Transmission Lines & E M. Waves
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Lecture - 37

In previous lectures we investigated a structure what is called a parallel plane waveguide. See, if you are having two conducting boundaries, we saw that the electromagnetic energy can propagate between these two boundaries in the form of some definite electric and magnetic field patterns which we call as modes. We also try to visualize the propagation of electromagnetic wave between these two boundaries as a superposition of uniform plane wave.

Now, this understanding of modal propagation in terms of the uniform plane wave gives you a correct picture of how the modes actually are established inside a waveguide. However, this approach that visualizing a modal pattern in terms of the fields of uniform plane wave is always not that easy as we saw in terms of parallel plane waveguide. In parallel plane waveguide, since the medium still was infinite at least parallel to the conducting boundaries, we still could use or visualize the uniform plane wave propagating between the boundaries.

However, if I have a structure, let us say like a pipe kind of structure; whether a rectangular pipe or circular pipe or a pipe of arbitrary shape and if I want to find out how the electromagnetic energy is going to propagate inside this structure, it will be extremely difficult to visualize the propagation in terms of superposition of uniform plane wave and in those situation, essentially we go more mathematically. So, we develop a mathematical framework and try to solve the problem of electromagnetic wave propagation in structure using that general mathematical formulation.

So in this lecture, essentially we develop now a general framework for analyzing wave propagation inside an arbitrary waveguide.

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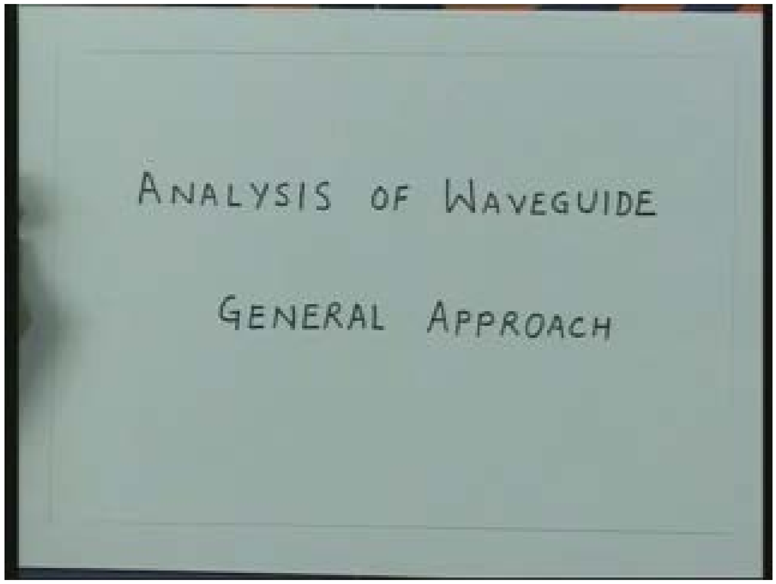

And then, whatever result we get now mathematically, we will try to correlate this results with what we have got for parallel plane waveguide which we visualize a superposition of uniform plane waves. So, that will give us confidence that the visualization which we had in terms of uniform plane waves for the (… Refer Slide Time 03:14) and that gives you much better physical inside into the problem.

However, since every time resolving in the uniform plane wave will not be possible, now we will develop a general mathematical framework and solve the problems in the following by using exit general mathematical approach. So, in this lecture essentially we do the analysis of a waveguide what is called a general approach. So idea now, here is as follows; let us see if I have some arbitrary shape waveguide, this is a pipe which I call waveguide. So, this is a conducting pipe and the electromagnetic energy we can flow along the axis of this pipe, so there is wave propagation in this direction. So, that is the way, the wave is going to propagate.

At the moment I am not even defining any shape for the cross section; it could be rectangular, it could be circular or in general it could be as arbitrarily shape as shown here. So, only information I want to use now for this analysis is that the wave is going to propagate in this direction which is the axis and then without loosing generality, I can say this direction is the z direction.

So, I can orient my coordinate system such that I may have this direction x, this is y and this is z; this is z, this is x, this is y. So, xy plane is like a cross sectional plane for this waveguide and the wave is going to propagate in the z direction which is the axis of this pipe what is called waveguide. So now, we develop a general framework starting from the Maxwell equations, what kind of fields would exist inside this and then we will later on we will deal with very specific cases rectangular waveguides and propagation of modal propagation inside rectangular waveguide.
Now, since this direction is a special direction, I call this direction as the longitudinal direction because this is a direction in which the wave is going to propagate and a direction perpendicular to this direction, we call as a transverse direction. So, we say this direction is longitudinal, so any field component which is in this direction, we call as the longitudinal field and any field which is perpendicular to this wave propagation that we will call as a transverse field. Any electric or magnetic field, now we will resolve into two components; one is along the wave propagation, that we will call a longitudinal component and other one will be perpendicular to direction of wave propagation that we will call as a transverse.

Since the longitudinal direction is the same direction as z, essentially we will say that any field now has two components; one is transverse component and other one is the z component. If we do that, then I can now write down the electric and magnetic field in general; so I have here electric field which is the vector, this can be represented by electric transverse E which is the vector because it can lie anywhere in the xy plane plus it has a longitudinal component and let us say its value is E_z and it is oriented in z direction. So, E_z is a scalar quantity and E transverse is a vector quantity and this vector lies in the xy plane which is the transverse plane.

Similarly, I can write down the magnetic field also that is the transverse magnetic field plus the longitudinal component of the magnetic field with unit vector in the z direction.

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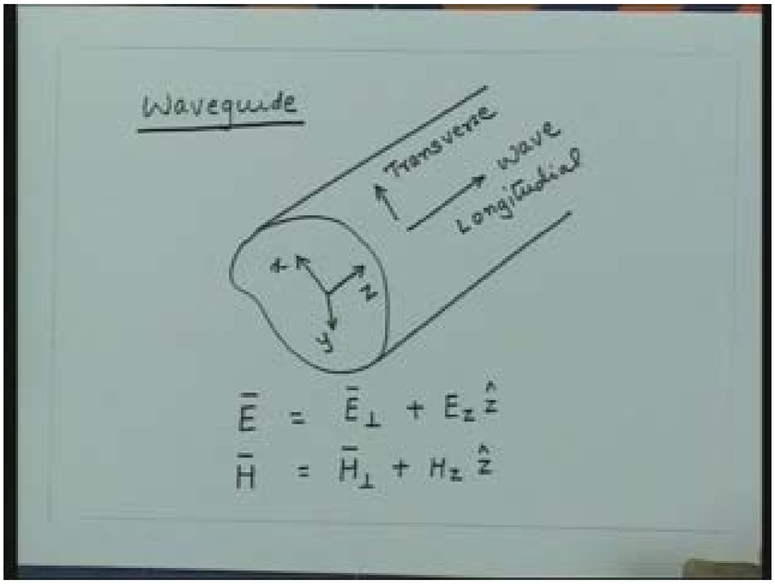

So, any electric field or magnetic field can be represented by combination of longitudinal component, this quantity is scalar and a transverse component which is this which is a vector. So by definition, then we will have E perpendicular, this vector will be the x component of electric field with unit vector x plus y component of electric field with unit vector y and same is true for the transverse magnetic field which will be having x component with unit vector x plus y component with unit vector y. Now, as we have defined the longitudinal and transverse fields, we can also define the del operator separated out for longitudinal and transverse component.

So, we can define this operator del also as del transverse plus the longitudinal part of that which is d by dz with unit vector z. So, this is the vector operator, this is a scalar operator d by dz with unit vector z and this quantity is del perpendicular is nothing but d by dx x plus d by dy into y. So, we define now the fields with their explicit transverse and longitudinal components, we also define the vector operator del in terms of this transverse and longitudinal component.

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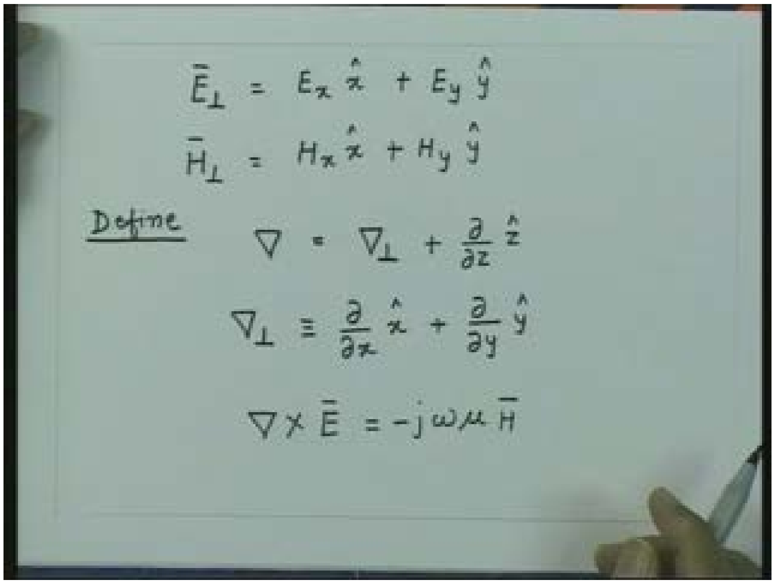

Now, we can just take this and substitute into any of the Maxwell equations and we can take Maxwell equation del cross E that is equal to minus j omega mu into H. So, I can substitute now for E which is E transverse E longitudinal, H is H transverse H longitudinal and del which is given by this. So, I get this equation returned explicitly for transverse and longitudinal components, that will be now del transverse plus d by dz z cross the electric field which is E transverse plus E_z z that is equal to minus j omega mu into H transverse plus H_z z.

I can open these brackets, from here I will get or this gives me del transverse cross E transverse plus del transverse cross E_z z plus this term d by dz z cross E transverse plus this term is d by dz z cross E_z z that is equal to minus j omega mu into H transverse plus H longitudinal with unit vector z.

Now, we can note the following things that if I take the cross product with z; so if I take with any cross product with z, that cross product will always lie in a transverse plane. See, if I take this quantity here, the del cross with z cross z unit vector; this quantity will be the transverse component, this will be a transverse component but this quantity if I look at, then this quantity will lie along the z direction because this vector del vector is in transverse plane is E transverse is in the transverse plane which is xy plane. So, the cross product of these two must lie perpendicular to that plane. So, this component essentially is the longitudinal component and similarly this quantity here which is z cross z, this quantity is identically 0.

So, what I now find is that this term is longitudinal, this is transverse because it is crossed with z, this is transverse - crossed with z and this quantity is identically 0. So, what we can do is, we can now separate out the longitudinal and transverse component and equate them on two sides. See, if I take a transverse component which is this and this, that should be equal to the transverse component on the right hand side. So, from here, I get minus j omega mu H transverse, that will be equal to this quantity which is del transverse cross E_z z plus d by dz z cross E transverse.

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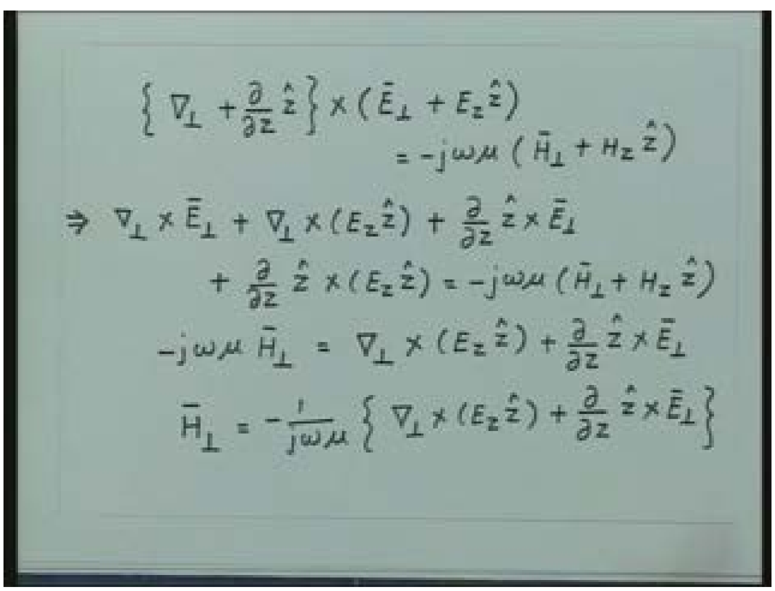

I can bring this minus j omega mu down here to get explicitly this quantity which is x transverse. So, this gives me H transverse that will be minus 1 upon j omega mu into this thing which is del transverse cross E_z z plus d by dz z cross E transverse. Similarly, if I take the other equation or the curl equation, instead of taking del cross E if I take the other equation and I can assume that the medium which is filling this waveguide is ideal dielectric, there is no conductivity I mean there is no conduction current; I can start with this equation that del cross H that is equal to j omega epsilon into E and again I substitute for E with transverse component and longitudinal component and same for the operator for the del, I will get on the line similar to this, I will get a transverse component of the electric field and that will be E transverse. That will be equal to 1 upon j omega epsilon del transverse cross H_z z plus d by dz z cross H transverse.

So see here, this expression is identical to this; this is the expression which we had H transverse, so minus j omega mew is replaced by j omega epsilon and E_z is replaced by z, E transverse is replaced by H transverse. So now, we got these two equations where the H transverse and E transverse are related.

Similarly here, the H transverse and E transverse are related. Now, we can substitute H transverse from this equation into this equation and if I do that and then bring the E transverse term on one side for substituting for H transverse from here into this and bring the terms on one side, essentially we get expression which will be omega square mu epsilon E transverse minus d by dz z cross d by dz z cross E transverse and that will be equal to minus j omega mu del transverse cross H_z z plus d by dz z cross del transverse cross E_z z. So, just we have taken this transverse component here and substituted into this to get expression like that.

Now, what we do? We use the vector identity and that is if I have a 3 vectors, vector triple product, so we can use identity for vector triple product that is if I have a 3 vectors A B C, then the triple product A cross B cross C is equal to (A dot C) B minus (A dot B) C. So, using this vector identity, then I can look at each of this product here; so I have a triple vector product here and I have a triple vector product here, so I can expand these are you increase identity.

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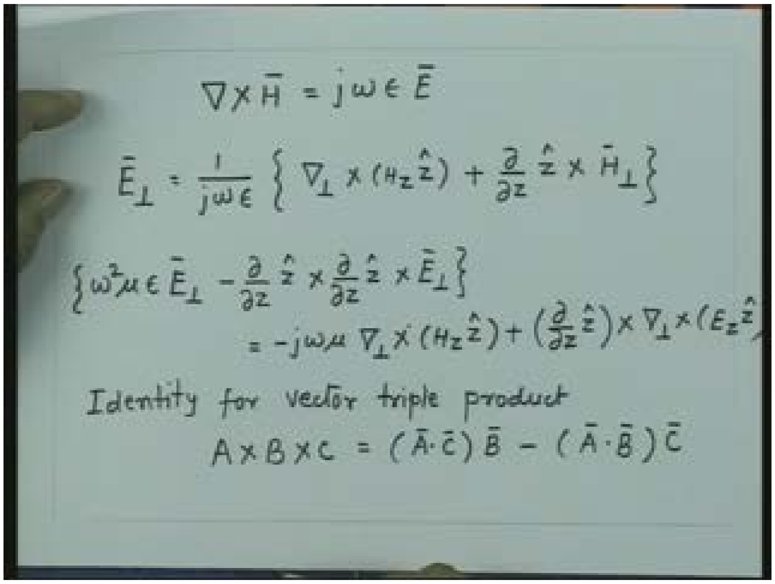

So, let us look at the terms one by one. So first, let us look at the term which is this term; if I write down this term here that is d by dz z cross d by dz z cross E transverse and I substitute now use this vector identity, so I get (A dot C) B minus (A dot B) C, this vector. So, this we can write as d by dz of d by dz z dot E transverse z minus d by dz z dot d by dz z E transverse. Now this quantity, this is the vector which is in z direction, the E transverse vector is in the transverse plane that is perpendicular to z direction, so this dot product is identically 0.

So this product, the first product here, this one, essentially is nothing but minus d^2 by dz square E perpendicular or E transverse; this is one simplification we got for this term. Same thing we can do for this term also; these are triple product again, so this term which is d by dz z cross del transverse cross E_z z that is equal to del transverse d by dz z dot E_z z minus d by dz z dot del transverse E_z z.

Now this is again, this del transverse is the vector which is in the transverse plane that means dot product of z and this will be again identically 0; as we have said, this quantity over 0, same is here this quantity is identically 0. So, what we get from here is this triple product that will be equal to del transverse and this is the dot product of z and z, this will be simply d by dz of E_z. So, this quantity is dE_z by dz.

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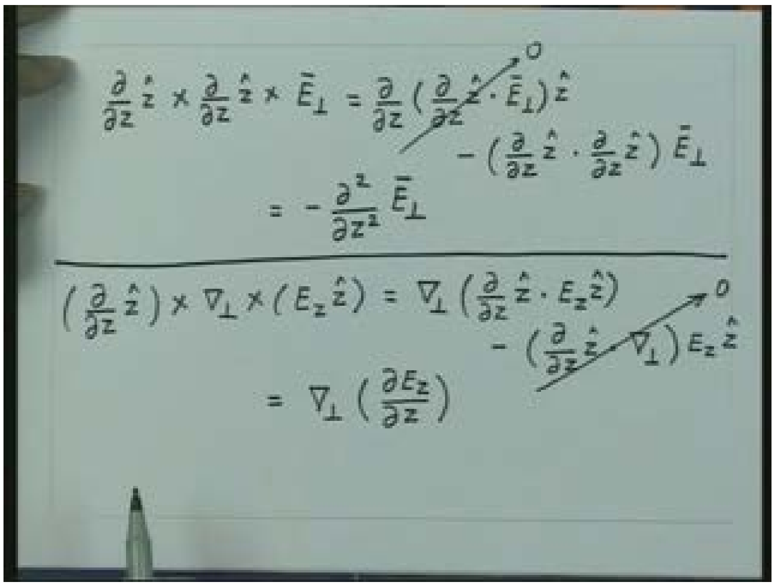

Once we get now this simplification, I can substitute now for this quantity here which is minus d^2 by dz square E transverse for this term and for this term, I have got this quantity. So, this equation which we had for the transverse component of the electric field, now can be written as omega square mu epsilon plus d^2 by dz square E transverse, this E transverse here that is equal to minus j omega mu del transverse cross H_z z plus del transverse d E_z by dz.

Up till now we have just taken two Maxwell equations and just applied some vector identities and derived now the expressions which are for the transverse component of the electric field in terms of essentially the longitudinal components. And, why we did that is that since the waveguide is a structure in which the wave is going to propagate in z direction, this is the special direction that is the direction in which ultimately the energy is going to flow.

We also know that since the 6 components of the electric and magnetic fields are related through Maxwell equations, all the 6 components are not independent components. So, we can choose two of the components as independent components and can represent the remaining 4 components in terms of these 2 components by using the Maxwell equations.

Now, since this direction is the special direction in which the wave is propagating, generally we take the electric and magnetic field components which are in this direction that is the longitudinal component as independent components and try to express the transverse fields in terms of the longitudinal components and precisely that is what we did. We now got the expression for the transverse key in terms of the longitudinal component which is E_z and z. So, we treat E_z and z are the independent components to solve essentially finally the problem for E_z and z. Once we get E_z and z, then we can find out the transverse field components by using this relation.

Now, since our problem again is the wave of propagation in the z direction, we are assuming essentially the wave is traveling in the z direction. That means the travelling wave behavior as we have seen earlier can be given as e to the power minus gamma z where gamma is the propagation constant in the z direction. So, the wave travels in the z direction with a propagation constant gamma.

So, since we are now interested in the traveling wave inside the structure, we assume that functional form of any field electric or magnetic in the z direction will be given as e to the power minus gamma. Once we get that then we know for a travelling wave in z direction the E and H, they will be varying e to the power minus gamma z where gamma is propagation constant and then all the derivatives which we have with this z can be equivalent to multiplying that quantity by minus gamma. So, from here we get d by dz is multiplying the quantity minus gamma and d^2 by dz square that is equal to gamma square.

So, this quantity gamma which still we have not found out, what we know is this is the one which is going to control the behavior of the wave propagation on the structure. So, we know, this is called the propagation constant.

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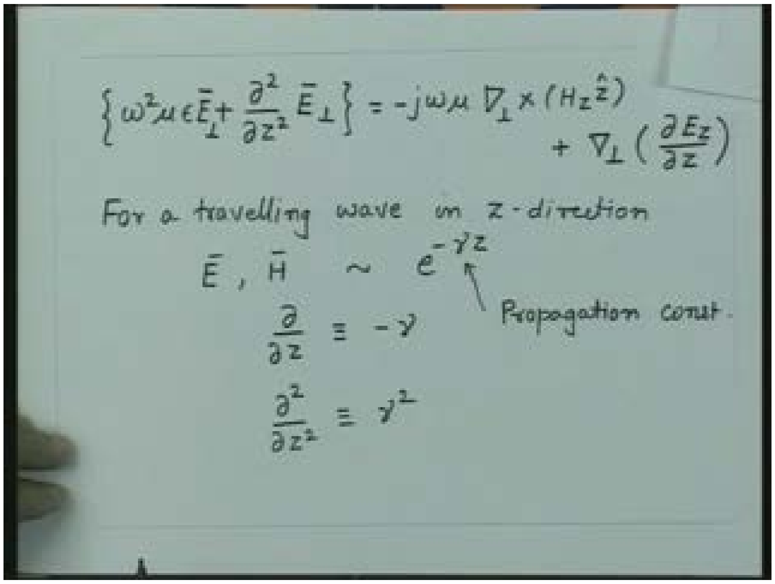

So, I can replace now d by dz by minus gamma and d^2 by dz square in the expression by gamma square. Once I do that, then I can get now the expression which will be omega square mu epsilon plus gamma square E transverse that is equal to minus j omega mu del transverse cross H_z z plus this quantity and d by dz is multiplying by minus gamma, so that is minus gamma del perpendicular E_z.

Now, for the brevity reason, let us define this quantity which is omega square mu epsilon plus gamma square as something H square. So, let us define omega square mu epsilon plus gamma square is equal to some quantity h square. We will assign some physical meaning later what this h would represent but at the moment just algebraically define this whole term at h square.

Once we do that, then I can now have final expression for the transverse electric field that will be equal to minus j omega mu upon h square del perpendicular cross H_z z minus gamma upon h square del transverse E_z and other similar lines if I use other equation for the transverse component of magnetic field, I would get the identical expression for the transverse component of magnetic field where minus j omega mu is replaced by j omega epsilon and E_z and H_z are interchanged, we get j omega epsilon divided by h square del transverse cross E_z z minus gamma upon h square del transverse H_z.

What is this quantity representing? This quantity is representing a curl in the transverse plane, whereas this quantity is representing the gradient of this quantity E_z which is the scalar quantity in the transverse plane. So essentially now, if you can solve the problem of wave propagation in terms of E_z and H_z, then we can find out the corresponding fields that in the transverse plane which are E transverse H transverse. So now, the approach for solving the problem in general will be as follows.

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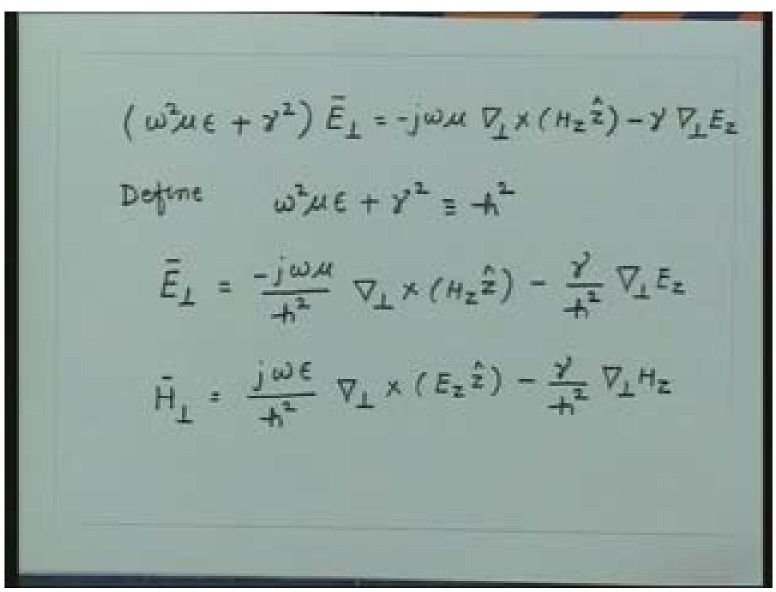

We take a geometry in which we want to study the wave of propagation, find the solution for E_z and H_z for the geometry, then substitute into these expressions and then you get the component of transverse fields that is E_x E_y H_x and H_y. Now, looking at this expression, we can make certain conclusions. Firstly, if you assume that this medium is completely lossless, then this quantity gamma which is the propagation constant is purely magnum because the wave is travelling and its amplitude is not going to change. So, for a lossless medium, we know that gamma should be j beta.

If I consider a medium which is lossless, this medium, that means the conducting boundaries are ideal from medium which is filling this pipe or this waveguide this also an ideal dielectric. So, for this we have gamma which is equal to j beta. So, your H which you have defined here, this quantity is minus beta square. So you have, giving the omega square mu epsilon minus beta square is equal to h square.

Now, this is the beta is the propagation constant in the z direction or let me put beta bar as you have taken earlier or a parallel plane waveguide. So, this is now the phase constant in the z direction; if you recall this quantity omega square root mu epsilon or the phase constant of the wave in that unbound medium so that you find this quantity h is nothing but the propagation constant of this wave in the transverse plane. So, h represents the propagation constant of a wave in the transverse plane that means in the xy plane.

Now, from this expression now, few conclusions can be drawn. Firstly, if the transverse fields have to exist, then both E_z and H_z need not be 0 but need not be non-zero. Even if any one of them is non-zero, still I can get the transverse fields. That means I can have the field distribution inside this waveguide with either E_z equal to 0 or H_z equal to 0 or both of them non-zero.

However, if both of them are 0, then the transverse fields do not exist in general except when h is equal to 0. So, now we have some interesting conclusions can be drawn from this result that is for fields to survive inside the structure, both E_z and H_z cannot be 0 except in the case when x is equal to 0. So, we see from here, a conclusion; one, both E_z and H_z cannot be 0 except when h is equal to 0.

Not only that, when h is equal to 0, to have the finite fields transverse fields, the E_z and H_z must be identically 0. So, when converses, that when h is equal to 0, E_z and H_z must be 0. So, let us say first, if we h is not equal to 0, then any of the quantities non zero will still give me the fields, so this field will survive.

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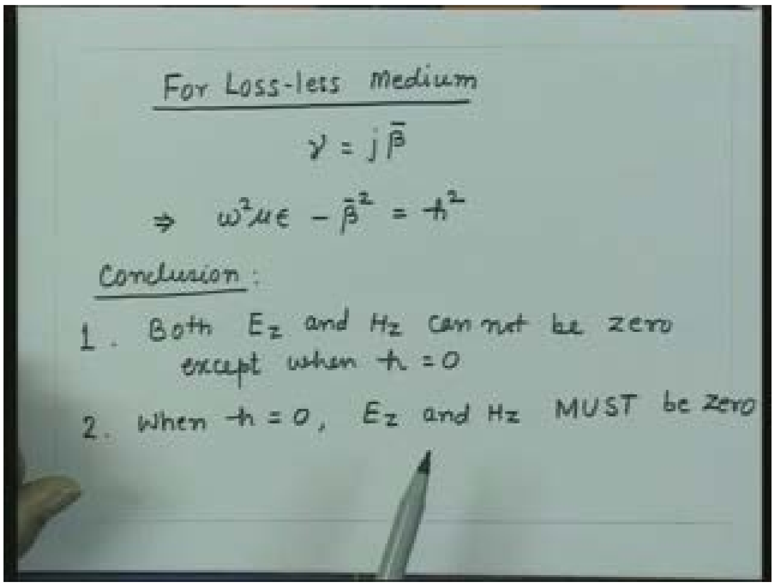

So now, if I take E_z equal to 0, I will get the fields which will be H_z and the transverse field which will be electric and magnetic fields or for this situation, the electric, since E_z is equal to 0, the electric field is completely transversed because there is only component of electric field which is E transverse and this field configuration, then we designate as a transverse electric configuration as we have done in the parallel plane waveguide.

So now, from the general analysis, we say when E_z equal to 0 and H_z is not equal to 0, we get a configuration for the fields which will be nothing but a transverse electric configuration. Similarly, when E_z is not equal to 0 but H_z is 0 that gives me now the magnetic fields for which the fields lie only in the transverse direction. So, that we get field configuration which is nothing but the transverse magnetic configuration.

The third configuration which is when E_z is equal to 0 and H_z is equal to 0, this will give me the field, transverse fields provided when h is equal to 0. But when E_z and H_z both are 0 that means we are talking about now the fields which are transverse electric and transverse magnetic field. That means these field distribution must be same as the transverse electromagnetic field.

So, what we conclude is a very important and very perform conclusion that when E_z and H_z both are 0 which represents the transverse electro magnetic wave, for this wave, h must be identically 0. So third case, when E_z and H_z both are 0, we get wave which is TEM wave transverse electromagnetic and for this, h has to be 0.

Now, if you look at this expression here which is giving me now the phase constant of this particular field distribution, I can take it on the side. So, from this expression, I get a quantity beta bar that is square root of omega square mu epsilon minus h square that is what we get from here. So, this is the phase constant of the medium and omega square mu epsilon is the square of the phase constant in the intrinsic medium which is filling this waveguide.

So, what we important conclusion which we draw here is that whether we take a transverse electric case or transverse magnetic phase, since in this case h cannot be 0 because otherwise the transverse field will go to infinitive, it will become infinite, it is goes 0. So, when any of these H_z and E_z are not 0, H cannot be 0 that means H is always a finite quantity or the velocity proportional to related to this that means the phase velocity will be a function of frequency. Whereas, if I take h is equal to 0, then this quantity beta bar will be omega square root mu epsilon and the phase velocity will become independent of frequency.

What that means is that TE mode or TM mode has to be always dispersive because the velocity will be function of frequency, whereas the TEM mode has to be essentially non dispersive because for this, now the phase velocity is going to be the velocity in that medium. So, we draw very important conclusions essentially from here and that is the TE and TM modes are essentially dispersive; that is one.

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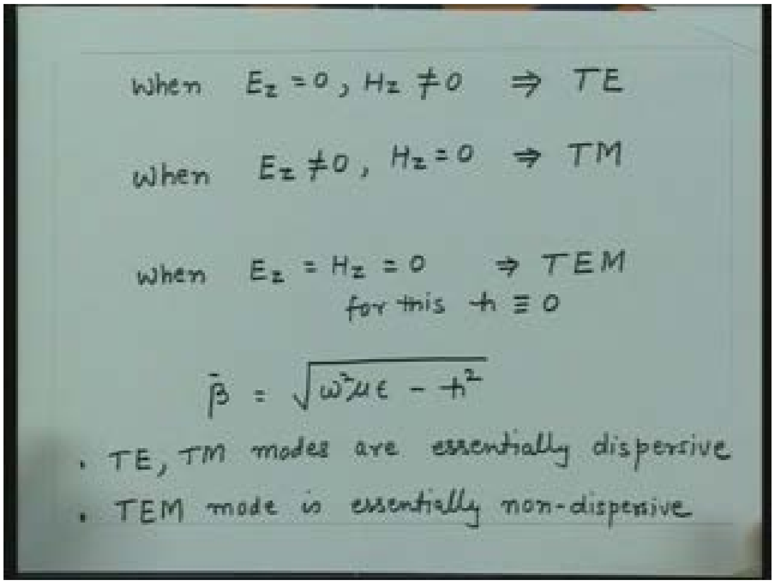

Second conclusion is TEM mode which corresponds to h is equal to 0 is essentially non-dispersive. That means whenever we try to exist the field distribution which will be either transverse electric or transverse magnetic, then we will always have their velocity varying the function of frequency; whereas when we take a mode if at all it exists and if the mode is transverse electromagnetic, then this mode will always travel with a velocity independent of the frequency.

Now, the question whether TA mode will always exist on a given structure that is the second reason. So, it is possible that every structure you may not get the transverse electromagnetic mode propagating. But what we are saying is if at all the transverse electromagnetic mode exist on the structure, then this mode will be non dispersive and it will be traveling with a speed which will always be the speed in the unbound medium which is spelling that waveguide.

So, by doing this general analysis, we reached to this very very performed conclusions and now using this expression which you have derived, now we can analyze a particular waveguide propagation. Before we take up the specific problem however, let us write down these quantities very explicitly which is del transverse for this components H_z.

See, if I have some quantity scalar quantity si which can represent either H_z or E_z, then I can write the del transverse cross some scalar quantity si, find the unit vector that is equal to x y z, d by dx d by dy 0, 0 0 si. On expanding, this quantity essentially would be equal to d si by dy with unit vector x minus d si by dx with unit vector y and similarly we have this gradient, transverse gradient of some quantity, quantity si that will be equal to d si by dx x plus d si by dy y.

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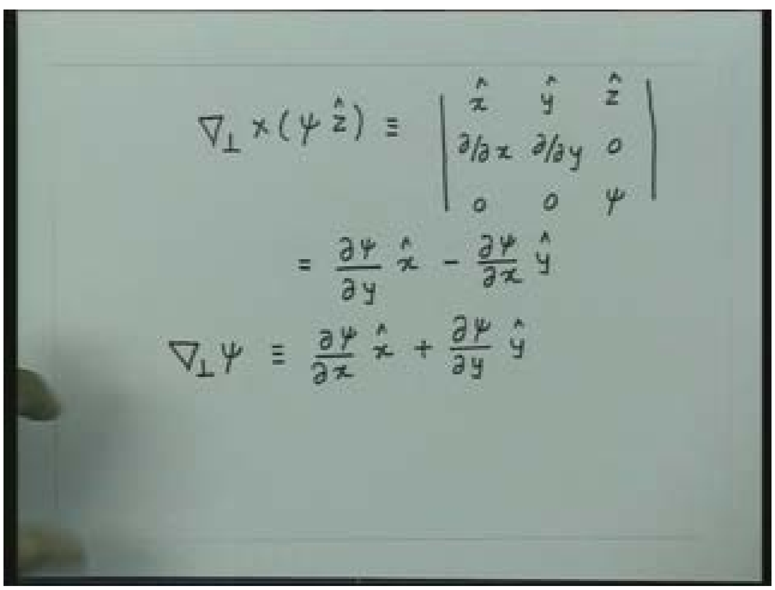

Suppose, you get these quantities now explicitly written in terms of the derivatives with respect to x and y, then we can calculate these transverse component E_x E_y H_x H_y in terms of the longitudinal components E_z and H_z. In Cartesian coordinate, then we can use this expression to get a transverse field component in terms of the longitudinal components and then the expression essentially would look like that. These expressions are quite symmetric in the sense if I look E_x here, you have minus j omega mu upon h square d H_z by dy minus j beta where beta is the phase constant of the mode upon h square d E_z by dz.

But if I compare the E_x and H_x, essentially minus j omega mu if you replace by epsilon and H_z is replaced by E_z, you get the expression for H_x. And similarly, you can see for E_y also that this is j omega mu upon h square d H_z by dx minus j beta upon h square d E_z by dy. So again, if I replace j omega mu by minus j omega epsilon or j omega epsilon by minus j omega mu, then this expression becomes this. So, interchanging H_z with E_z and replacing minus j omega mu by j omega epsilon, essentially we get the expression for the magnetic field.

So, the expressions are quite symmetric in that sense and they are very easy to remember also. So with this, now development, general development for the propagation of a mode inside a waveguide; now we are prepared to take a very specific cases and as we saw that there are two types of modes which can exist, the transverse electric mode and the transverse magnetic mode, we also saw a special case which for the transverse electromagnetic mode and now we take a very specific case that is what is called a rectangular waveguide. That is if the cross section of this pipe which we talked about is rectangular; then what kind of fields are going to propagate inside this waveguide?

So, in the next lecture, essentially we investigate the propagation characteristics of TE and TM modes inside a rectangular waveguide.