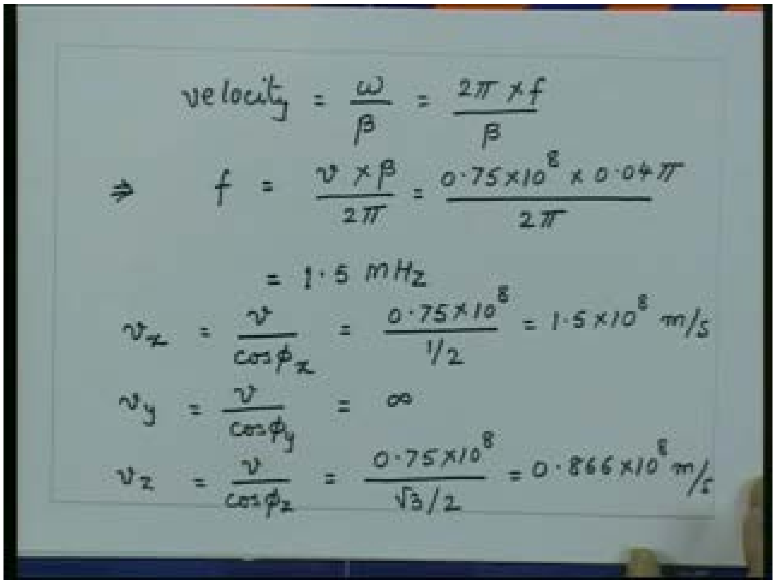

Now, to find the state of polarization of this wave one way is to find out the equation of the ellipse which the electric field vector draws but in this case what we can do is we can look at the electric field and draw... find out the magnitude of electric field at different times.

So let us say if I write down the electric fields... in this the electric field has three components: the x component is given by this, the y component is given by this and the z component is given by that (Refer Slide Time: 25:00). So here the magnitude is root 3 and there is a minus j that means this x component can be written as: E of x can be written as root 3 cos of omega t minus pi by 2 that corresponds to minus j, the y component has an amplitude 2 and it has a zero phase so we can write down the y component E y that is equal to 2 cos of omega t at the z component E z that is equal to this j so it has magnitude 1 and phase plus pi by 2 so that is cos of omega t plus pi by 2 so this gives us root 3 sin of omega t and this will give us minus sin of omega t.

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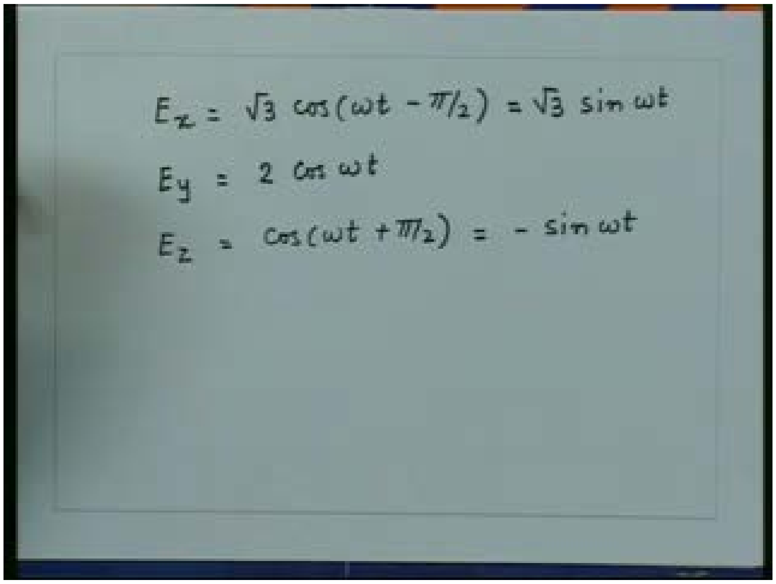

Now we can calculate the the field magnitude at different instant of time. So let us consider the time which is t equal to 0, t equal to quarter period t by 4 t equal to t by 2, t equal to 3 t by 4 and so on and find out what is the magnitude of the electric field. So at t equal to 0 if I take then this this field will be 0, this field will be 2 (Refer Slide Time: 27:01) and this will be again 0. So you will have E x equal to 0, E y is equal to 2 and E z is equal to 0 giving mod of E equal to 2.

If I go after a quarter period which is t by 4 then this quantity will be plus 1, this will be minus 1 and this quantity will be 0 so in this case we will get E x which is equal to root 3, E y which is equal to 0 and E z is equal to minus 1 that again gives the mod of E which is square root of E x square plus E z square that will be equal to 2.

Similarly, if I go one more time which is which is t by 2 so again this will be equal to pi, you will get the magnitude which will be the same so let us take a time which is not multiples of pi by 2 but let us say take a time which is t by 8. So if I substitute for this now this angle will be equal to 45 degrees so you get root 3 upon root 2 so for this time you get E x is equal to root 3 upon 2, E y will be equal to 2 upon root 2 2 upon root 2 and E z will be equal to minus 1 upon root 2.

Again if we calculate the magnitude of the electric field that will be equal to 2. So what that indicates is as the time varies from 0 to t by 8 to t by 4 the magnitude of the electric field remains constant which is equal to 2 that means this wave represents a circularly polarized wave.

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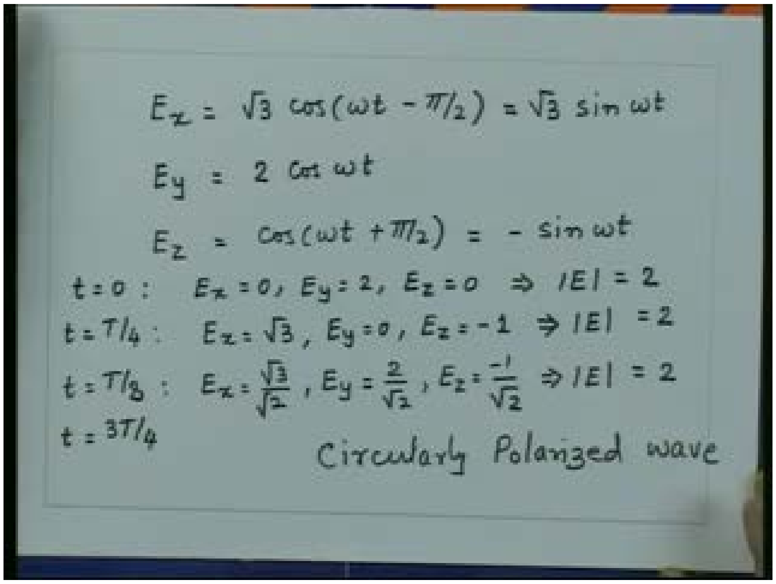

Now, to find the sense of rotation essentially we can look at this diagram (Refer Slide Time: 29:45) and the wave is traveling essentially in this direction. So if I consider at t equal to 0 the electric field is in the in the y direction so the electric field is along y direction which is like that and that is what is given by this. If I go at a t by time little later the E x is root 3, E y is 0 and E z is is minus 1. So initially the field the electric field was it was like that and after the time t by 4 the E x is become root 3 and the z component is negative. So essentially the electric field vector has moved like that. So since this since this quantity z is negative in this so the point which has moved here the electric field vector it will be root 3 in this direction and minus 1 in the z direction so the point will come somewhere here (Refer Slide Time: 30:50) and that is where the electric field would be so essentially the electric field has gone from here to here something like that in this direction.

Now since the wave is moving this way if I now look in the direction of the wave the electric field vector is rotated like this from here to here that means this wave essentially represents a left handed rotation. So we have the sense of rotation for this wave which is left handed.

So this problem clearly demonstrates how knowing the electric and magnetic fields one can find out the various parameters regarding the medium, the velocity, the frequency, the state of polarization and so on. So if we systematically proceed with the concept which we have developed for uniform plane wave, then by knowing the fields we can essentially find out the various parameters of the medium, as well as the properties of the electromagnetic wave.

Let us us take a second problem.

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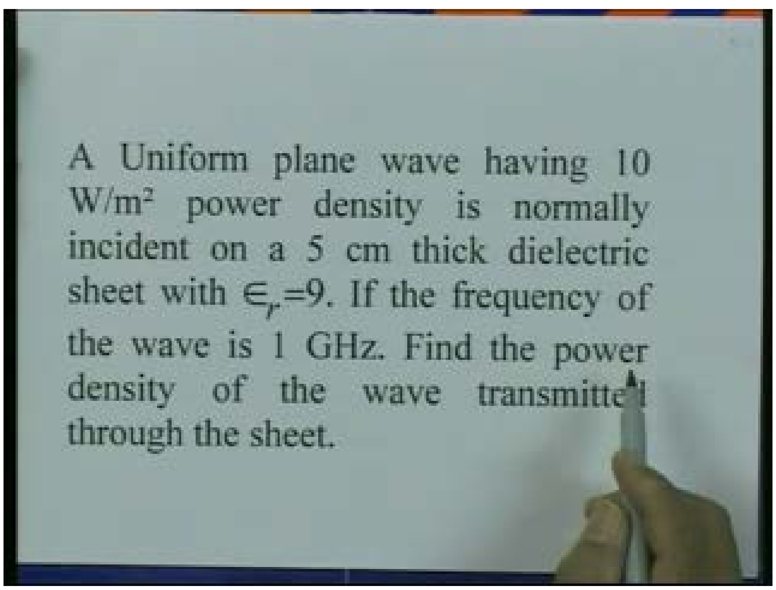

Here we have a uniform plane wave having 10 watts per meter power density is normally incident on a 5 centimeter thick dielectric sheet with epsilon r equal to 9. If the frequency of the wave is 1 gigahertz, find the power density of the wave transmitted through the sheet.

So essentially what we have here? We are having a medium which is having a finite thickness and then we have to find out how much field is transmitted on the other side of the sheet which is having a dielectric constant nine and a thickness of 5 centimeter. So this problem essentially can be can be solved as follows:

We have a sheet which is like that (Refer Slide Time: 33:01) whose dielectric constant epsilon r is equal to 9, on this side you have dielectric constant epsilon r equal to 1 and on this side also you have epsilon r equal to 1. Now the wave is incident from this side is normally incident on the dielectric slab which is having a power density of 10 watts per meter. So the power density p is equal to 10 watts per meter square for this wave and it is having a thickness of 5 centimeter.

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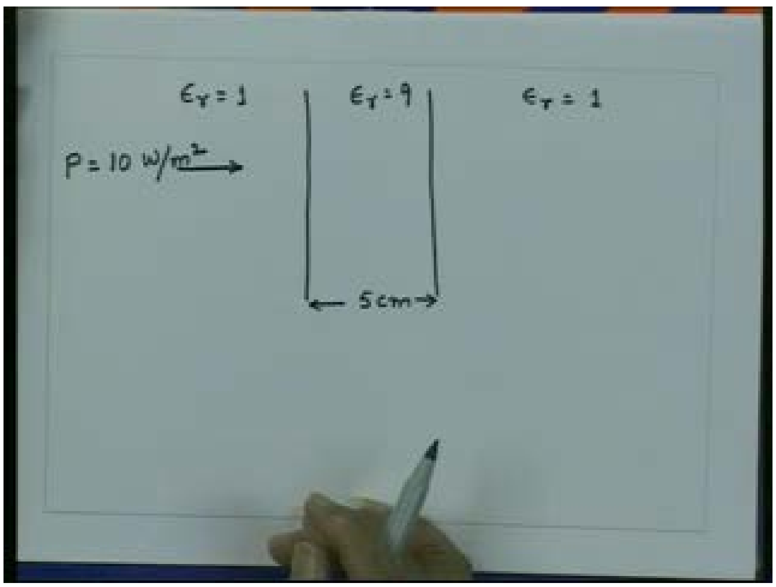

This problem essentially can be solved in many ways. one is which... since this is a normal incidence we can treat it like a one dimensional problem. So we can treat it as if you are having a medium here which is having certain impedance since it is like a like a long transmission line, then you are having another section here whose medium parameter is this so it has a different characteristic impedance and then in this side again you are having a infinite medium so you have another line which is having a impedance which corresponds to epsilon r equal to 1.

So one of the ways to solve this problem is by analogy with the transmission line which says this is equivalent to the transmission line then there is another transmission line here (Refer Slide Time: 34:40) and then the same transmission line again which continues up to infinity.

Now the characteristic impedances of these lines would be the intrinsic impedances of this media. So here we have the intrinsic impedance which will be eta which is same as eta 0 which is equal to 120pi. Here again you will have a impedance eta which will be eta 0 which will be 120pi and in this case you have an impedance eta which will be eta 0 divided by square root of epsilon r. So we will have here eta which is equal to 120pi divided by square root of 9 so that is equal to 40pi.

So now the wave is incident from this side, you have a section here which is having an impedance characteristic impedance of 40pi and you have the length of this transmission line which is 5 centimeter so this is since this lies infinite now it will see an impedance which is equal to a characteristic impedance of this so I can I can find out... so this is equivalent to, this section is equivalent to... we have a transmission line of 120pi then we have a transmission line which is 40pi characteristic impedance and this is now this line is terminated in an impedance which is equal to 120pi.

So one of the ways to solve this problem now is we have to find out what is the power transmitted through this sheet on the other side? So essentially we want to find out what is the power density which is gone to this resistance which is 120pi.

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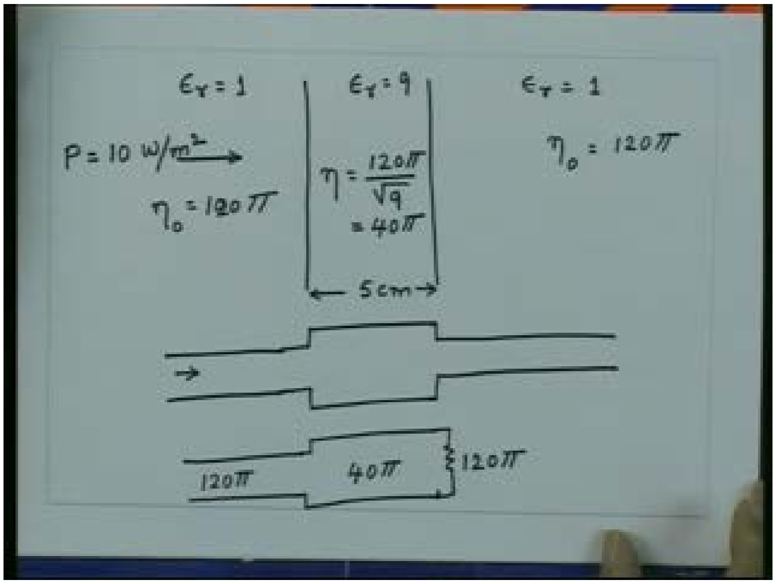

So if you go by transmission line approach essentially what we can do is we can find out what is the impedance which would be seen at this location, then how much power has been transmitted transferred to that impedance that is the power which is finally will be delivered to this impedance so we can get the power and from there we can find out the power which is transmitted to this medium. For this we have to find out now this length which is 5 centimeter which we have to find in terms of the wavelength and now the wavelength depends upon the dielectric constant of the medium so here the velocity we can calculate and from there we can calculate the wavelength of the wave in that medium.

So since the epsilon r is 9 the velocity of the wave in that medium on the sheet that is equal to the velocity in the air divided by the square root of epsilon r which is root of 9 so that is equal to 10 to the power 8 meters per second and the frequency is given for this problem is 1 gigahertz so the f given this way is equal to 1 gigahertz that is equal to 10 to the power 9 hertz. So from here as we have calculate the value of lambda which is the velocity divided by frequency that is equal to 10 to the power 8 divided by 10 to the power 9 so that is equal to 0.1 meters or that is equal to 10 centimeter.

So in this medium (Refer Slide Time: 39:17) the wavelength lambda is 10 centimeters. That means this length here 5 centimeter essentially is the distance of lambda by 2. So in this case the problem becomes very simple that now this distance is equal to lambda by 2 and the impedance which you see the normalized impedance is transformed here at a same value at a distance of lambda by 2. So essentially we can we can calculate this this impedance and transform this impedance on the other side so you can get essentially the impedance which is 120pi which is on the other side.

Once you get that impedance which is transformed here then you can find out what is the power which is delivered with this load and from there then you can find out the power density which is gone to the second medium; that is one approach.

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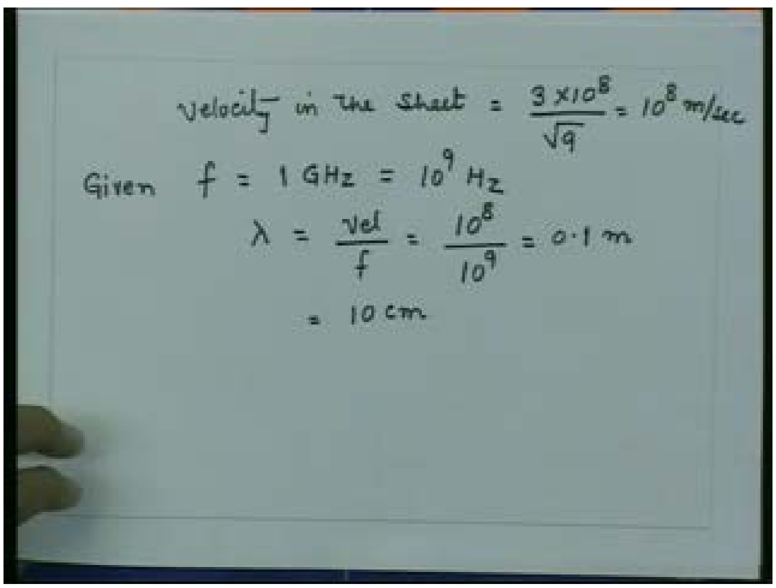

The second approach is which is more like a wave approach and that is... consider now this medium and the wave incident on this. So let us say this boundary is is boundary... this medium is 1, this medium is 2 and this medium is 3 (Refer Slide Time: 40:47). So we have now the two media interfaces: one is from 1 to 2 then the other one is from 2 to 3 so the wave is incident on this, on this boundary it sees the reflection and part of the energy is transmitted here, this energy travels all the way up to this boundary then part of the energy is reflected from here, part gets transmitted, this energy then travels all the way up to this point since this is again medium discontinuity part of the energy is transmitted here and reflected here. This energy again reaches at this point it gets reflected or part gets transmitted, this when reaches here, again part gets reflected part gets transmitted so what we can find essentially that the net reflected wave will be sum of all these reflections; this first reflection plus this second reflection plus the third reflection, if we sum up all of them together that gives me the total reflected wave in this direction.

Similarly, the transmitted wave would be this transmitted wave plus this plus this. If I sum up all these waves together that essentially give us the total transmitted wave on the other side.

So now what we can do is we can essentially find out the the reflection which are coming from here so we can write down the reflection coefficient for each of these boundaries and the transmission coefficient on each of the boundaries and then find out the total summation and from there essentially we can get the total reflection and the total transmission.

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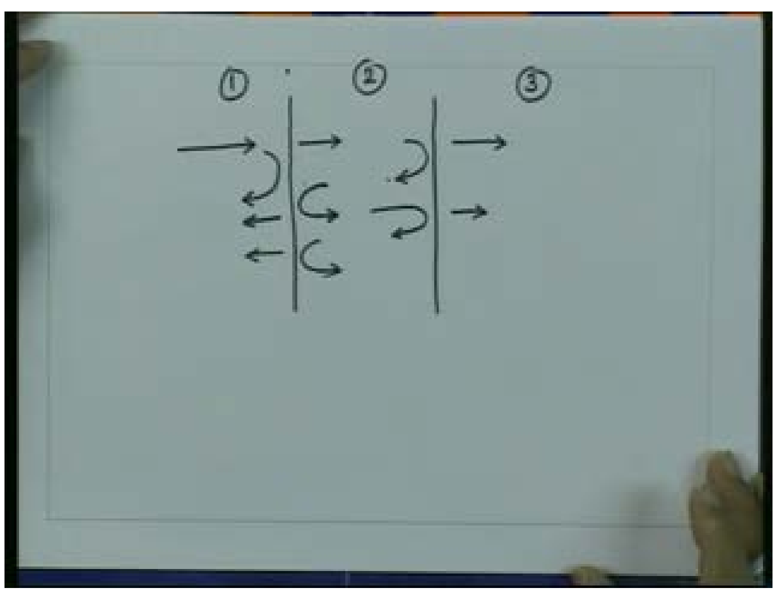

Now if I say that the reflection at i j boundary so that means 1 2 boundary if I take that would be gamma 1 2, so from here let us say reflection I write down at an interface which is given by i j that is nothing but impedance of the medium j minus impedance of the medium i divided by impedance of the medium j plus impedance of the medium i and from here essentially we get gamma ij is equal to minus of gamma ji.

So, if the wave is going from here to here the reflection coefficient is gamma 1 2. However, when the wave comes here the reflection coefficient this would be gamma 2 1. So this one is gamma 1 2, this will be gamma 2 1, this will be gamma 2 3 and so on.

Similarly the transmission coefficient you can get tau ij that is equal to two times eta j divided by eta j plus eta i. So this transmission coefficient here would correspond to eta 2 divided by eta 1 by eta 2, this transmission coefficient would correspond to eta 3 2 times eta 3 divided by eta 3 plus eta 2 and so on.

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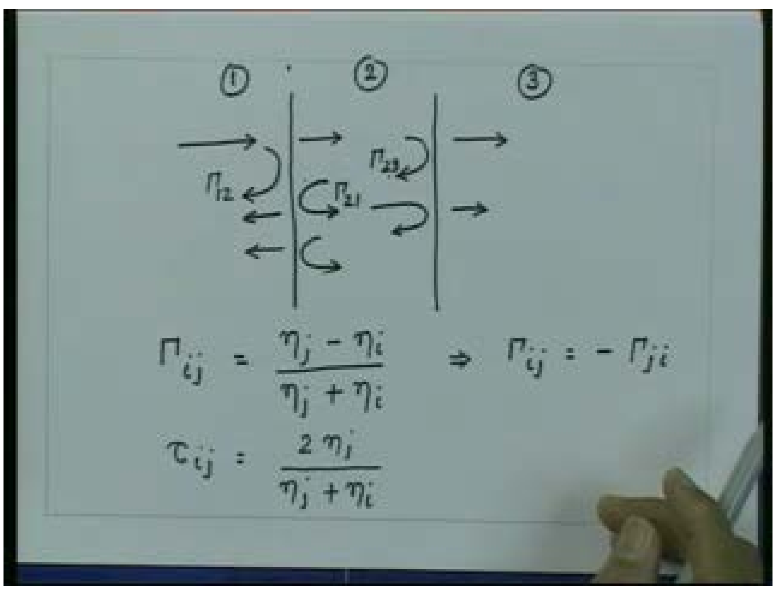

So now if I if I write down and you are having this distance d and the phase constant in this medium let us say is given by beta so you are having the phase change when the wave travels from here to here which is equal to corresponding to beta d. So this wave travels a distance which is beta d phase bet beta d, when it comes here it has traveled a distance 2 times beta d so now what we have here is the first reflection plus the second quantity here which is one reflection here one transmission one reflection and one transmission which is this wave.

So, if I write down now the the the total reflected field which is going to come from here that would correspond to your E reflected that is equal to gamma 12 E incident plus tau 12 then gamma 23 and tau 21. So we can write here tau 12 E i gamma 23 tau 21 with a phase term of which is e to the power minus j 2 beta d. The next term would be whatever the whatever has come here (Refer Slide Time: 46:24) this will be tau 12 plus gamma 23 gamma 21 gamma 23 and tau 21. So we can we can write that so that is equal to tau 12 E i gamma 23 gamma 21 gamma 23 tau 21 e to the power minus j 4 beta d plus and so on.

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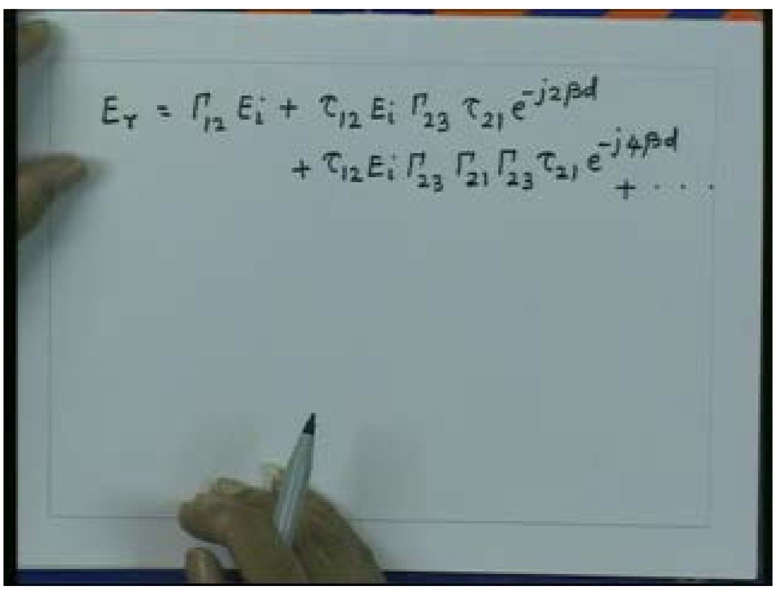

So essentially you see if I take this quantity common you will have now a geometric series for this. so we can write down now the reflected wave E r that is gamma 12 E i plus tau 12 gamma 23 tau 21 e to the power minus j beta d into 2 and summation of the geometric series which we get for this term will be multiplied by E i upon 1 minus gamma 12 gamma 23 e to the power minus j 2 beta d.

So if I know now the field which is incident on this on this wave then i can find out what would be the the net reflected wave which will be essentially given by that. So precisely same thing we can do for the transmitted wave also which is going to going to this. So this transmitted wave will be... this transmitter multiplied by this transmitter that will be two reflections and then this transmitted. If I sum up again all these fields essentially we will get the transmitted wave also on the line similar to this that will be equal to tau 12 tau 23 e to the power minus j beta d into E i divided by 1 minus gamma 21 gamma 23 e to the power minus j 2 beta d.

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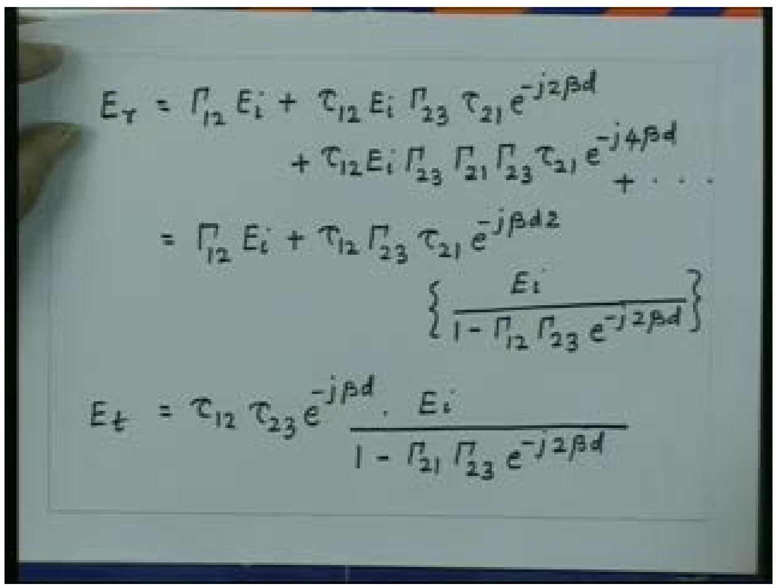

So I can substitute now this value for beta d for 2 beta d and I can substitute the value for gamma 21 and gamma 23 and from here then I can find out what is the transmitted wave. So the power density which will be transmitted will be proportional to mod t square so I can find out this quantity and can get actually the the power transfer. So in this case since now the quantity which are given are... the situation is this that this is equal to epsilon r is 1, epsilon r is 1, and this is epsilon r is 9 essentially we have the eta 1 which is equal to eta 0, eta 2 is eta 0 is square root of epsilon r is 3 eta 3 is equal to eta 0 again.

So from here we can get essentially these quantities which is gamma 12 which is eta 0 by 3 minus eta 0 upon eta 0 by 3 plus eta 0 so that is equal to 1 upon 3 minus 1 upon 1 upon 3 plus 1 so that will be equal to minus 2 upon 4 is equal to minus 1 upon 2 and gamma 23 will be eta 0 minus eta 0 upon 3 minus eta 0 plus eta 0 upon 3 that will be equal to 2 upon 4 equal to 1 upon 2.