Transmission Lines & E M. Waves
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Lecture - 44

In the last lecture, we started discussion on a very important topic - that is, radiation or antennas. The idea here is to establish a relationship between the electric and magnetic fields with their sources which are current and charge.

We saw that if the charges are accelerated, then there is a possibility of radiation. So, if we create a structure where charges are accelerated, then we might get radiation from that system. Going to the basics then again, we formulated the problem for Maxwell's equations. For mathematical convenience, we introduce a quantity what is called the magnetic vector potential which is related to the magnetic flux density; and then, substituting the magnetic vector potential in the Maxwell's equations and by using some vector identities, we essentially got the relation between the current and the magnetic vector potential and the charge and the electric vector potential.

We also established a relationship between the magnetic vector potential and electric scalar potential with the help of what is called Lorenz gauge condition. So essentially, these two quantities - the magnetic vector potential and the electric scalar potential - are not independent, but they are related to the Lorenz gauge condition.

So today, we will take off from there and then we will try to solve the equation for the magnetic vector potential or electric scalar potential; and then from there, we will try to derive the electric and magnetic fields. Yesterday, we saw that we have this condition what is called the Lorenz gauge condition, which relates the divergence of the magnetic vector potential to the time derivative of the electric scalar potential and by substituting this, then we got this equation which is same as the wave equation; and then we also put forward the logic for choosing this condition – that, if you take this condition, then the equation essentially reduces to the wave equation which when the sources are zero, that means when the j is 0, will give me the wave equation in the source free medium.

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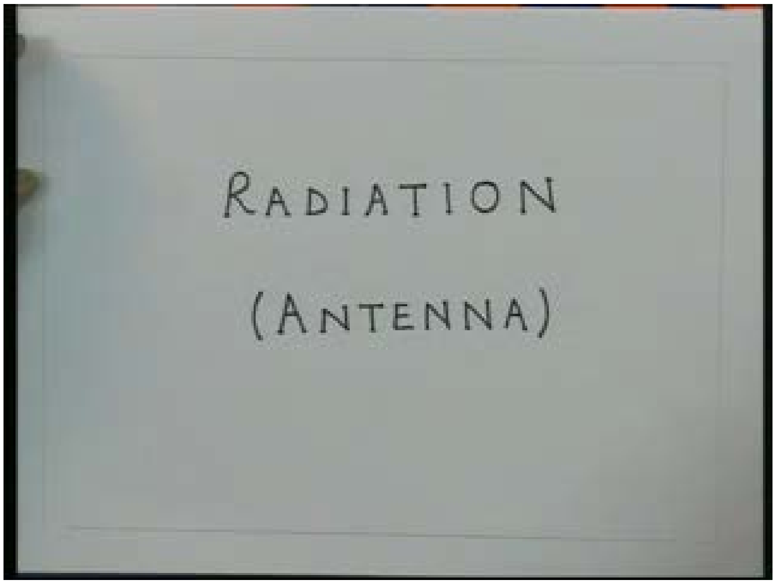

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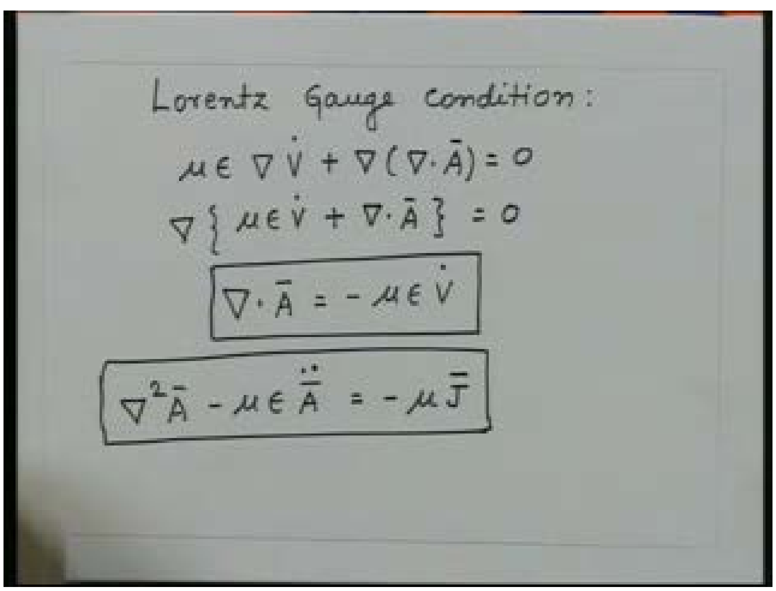

So, now our task essentially is to find the solution to this equation or substituting for Lorenz gauge condition into another Maxwell's equation the solution for this equation.

So, as we noted, both these equations - whether you got for the scalar potential or vector potential, both of these equations are the wave equations. However, the electric scalar potential is related to the electric charges; that is, rho volume charge density and the magnetic vector potential is related to the conduction current density j.

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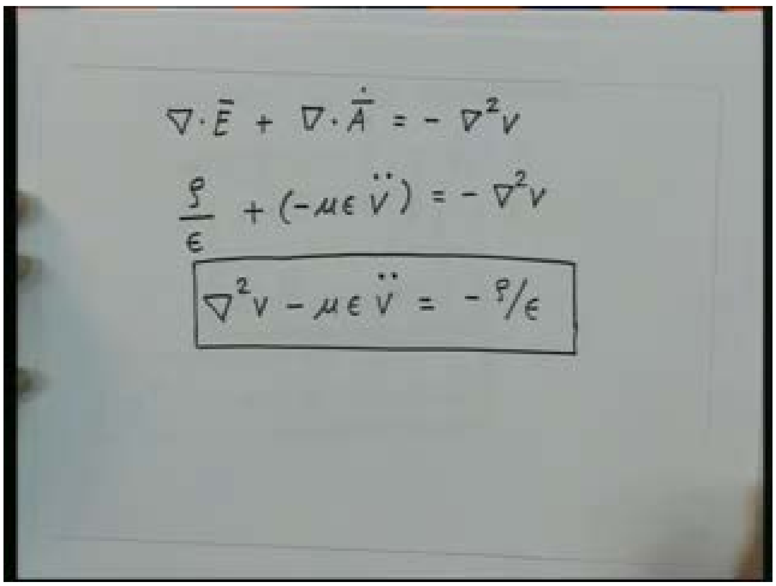

So, as we said, we can solve for either of these two either we can solve for v by using this equation or we can solve for a in this equation and substituting into this condition. Then we can get the other quantity. However, if you solve for v, then solution of a will require integration over space which is rather tedious.

So, it is better to find a solution for a from this equation and then from here we can get a solution for v, because that will require only an integration over time; and for time harmonic fields, the integration over time is extremely straight forward thing. Second thing what we note from this equation is that this operator del square is a scalar quantity. So, this equation as such is a scalar equation. The vector nature is coming only because of the vector magnetic potential and the conduction current density which is the vector quantity. Said differently, essentially this equation is satisfied by all the three components or which is ax, ay, az or in coordinates, spherical coordinates, a r, a theta a phi and each component is related to the corresponding component of j.

So as such, this equation is a scalar equation, only the vector nature is coming because of the source and the magnetic vector potential and essentially that is what, that is what that is the phase that we are going to make use of when we solve this equation and try to get the solution to this equation. So today, we solve, try to solve this equation and the technique which is used for solving this equation is what is called the Green's Function technique.

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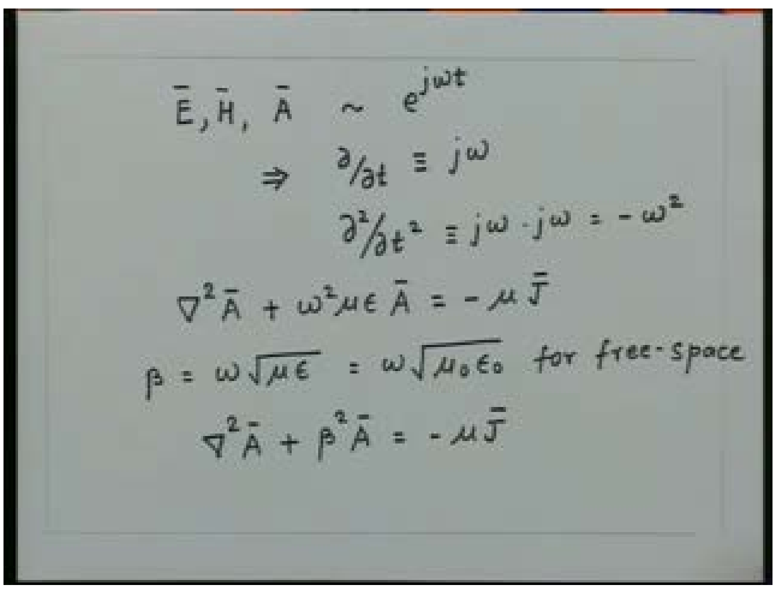

So before that, let me take explicitly the time function which is e to the power j omega t. Say, as we have always taken all the quantities, whether you take e or h and now even the magnetic vector potential a, all these quantities have a time variation e to the power j omega t that gives all time derivatives d by dt equivalent to multiplying the quantity by j omega or d 2 by dt square. That is equivalent to j omega into j omega, which is minus omega square. So, the equation which we have got for this, I can substitute now for the time derivatives, the double derivative.

So, this is equivalent to multiplying this quantity by minus omega square. So, this equation then becomes del square a plus omega square mu epsilon a. That is equal to minus mu into j, noting that this quantity is nothing but the square of the propagation constant in the medium. And when we talk about the antenna problem generally, the medium is the free space.

So, mu and epsilon are the permeability and the permittivity of the free space. So, this quantity - omega square mu epsilon is nothing but the square of the phase constant in free space. So what we have that beta which is omega square root, mu epsilon which is equal to omega square root mu 0, epsilon 0 for free space. So, this equation we can write now for the free space. That is del square a plus beta square a that is equal to minus mu into j.

So, this is the equation essentially we have to solve now and this equation we will solve by using, as we said, a technique what is called the Green's Function technique now what is the Green's Function the Green's Function essentially is the spatial impulse response of a system which is described by this differential equation.

So, if I replace the source by an impulse in space, that means whatever is the current distribution if they have if the right hand side if I replace this quantity by a delta function in space and find the solution to these differential equation, that solution essentially is called the Green's Function. And as we already said that since the vector nature in this equation is due to these quantities and intrinsically this equation is a scalar equation.

We can essentially solve this equation by replacing a by a scalar quantity and then multiplying the scalar quantity by appropriate vector which will be a. We can get the vector solution. So first, what we do is we just convert this equation by replacing this quantity a by a scalar quantity and right hand side by a delta function in space and whatever solution we get to that system. We say that is the impulse response of this system, spatial impulse response; and once we know the impulse response, then we can always find the response for any arbitrary source and that will be minus mu into j.

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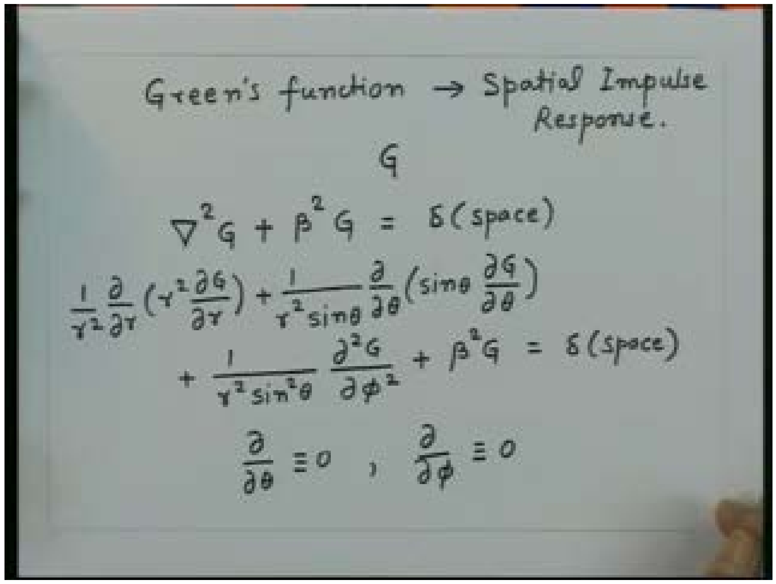

So, the Green's function is the spatial impulse response. So, if I take this equation as we said and if I replace this quantity a by a scalar function, let us say I have the scalar function which is denoted by psi. So, this equation essentially can be greater in terms of psi and if I replace this right hand side by delta function which is in space, then the solution which I am going to get to that equation we called as Green's function.

So, let us say I denote that quantity by g which is a function of space is the Green's function and this Green's function should satisfy the equation which is del square g plus beta square g. That is equal to the delta function in space. So, at some location in space if I have an impulse, then the solution which will get in space which will be consistent in this differential equation will be called the Green's function for the system. So essentially, now we have to solve this equation in the appropriate coordinate system and as we have mentioned earlier for the radiation problem, the appropriate coordinate system is the spherical coordinate system.

So essentially, we expand this del square in spherical coordinate and try to find the solution to this in the spherical coordinates. So first, expand this in the spherical coordinates. So, what we will get is 1 over r square d by dr, r square dg by dr plus 1 upon r square sine theta d by d theta, sine theta dg by d theta plus 1 upon r square sine square theta d 2 g by d phi square plus beta square g. That is equal to delta as the function of space.

Now, since we have to locate a delta function in the three dimensional space without losing generality, I can say that this is an impulse which is located at the origin of the coordinate system. So that means the delta I can locate at the origin of the coordinate system that is r equal to 0.

Now since these are impulse and these are scalar quantity now was this equation now we are this g. We are putting taking a scalar quantity because the vector nature is with the vector potential and the currents. This is putting a point on the origin. That is what the source is; you are having a delta function and the origin.

So essentially, it does not have any directional dependent. So, now the equation whatever system we are talking about for which the source is located at the origin of the coordinate system and therefore from no matter from which direction you see, you see the same for source. So that means the solution for this equation now is spherically symmetric. It does not depend on theta; it does not depend upon phi.

So, in this case, you will have d by d theta will be identically equal to 0, d by d phi will be identically equal to 0. So, if I substitute now in this equation, this is d g by d theta. So, this will go to 0, this will go to 0. Only this is the term which will remain. So, this wave equation in the spherical coordinate system will reduce to 1 over r square d by dr, r square dg by dr plus beta square g is equal to delta function which is located at r equal to 0. Note here that the partial derivatives, now we have converted to the full derivative because g is now only a function of r. It does not depend upon theta and phi because this is the spherical symmetric case for delta function which is located at the origin. We can expand this thing and then we can get an equation which will be in terms of g where you will have a second derivative of g and so on.

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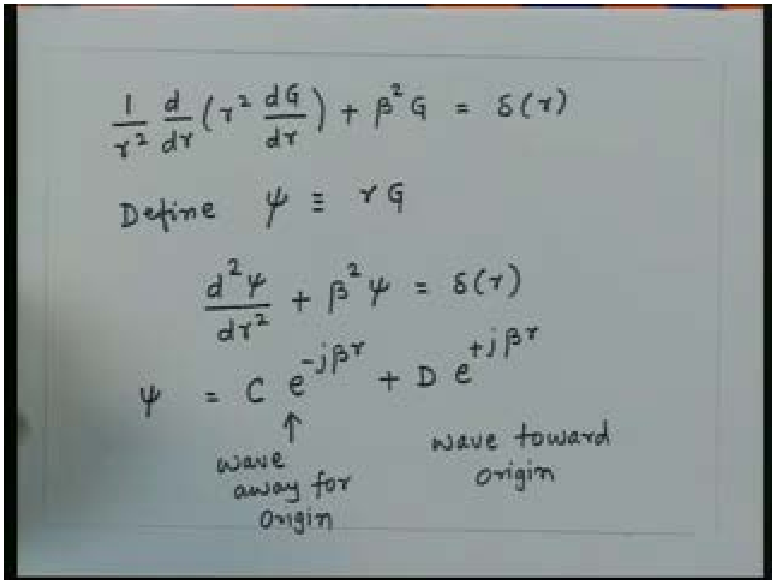

However, what we do before we do that? We define another variable, let us say psi, which is r times g and by doing this, this derivative gets significantly simplified. So, define some quantity say psi equal to r times g. So, if I differentiate this quantity now and replace this g is equal to psi upon r, this equation will reduce to d 2 psi upon dr square plus beta square psi. That is equal to delta into r. This equation is a very familiar equation because this is exactly identical to what we have got in transmission line case, what we have got in one dimensional wave propagation problem.

So, if I put that equal to zero, then this is the equation which is same as your transmission line equation and for this, the solution we have already derived and we have also seen the physical interpretation to the solution. Also, now our basics signals and systems knowledge, we know that the impulse response to this system is nothing but the complementary solution to this equation with boundary condition appropriately satisfied at the location of the source which is at r equal to 0.

So, the solution to this general problem now, the complementary solution which is the impulse response, that will be psi with the function of r will be equal to some constant, let us say c, e to the power minus j beta r plus d e power plus j beta r. And if I appropriately satisfy the boundary condition for this expression at r equal to 0, then that will be nothing but spatial impulse response of the system. Before getting into that however, we know these two terms which we have got here, they essentially represent the traveling waves and in this case the traveling wave is in the direction of r and r is always positive in the spherical coordinate system. The source is now located at r equal to 0.

So, this term essentially represents a wave which is traveling in all directions positive; all directions that means it is going away from the origin. Similarly this quantity represents a wave which is traveling a negative r direction but r is always positive for the spherical coordinate system.

What that means is this wave essentially represents coming towards the origin. So, this wave tells wave, away from origin and this wave is coming towards the origin wave. So essentially the solution which we get for this differential equation for this quantity psi, that will be representing two waves; one which goes away from the origin in the direction r and other one comes radially inwards and essentially collapse on the origin.

Since our source is at r equal to 0, that is at origin, there is no energy source anywhere in the space. So, there is no reason for this wave to exist because we are talking about infinite medium. There are more boundaries, so there is no possibility of deflection. So essentially for this problem, when we are having a source only at the origin of the coordinate system and the medium is infinite, this wave will be non-existent. So the solution essentially will be one wave which will be going outwards from the origin.

So, for the infinite medium then, our solution for psi will be some constant which we have to evaluate from the boundary conditions e to the power minus j beta r; and now substituting for in terms of g, the psi is equal to r g. So g will be psi divided by r. So we get the Green's Function for this case g. That is some constant e to the power minus j beta r upon r. Two things should be noted in here at this point and that is the r equal to constant essentially represent the constant phase surface.

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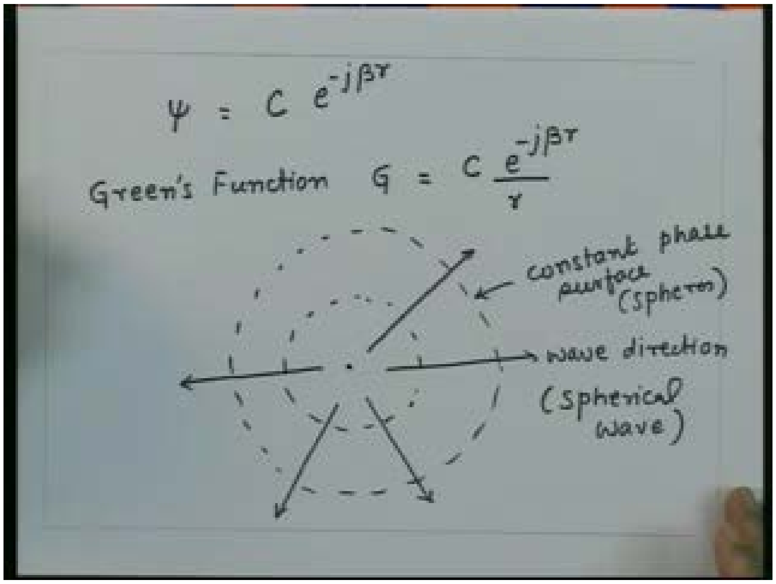

As we saw in case of parallel plane, plane wave propagation, when the wave propagates the constant phase fronts where planes in this case the constant phase fronts are defined by this quantity, r equal to constant. And as we know, r equal to constant defines a surface which is a sphere. So that means, if this is the origin then the constant phase surfaces are the spheres which are around it these are the constant phase surfaces and the wave is traveling radially outwards it is going in all directions that means the wave is traveling essentially in this direction radially outwards.

So, this is the wave direction and these surfaces are the constant phase surfaces which are spheres. So, since now the constant surfaces are spheres, this wave we call as the spherical wave. So, we got this phenomena, the wave phenomena, which will be the spherical wave phenomena. So, if you take the coordinate system with a spherical coordinate system that means if I visualize the 3 dimensional space as a sphere, then the solution to the wave equation is the spherical wave.

Recall when we defined this 3 coordinate system. We had visualized the 3 dimensional space in three different ways. We had visualized the three dimensional space like a rectangular box. We could visualize a 3 dimensional space like a cylinder or we could visualize the 3 dimensional space like a sphere. So, if we had taken the three dimensional space like a box, then we get a solution to that wave equation which is the uniform plane wave solution. If you visualize the 3 dimensional space as a sphere, then we will get the solution to the wave equation which will be the spherical wave solution. And later on we will see that even the spherical wave has the same properties as the uniform plane wave has, except now the phase fronts are no more planes but they are the spheres.

One more thing we should note from here is that when we talked about the uniform plane wave, the amplitude of uniform plane wave, they do not change in the function of distance. So, the phase fronts are planes and the amplitude does not change as the wave propagate. However, for a spherical wave, the phase fronts are spheres and the amplitudes reduces as 1 over r. So, as we go further and further essentially, the amplitude will become smaller and smaller for this wave. Later on, essentially we will try to calculate the power of which is coming out with this wave and they are trying implication of this will become little more clear.

So at this point, essentially we conclude that the general Green's Function or general solution to the wave equation in the spherical coordinate system gives me a wave, what is called a spherical wave. That means the phase front for this wave are spheres, wave radially goes outwards and the amplitude of this wave reduces as 1 over r.

The solution to this equation still is not complete because as we mentioned, we have to evaluate this constant c by applying appropriate boundary conditions. So essentially what we do, we just go back to our original differential equation which is this equation and in this we substitute. Now the solution which we have got, general solution for this expanding wave and then integrate over the volume around the origin and then take a limit when r tends to 0.

So this delta function, the boundary condition, will be imposed and then we can evaluate this quantity c which is the constant for this Green's Function. So we will do now, we just take this solution, substitute into this equation and integrate both sides over a volume enclosing the origin and then you take the limit r tends to 0.

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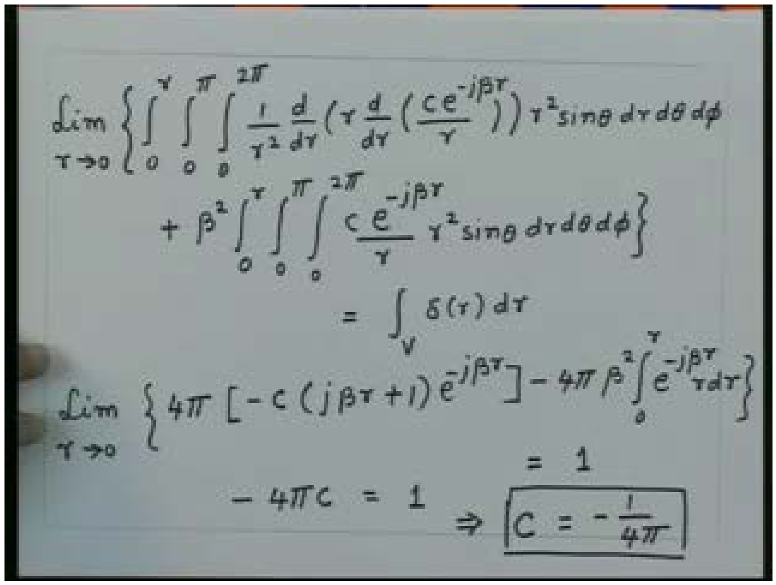

So, let us do that. If I substitute now for t, essentially we want to take the limit when r tends to 0 and the r limits now are 0 to r, theta limits we know are from 0 to phi and the phi limits are from 0 to 2 phi. We have this quantity 1 upon r square. So, we can substitute here. This is 1 upon r square d by dr, r square d by dr of the solution which is c e to the power minus j beta r upon r, r square sine theta dr, d theta d phi. That is the incremental volume. So the first term integration over a volume will be essentially this, the same thing. We can do for the second term. So, second term will be plus beta square. Again, same integration limits is 0 to r for theta 0 to phi, for phi 0 to 2 phi e to the power c times e to the power minus j beta r upon r, r square sine theta dr, d theta d phi; and that is equal to the integral over the volume delta r dr.

Now by definition, the integral of delta function over a volume which is encloses the location of delta function that will be equal to 1. So, this quantity by definition of delta function is equal to 1. I can differentiate this quantity here and write down this integral and that will be essentially can be written as limit r tends to 0, 4 phi minus c j beta r plus 1 e to the power minus j beta r minus 4 phi beta square integral 0 to r, e to the power minus j beta r, r dr. That is equal to 1 because this integral by definition of delta function is equal to 1.

So now substituting the general solution to the wave equation, essentially now we are trying to evaluate this constant c which is consistent with the source which is delta function. And once you get the solution, this constant c, we get the complete solution to this wave equation with driving source as delta function and then that we will call as the Green's Function.

So, now as r tends to 0, this quantity e to the power minus j beta r that quantity will tend to 1 because the phase will go to 0. This is r dr also when r goes 0, this integrate will go to 0. So in the limit when the r tends to 0, this quantity will go to 0 in this expression. When r tends to 0 again, this will be this quantity will go to 1, this quantity will go to 0; but this quantity will remain as minus c.

So in the limit when r tends to 0, essentially we get minus 4 phi into c. That is equal to 1 or we got this arbitrary constant c which is minus 1 upon 4 phi. So this gives now this arbitrary constant c. That will be equal to minus 1 upon 4 phi. Substituting, now this constant, arbitrary constant into Green's Function which is this, then we got the complete Green's Function for this problem and that will be g to the function of r. That will be equal to minus e to the power minus j beta r upon 4 phi into r.

So this is the Green's Function for the problem which we are having under investigation. So that means these spherical coordinates system taking the source located at the origin. That is what will be the solution to the wave equation which we call the Green's Function. So, once we get the Green's Function or in normal system once we get the impulse response, that is what this Green's Function is with a spatial impulse response.

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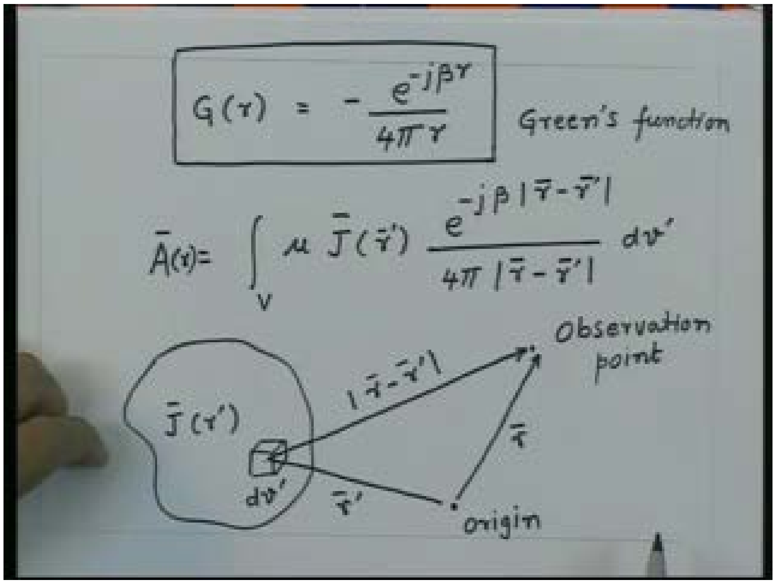

Then, finding the output of any system is the convolution of the impulse response with the driving function. So now in this case, the driving function is nothing but the right hand side of the wave equation which is minus mu times j. So, if I convolve this quantity, the Green's Function, with the spatial distribution of mu times j, then I can get the vector potential at arbitrary location in the space.

So essentially now is a convolution of the two. So the vector potential 'a', that any location now a will be essentially given by the convolution over the volume j(r) e to the power minus j beta r minus r prime here r prime divided by 4 phi r minus r prime into dv integrated over the volume.

So this is essentially the convolution integral where this r prime denotes the location of the current. So now we are having region where the sources are located and r is the location where we are trying to find out the vector potential. So essentially we are having the source phase here where the j are there which are function of r prime. You are having a small infinite decimal volume in this. If you call as dv prime integrating over the source phase and we are having the observation point here and this is my origin. So, this distance from origin here is r which is a vector. The distance from here for a origin to this location where the current is this one is a vector which is r prime. So the distance between this location, the source location and the observation point is the difference of these two vectors.

So this is r minus r prime vectors magnitude. So in this case, the r are the vector quantities. So we can put the vector signs here r and r prime are the vectors. However, what we will note here in the normal convolution, you would have simply had r minus r prime without putting anymore sign. In this case, since r in spherically coordinate system is always positive, you have to make sure that this quantity always remain positive because there are no negative r is defined. So if you have done the convolution let us say a Cartesian coordinate system we would have just got the value which could be negative or positive. However in this case, the distance is always positive between the source and observation point because in spherical coordinate system, the distance is always positive. That is the reason we put this mod sign here.

So knowing now the current distribution at this location, essentially we can calculate the magnetic vector potential at any point in the space which is this observation point. So now, it is a matter of getting the current distribution which should be a priory known and once you know this distribution, mu times this j where this current are distributed, then substitute into this integral, we can find out the vector potential at any location r. So, we can say that this is at any location r which is essentially given by this. So, let us see what we have done right from the beginning.

We took the wave equation, replaced the source term in the wave equation by a delta function in space, found a solution to that equation. We call that solution as the Green’s Function which essentially represents the spatial impulse response of that wave equation. Then using the simple convolution property, we have got the magnetic vector potential at some location r which is convolution of the Green’s Function and the source distribution, the driving function, which is minus mu times j. And once we get this magnetic vector potential at this location, then the problem is straight forward. Then we can go back to the definition of magnetic vector potential from which we can calculate the magnetic fields or the magnetic flux density. And once we get the magnetic field, then we can go to the, any of the Maxwell's equations and can solve for the electric field and essentially we got the solution to the problem.

So now we will take very specific cases and solve the problem. That means find the electric and magnetic fields for given current distribution. However, keep in mind that first in the whole thing which we have got done so far, we should know the distribution of the current. Just somebody should be giving you a priory. Once the distribution is given to you, then the problem is straight forward; but finding the current distribution which also should be consistent with the Maxwell's equation is the rather tedious case.

So generally, when the current distribution is not given to you and somebody gives you only some excitation point on a structure, then we have to solve the problem in a self-consistent manner which will give the current distribution as well as the fields around it through the Maxwell's equation and satisfying the boundary condition. However, in this course, we do not get into that which is very complicated. We take the simpler cases and we assume that the current distribution is a priory given and now we are interested in finding out what kind of fields will be generated by this current distribution. That is the problem of the antenna.

So in general simpler antennas, the current distribution is obtained in some other way, independent way and then from the knowledge of the current distribution, then we can go to this and find out the vector potential and then find the electric and magnetic fields. So now, let us take the simplest possible case of the current distribution and that is if I have a very small element of current where once I say current it has a direction. So the source which we have taken for Green’s Function, the source was a point which was not vector. It was located at the origin; but as soon as I say I have a current, then I have a direction.

So, I must put a small current element. So the current will be flowing over a very finite length. That this length is extremely small in fact is tending to 0. So we call this as an infinitely a small current element. That is the smallest current or simplest current distribution you can think of. However, if you can get the current distribution for the small current element, then we can visualize any other current distribution as superposition of these small, small element. What one can do if I have a structure? I can divide the structure into small, small current elements and then simply applying the superposition.

If I know the fields due to one current element, I can just simply find out the fields due to all the current element and I get the electric and magnetic fields for any arbitrary current distribution. So the analysis of the small current element is an extremely important analysis because that gives you the foundation for finding electric and magnetic fields for arbitrary current distribution. So, that is the reason we carry out that analysis in detail.

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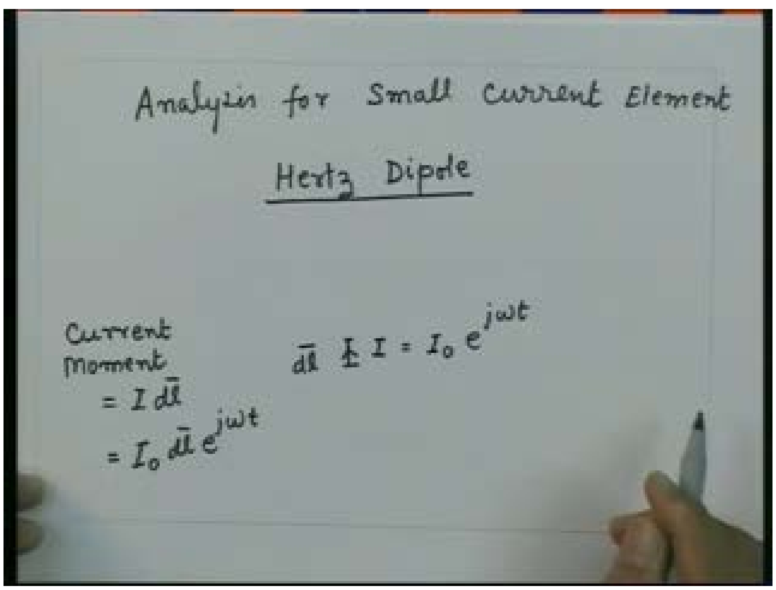

So, we carry out the analysis for small current element. This small current element many times as referred as the hertz dipole and the meaning of this will become clear in a short while from now. So, this thing is also called the hertz dipole. So, without worrying about how the current is again excited, in the current, in this small element, let us say I have a small current element which is carrying current in this direction which is “I” and the length of this current element is, let us say, given by some dl “I” current is the scalar quantity; but this current is flowing in the direction which is along the length. So, I can say dl is the vector quantity.

So, a current element which is also called as a hertz dipole is characterized by what is called a current moment which is product of the current and the length of that element. So, we have a quantity what is called moment which is equal to I dl. Now, assume that this current is now varying sinusoidally as a function of time. So, this I is equal to some current amplitude e to the power j omega t. So, we have here the current moment which is I 0 dl, e to the power j omega t.

Now, this current which we have got here can be imagined as if as there is a small rod of area of cross section which carrying current j. So, if I integrate j over the cross section of the rod, I will get the current which if I have multiplied by the length and assuming that the current is constant across the length of wire.

You get this quantity I times dl which is nothing but the integral of the current or integral of a current density over the volume. So now for this current element, if I want to find out the vector potential at some location at this point, firstly we will note here, since the distance or the length is very small, the quantity which we have here this is e to the power minus j beta r minus r prime. That quantity is practically constant.

So, r minus r prime is the distance between the center of the current element and the observation point. And since their length is very small, we assume that this term practically remain same for all points on this element. So, we can now put this current element at the origin of the coordinate system. So, let us say we have this coordinate system and the current element is located here. I have enlarged it little bit. So, this is I dl and the current at some instant of time let us say flows like that and then, you have a observation point here which is at a distance r. This angle is phi, this is x axis, this is y axis, this is z axis and this angle is theta.

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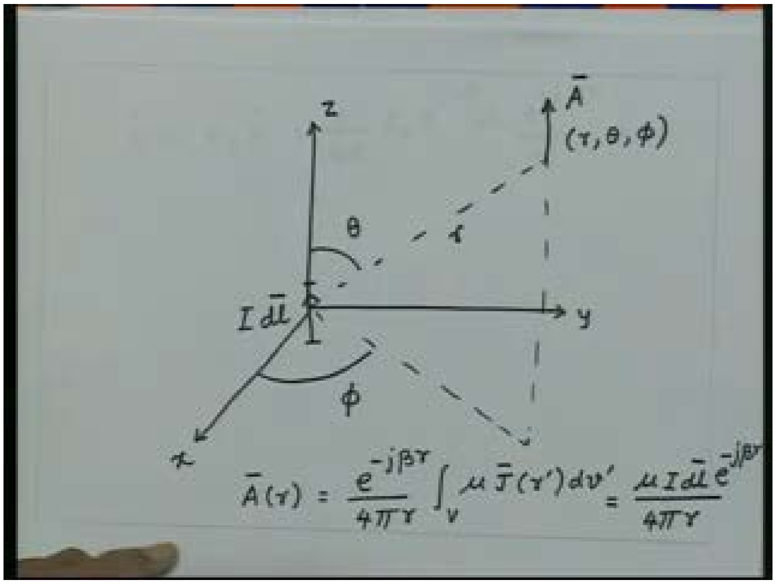

So, this point here is r theta phi where we want to find out the magnetic vector potential. So, if I substitute now so this distance r is r minus r prime because r prime is the origin where the current element is located. So, since the length of this one is much smaller compared to this r, this quantity e to the power minus j beta r upon 4 phi r is practically constant. So, we can take this over the integral sign. So we can get for this one, a of r equal to e to the power minus j beta r upon 4 phi r integral over this volume mu j function of r prime dv prime and as we saw that we can take mu outside; mu is not varying as the function of space.

This integral over the volume for j current density will be same as I into dl because integral over the cross section of this will give me the current multiplied by the length will give me the integral of j over the volume. So essentially, this quantity will be equal to mu I dl e to the power minus j beta r divided by 4 phi r.

So, I get now the vector potential due to this current element at this location and as we mentioned earlier, the vector nature is only because of this current element. That means the direction of the current and the magnetic vector potential is same because that is the quantity which is the vector quantity; all other quantities are scalar quantities. So, we have here a which is in the direction dl and dl is oriented in our case in the direction z.

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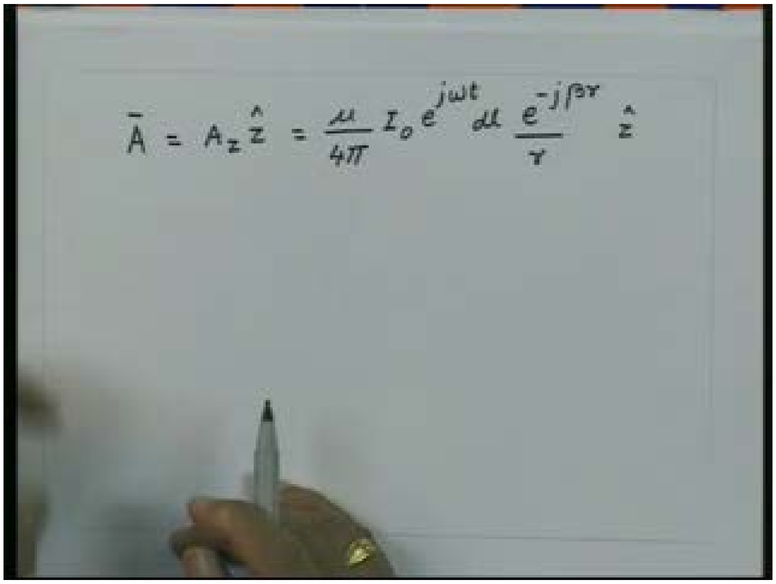

So, I can write this one explicitly now. Taking that unit vector out, I can get a, which will be only in z direction because the current is in z direction vector. That will be equal to mu upon 4 phi I 0, e to the power j omega t dl, e to the power minus j beta r upon r, z direction. So, it is interesting thing now to note that since the current element is oriented in this z direction, the magnetic vector potential also is oriented in z direction. This is a and no matter where I go in the space, anywhere I go, everywhere the magnetic vector potential will be oriented in z direction because here the current is oriented in z direction.

Later on, we will see this property is not true for the electric and magnetic fields and then we appreciate the use of the magnetic vector potential because this is directly related to the current distribution. So, if we know the vector currents, we can immediately find out magnetic vector potential and the directions.

But the electric and magnetic field will have arbitrary direction that we go from different point in space. So, what we see for this case, what is called the hertz dipole that everywhere in the space the magnetic vector potential will be having the direction same as the hertz dipole and it will be varying inversely as the function of r and will be proportional to the, what is called the current moment which is the product of the current and then length of the current element. With the knowledge of this magnetic vector potential, now we can go to the Maxwell's equation and we can find out the electric and magnetic fields.

So, when we meet next time, essentially we make use of this magnetic vector potential and then we try to derive the electric and magnetic fields and then we investigate what kind of fields are generated by the hertz dipole.