Transmission Lines & E. M. Waves
Prof. R. K. Shevgaonkar
Department of Electrical Engineering
Indian Institute of Technology, Bombay

Lecture – 21

In previous lectures we discussed the Maxwell's equations and the boundary conditions. Now we investigate the solution of the Maxwell's equations. Let us take a simplest possible problem that is if we consider a medium which is completely source free that means there are no charges, there are no currents and if the medium is unbound then we can ask the question in what form the electric and magnetic fields would exists in this medium. At this point of time we differ the question of how the electric and magnetic fields would be generated in a medium.

Let us assume somehow we have got electric and magnetic fields and then we can ask the question, what will be the relationship between the electric and magnetic fields in this unbound source free media. We also believe that the solution which we get from this equation will be the simplest possible solution which is consistent with these equations. So, we believe there is the nature of hollow the simplest possible path which is consistent with the constraints. What I mean by this is that we look for a simplest solution which is consistent with the constraints.

If I take a simple example that if the function was known at one point then the simplest solution would be the function constant passing through that point. If I know the function value at two points then it would be the linear function passing through these two points and so on. Of course when we have two points there are infinite function which can pass through these points. However, as we said we will accept the simplest solution which will be the linear solution which is consistent with the values of the functions at these two points.

Precisely, the same thing we are going to do for solution of the Maxwell's equations that we take the simplest solution task whether this solution is consistent if there is a

Maxwell's equations. If it is not then we increase the complexity of the function and then see whether the function becomes consistent with the Maxwell's equations.

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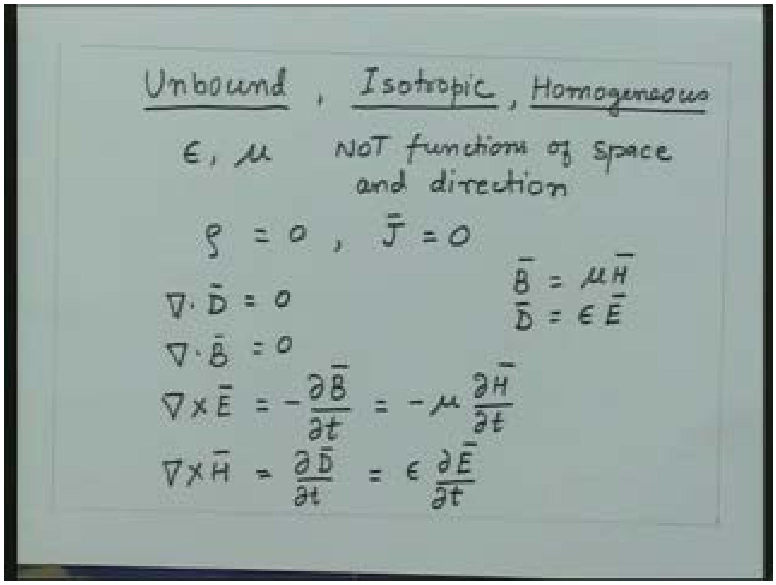

So, now let us consider a medium which is unbound, so we are now investigating a medium which is unbound that means there are no boundaries, there are no boundary conditions to be applied. So, the medium is unbound, also assume the medium is isotropic that means the permeability and permittivity of the media has scalar quantities. We also assume that the medium is homogeneous that means the medium parameters like permittivity and permeability are not functions of space and we also assume that there are no charges and currents in the medium. So, we have epsilon and mu which are not functions of space and direction.

The homogeneousness is not a function of space. The isotropic nature is that these quantities are not function of direction. Then we assume there are no sources, so the charge density is rho is equal to 0 in this medium also the current density is 0, so we have j is also equal to 0 in this medium then we can write the Maxwell's equations under this constraints and that will be del dot d that will be equal to 0 because there are no charges now, so the volume charge density is 0. Del dot b is always 0, del cross e is minus db by dt and del cross h that is equal to since j is 0, it will be dd by dt. Now, using the constitutive relation that b is equal to mu times h and since mu is scalar quantity is isotropic, so h and b are in the same direction. Similarly, d is epsilon times e, epsilon also is a scalar quantity.

However, the mu and epsilon the medium which we are talking about is a unbounded medium, so this medium is similar to like free space. However, the permittivity and permeability for the free space are denoted by epsilon 0 and mu 0. Let us consider hypothetical medium which is unbound but not necessarily free space, so we are saying that if the medium had some permeability mu, permittivity epsilon but there are no boundaries to this medium, what would be the relationship between the electric and magnetic fields in that medium and then by substituting mu equal to mu 0 and epsilon equal to epsilon 0 we can get the solution corresponding to the free space.

So, once I substitute into direct and mu and epsilon are also not time varying, so these equations can be written as minus mu dh by dt and this will be epsilon de by dt. In the electrical engineering invariably we solve the problems for the periodic signals and a periodic signal can be decomposed into this Fourier series. So, essentially if you find out the behavior of a system for a sinusoidal signal we can always find the response of the system for any periodic signal.

So, let us assume without losing generality that we investigate the problems here for the time harmonic fields that means the electric and magnetic fields both are sinusoidal functions of time. So, e and h both vary as a function of time as e to the power j omega t where omega is the angular frequency of the electric and magnetic fields.

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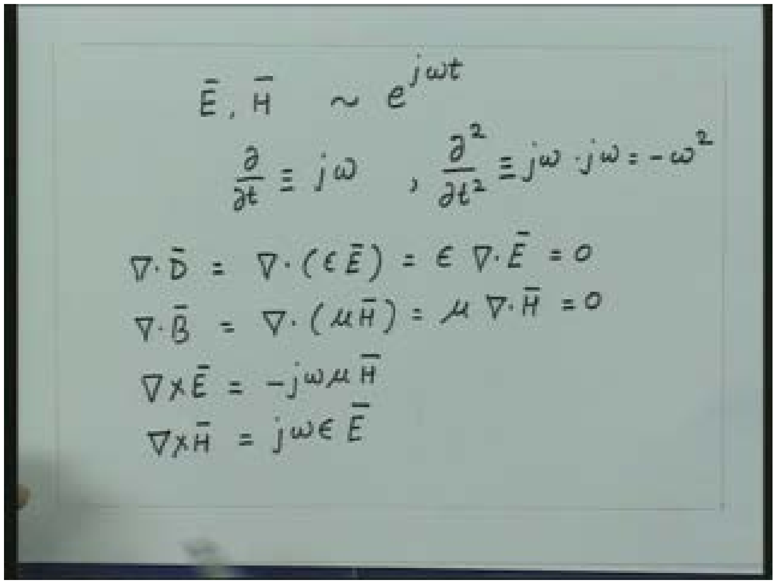

For the time harmonic field then the time derivative will be multiplying this quantity by j omega. So, any time derivative which we have in this equations d by dt can be replaced by j omega. So, a time derivative is equivalent to multiplying that quantity by j omega and the second derivative d 2 by dt square is equivalent to j omega multiplied by j omega that is equal to minus omega square. So, all the time derivatives which we have here in this can be now converted by multiplying this quantity by j omega. So, this quantity now can be minus j omega mu times h, this will be j omega epsilon times e.

So, for the time harmonic case there we can write down the equations which is del dot d equal to 0 and again substituting for d equal to epsilon times e in this equations and b equal to mu times h in this equation and since mu and epsilon are not functions of space, they can be taken outside the del sign. So, we can now finally write the equation del dot d is equal to del dot epsilon times e that is equal to epsilon del dot e should be equal to 0. And since epsilon is not equal to 0, essentially del dot e is equal to 0. Similarly, for del dot b is equal to del dot mu times h that is equal to mu into del dot h that is equal to 0 and since permeability is not zero essentially del dot h should be is equal to 0. Del cross e now is here if I substitute now for time derivative that is equivalent to multiplying by j omega, this equation is minus j omega mu into h and del cross h that is equal to j omega epsilon into e.

So, for a unbound homogeneous medium essentially the equations which we have the Maxwell's equations that is del dot e is equal to 0, del dot h also is equal to 0, del cross e is minus j omega mu h and del cross h is equal to j omega epsilon e. So, essentially now we are looking for solution of these equations, specialized equation for this unbound medium.

Now, if I look at these two equations here, here this is the space derivative of the electric field is related to the time derivative of the magnetic field. This equation if I recall is similar to what we are got for transmission lines. In transmission line if I replace electric field by the voltage and magnetic field by current then in transmission line I had derivative of voltage was equal to minus j omega l time’s current. So, mu which is permeability it has units Henry per meter, electric field having volts per meter, h is amperes per meter.

So, if I take that per meter out then this will be volts, this will be Henry and this will be amperes. So, this equation is identical to equation which we got for a transmission line, exactly similar thing is here these are space derivative of with the current that is related to j omega epsilon, so its like a capacitance multiplied by the voltage. So, hence we got the equation in case of transmission lines the coupled equations for voltage and current, this is essentially the generalized form of those equations. So, here we are having a general three dimensional space also we are having quantity here electric and magnetic field which are now scalars as we had in case of transmission line. So, we have these vector quantities which have a behavior in three dimensional space, so the space derivative is now denoted by the del operator.

So, transmission line case which we have discussed is the special case of this generalized three dimensional case, also else we have noted that time when we are analyzing transmission lines that these equations are coupled equations. So, here electric field is related to magnetic fields and magnetic field is related to electric fields. So, if you want a solution for the electric and magnetic fields we have to decouple this equations and if you recall incase of transmission lines, we have done that by taking derivative of each of this equations and substituting from the another equation we could decouple the equations. Same thing essentially we have to do in this case also, of course the derivative operation now will be in terms of this del operator. So, you have to appropriate detect that operation but principally we have to take a derivative of this space derivative of these equations and substitute from other to get the decoupled equations.

We should also note at this point that these quantities are the vector quantities and that's why when we define here the space derivatives we should ask what is the meaning of this space derivative now because this quantity is a vector quantity, in case of simple scalar functions the derivative was d by dx or d by dy or d by dz. However, when I am now talking about the space derivative in three dimensional space, I can operate space derivative on this which could be like a divergence operation or I can operate on this which could be like a curl operation. So, one possibility I take divergence of this quantity on both sides at the space derivative but if I do that that this will be del dot del cross h which will be identically 0. Well, this will be del dot h this will be equal to 0 because mu is not equal to 0, omega is not equal to 0 which is the same equation at this equation.

So, if I take divergence of this equation then I will get delta dot h equal to 0 which is same as one of the equations which you get from the Maxwell's equations. Exactly same thing if I do on this, if I take del dot del cross h that will be equal to 0. So, I will get again del dot e equal to 0 which is same as these equations. So, by having a divergence operation on this, we don’t get anything new, we simply get the set of equation which we already got from the basic Maxwell's equations. So, we can take curl of this equation. So, if I take a curl of this equation, let’s say I take the curl of the first equation which is del cross e, I get del cross del cross e that is equal to minus j omega mu times del cross h.

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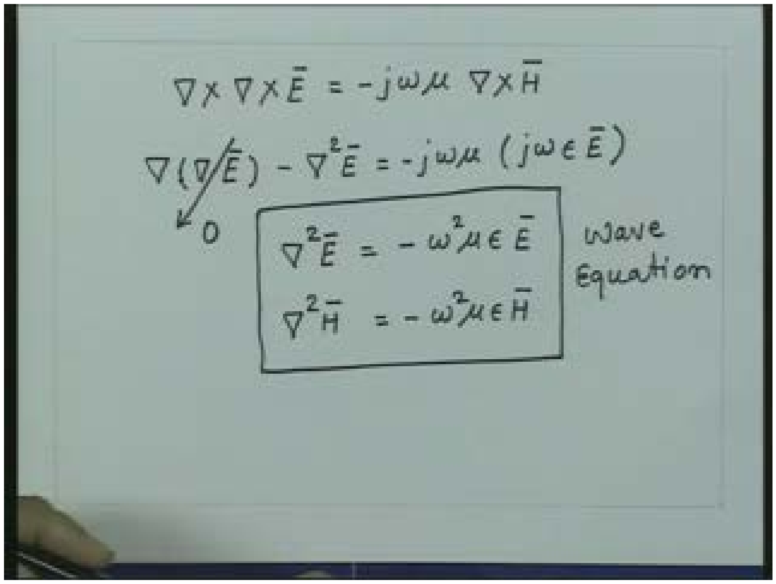

We can substitute for del cross h from the second equation from here and also can use the vector identity to expand a triple cross product which is del of, del of del dot e minus del square e that is equal to minus j omega mu and substitute for del cross h which is j omega epsilon e. Now for this medium which is the homogeneous medium and charge free medium, del dot e is identically 0. For this particular case this quantity is 0. So, this quantity is 0 in this case. So, I get the equation now finally which is del cross del square e that is equal to minus omega square mu epsilon into e.

If I attend the identical things on the second equation that if I had taken the curl of this and substitute there for del cross e from this equation, I get again the identical equation for the magnetic field and that is del cross h is equal to minus omega square mu epsilon into H. So, we have now the equations which are decoupled equations. So, this equation governs the electric field, this equation governs the magnetic field and both equations are identical equations.

Again if you recall exactly similar equation we had got in case of transmission lines, there also the voltage and current was governed by the same differential equation. So, here also we have a differential equation which is identical, there these equations where single equation because the quantity, the voltage and current they were the scalar quantities. So, this equation represented the one equation, these equations represented the one equation.

However, in this case the electric and magnetic fields are vector quantities. So, each of the equations consist of 3 equations corresponding to the components of the electric and magnetic fields. So, what that means is that each component of electric field ex, ey, ez in Cartesian coordinate satisfies these equations. The three components of h, hx hy hz satisfy these equation and del square operator as we know is a scalar operator.

So, essentially we can decompose the electric field in each of its components and every component of the electric and magnetic field must satisfy this equation. This equation recalling that the similar equation we had for transmission line and a solution of that equation physically used to represent a phenomena like wave kind of phenomena. These are the generalized form of the wave equations. So, we have this is the wave equations in the three dimensional space.

So, essentially what we are now interested is the solution of the wave equation in the three dimensional space for a time harmonic fields which are having frequency omega and for a medium which is having permeability mu and permittivity epsilon that is the problem.

Now, as we mentioned earlier we want to proceed from the simplest possible solution and ask whether the solution is consistent with this equations that is the set of Maxwell's equations and also the wave equation which you have derived from the Maxwell's equations. So, at the moment if you do not have any idea about the solution let us check whether though simplest possible solution which is the constant solution in three dimensional space will be consistent with these solutions. So, you can consider case one that the electric, let us look for the solution for the electric field to start with and the similar arguments will be valid for the magnetic fields also.

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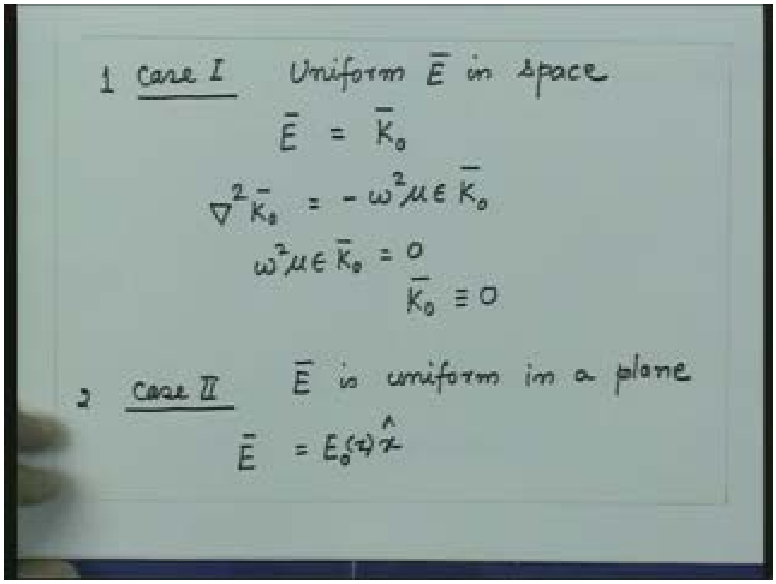

So, let us say that the electric field is uniform in the three dimensional space. So, case one is uniform E in space that means your electric field E is given as some constant k 0, E to the power j omega t is always implicit in all this quantity which we are defined. So as and when we require instantaneous values we can multiply this quantity by E to the power j omega term. If this is a solution is consistent with the Maxwell equation is must satisfy the wave equation which you derived. So, I can substitute for E equal to constant in this equation. Since, E is constant is del square is 0, these are second space derivative. So, essentially what we get is del square k not that is equal to minus omega square mu epsilon into k not and this quantity is 0, so we have here omega square mu epsilon k not that is equal to 0.

Since, we are now interested in time varying fields, the omega is not equal to 0, mu and epsilon are not equal to 0 anyway. So, the quantity which has to be 0 is cannot equal to 0. So, what this implies that k not should be identically equal to 0 for this or in other words it means that a uniform electric field in three dimensional space is not consistent with the wave equation, is not consistent with the Maxwell's equation.

So, if you are talking about the time varying fields than a uniform time varying field in three dimensional space cannot exist. One can then go to the next level of complexity that if the field is not uniform in all three dimensional space, can the field which is uniform in one plane be consistent with the Maxwell's equations. So, one can ask take the case second case that E is uniform in not three dimensional space but uniform in a plane. So, E is uniform in a plane. Without losing generality let us assume that we orient our coordinate system in such a way that the electric field is oriented in x direction.

So, up till now we are not defined which we were the coordinate system is and which direction the electric and magnetic fields are oriented. So, we have a freedom to choose the coordinate system and we decide to choose the coordinate system such that the x axis of the coordinate system is oriented the electric field direction that means by our choice the electric field is given by a some constant E 0 which will be oriented in the x direction, this is the scalar quantity I can put the unit vector which is in the x direction. So, this is the quantity now which is representing the electric field, which is oriented in x direction. And now we have to choose we are assuming that the E is uniform in a plane, once the E is oriented in x direction, there are two possibilities that E is constant in a plane which is containing the x axis or it is constant in a plane which is perpendicular to the x axis that means we are assuming that the electric field is constant in a plane perpendicular to its direction or electric field is constant in its plane.

So, in this case we are having two choices considering the plane which is perpendicular to this and that will be the yz plane or any plane which contains the x axis. So, we can have the plane which could be xz plane or xy plane. So, variation of the electric field in yz plane is one possibility, this variation in the xz plane or xy plane is the other case and these two cases are similar because now the plane is passing through the x axis. So, if I consider now a case that though electric field which is oriented in x direction is constant in a plane containing the x axis, so let us say I consider that the electric field now is varying in the x direction in the direction of its orientation. So, let us say this quantity now is only a function of x.

So, the electric field is oriented in x direction and we are assuming is varying in x direction. If this field is a solution of the Maxwell's equation, it should be consistent with the Maxwell's equations. So, this should we can take any of the Maxwell's equations and substitute this electric field in that. So, let us consider the equation del cross e that is equal to minus j omega mu into h. We can write down this curl, so it is x y z. Let us write down this in the Cartesian coordinate system which is easier to write d by dx d by dy d by dz and we have this quantity now E which is only x component which is E 0 x. So, this is E 0 function of x, 0 0 that is equal to minus j omega mu into h.

Now, since we are assumed that the field is not varying in y and z, it is only a function of x, the derivative with respect to y and z are identically 0. So, in this case we also have d by dy which is identically 0, d by dz is identically 0. So, if I substitute into this the equation xyz d by dx 0 0 E 0 (x) 0 0 that will be equal to minus j omega mu into h.

Now, this determinant is 0, if you solve this we will get all the components equal to 0. So, what that means is therefore this field which are oriented in the x direction and was constant in a plane perpendicular to that, so it doesn’t have any variation in this direction.

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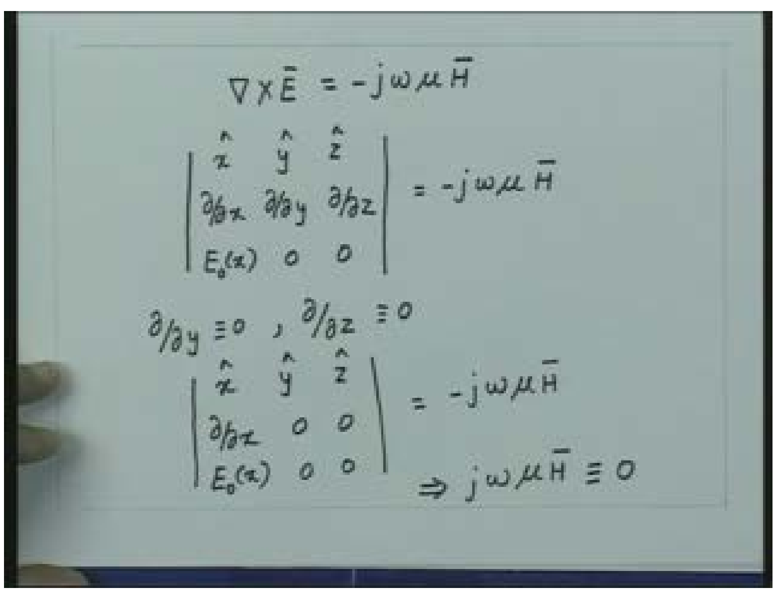

For this field though minus j omega mu into h is identically 0, so this gives me that j omega mu h is equal to 0. Again since the frequency is not zero we take time varying fields, so omega is not equal to 0, mu is not equal to 0. So, essentially it means the magnetic field is equal to 0. So, this gives me that for this field which is oriented like that and is constant in this plane, for this electric field the magnetic field has to be identically 0 so that means this is the electric field which will exist without any magnetic field.

However, we will know from our Maxwell's equations that the electric and magnetic fields are coupled, so you the magnetic field doesn’t exists in time varying case, the electric field also cannot exist. So, since there is no h in this case the electric field also cannot exist, so this field has to be identically 0. So, what we conclude in the second case that if the electric field is uniform in a plane perpendicular to its orientation then this field has to be identically 0 because this field says that the magnetic field will be identical 0.

So, now we have second conclusion that a electric field which is uniform in a plane perpendicular to its direction cannot exists in this unbound medium. So, then we can take the third possibility in this that now I have the electric field which is like this oriented in x direction. It is not constant in this plane, so it is constant in any of the plane which is passing through this and without losing generality I can take any plane, I can take either this plane or I can take this plane. Now, assume that the field is constant in this direction. So, I can assume now that the xy plane, the E 0 now is the function of z only. So, it is the constant in a plane which is the xy plane. So, I consider this electric field which is x oriented like this and I assume now that this electric field is constant in a plane containing the x axis that is the xy plane and it can have variation only in z direction.

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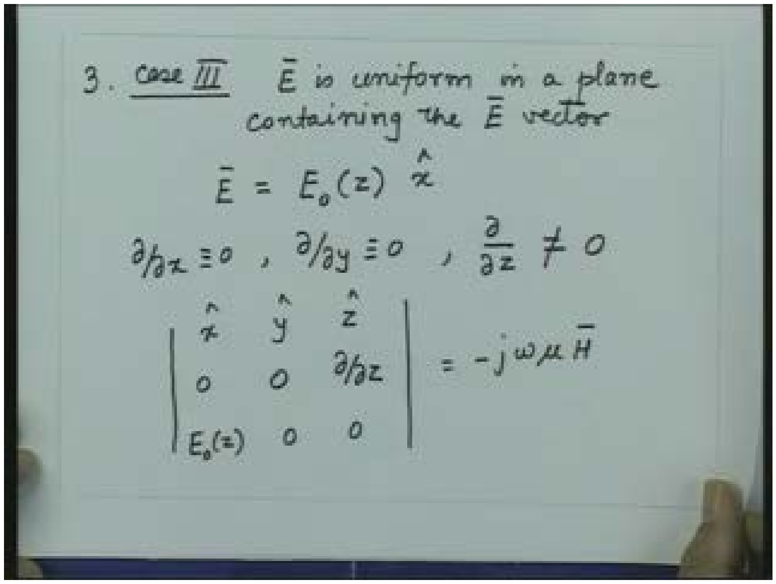

So, I have third possibility case three that E is uniform in a plane containing the E vector and in this case we are taking the E vector which is oriented in x direction, so it is uniform in a plane containing x axis. So, again without losing generality we can assume that this is constant in the xy plane and it has a variation only in the z direction. So, we can say that your electric field E is some vector E 0 which is now function of z only and it is oriented in x direction. Again we can do the same thing which we did in the previous case, we can go to the curl equation and substitute for now electric field as we did here.

However, in this case now since E is the function of z, the derivative d by dz is not zero but d by dx and d by dy are 0. So, in this case we have d by dx are 0, d by dy are 0 and d by dz is not zero. I can go back to the same curl equation and substitute, so I get here x y z, d by dx is 0, d by dy is 0, d by dz, x component of electric field is E 0 (z), this is 0 0 that should be equal to minus j omega mu into h. I can expand this determinant, so we can see here the x component of this will be minus d E 0 (z) by dz.

Similarly, the y component will be d by dz of E 0 (z) and the z component will be 0. So, what we see from here if you expand I will get the x component from here that is this minus this which is 0, so this quantity is 0. The y component will be plus y component will be d by dz of E 0 (z), so this will be d E 0 (z) by dz that is equal to minus or and z component sorry which will be this minus this which is 0, so the z component is 0 that is equal to minus j omega mu into h. So, this quantity is 0, x component is 0, the z component is 0, only there is a y component here so which gives me now that E is 0 (z) by dz y component that is equal to minus j omega mu into hx x component plus hy y plus hz z.

So, from here we see that the x component is identically 0, the z component is identically 0. I have only the y component of the magnetic field, so from here I get hx equal to 0, I get hy is equal to minus 1 upon j omega mu d E 0 (z) by dz and hz is equal to 0. So, if I consider a electric field which is oriented in x direction and if I assume that it is constant in the plane xy then the corresponding magnetic field is oriented in y direction but the important thing to note here is the magnetic field exists.

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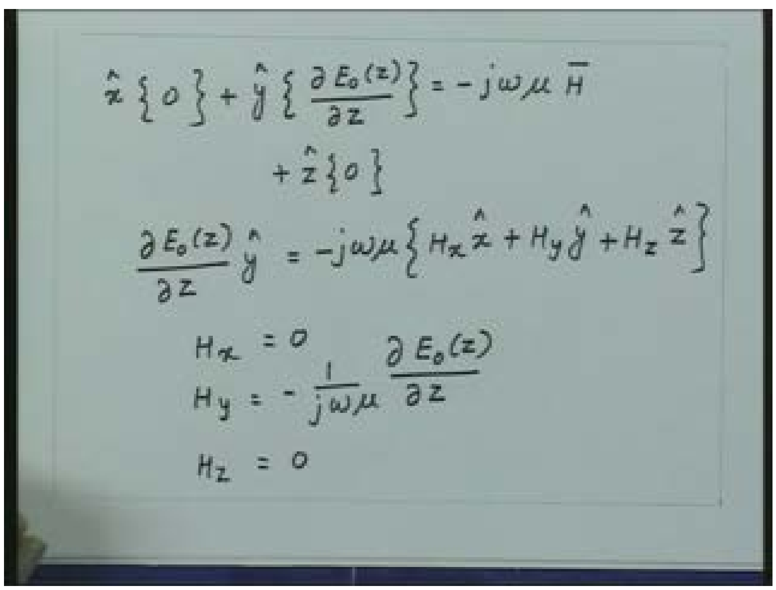

In the previous two cases we have seen that the magnetic field did not exist. In the first case we saw that the electric field does not satisfy the wave equation for the time varying fields so E 1 identically 0. In the second case when the electric field was constant in the plane perpendicular to its direction did not have corresponding magnetic fields because the magnetic fields are identically 0 and thus the reason the corresponding electric field for time varying fields at to be 0. In this particular case however we find that for the electric field which is oriented in x direction another variation in the z direction that means constants in the xy plane have the corresponding magnetic field. And since the E 0 is the function of z, it is not constant to the function of z, this quantity is a finite quantity.

So, we have now the magnetic field associated with the electric field which is x oriented. So, we now have three very important conclusions. Firstly, for time varying fields the uniform fields in three dimensional space cannot exists. We also see that a field which is constant in a plane perpendicular to its direction also cannot exist if the fields are time varying fields. So, the simplest possible fields which can exists in this medium are the once which are constant in a plane containing the direction of the field vector. So, this is the simplest solution we get for this problem. The exactly what we have did, the plane if we have considered here the xy plane in that case in that plane the fields were constant.

We could have done the similar thing for the xz plane also and we would again get a solution for which the field will be varying now that the function of y and that's why you will get again the similar kind of expression for the magnetic fields. So, we can say now in general that if the fields were constant in any plane containing the field vector. Then perpendicular to that plane the fields have a variation and that field electric and magnetic field will coexist in this medium and that's why for time varying fields this is the simplest possible solution.

Once we get this now then we can proceed and now try to get the solution of wave equation in this particular case. So, let us now go to the solution of the wave equation with this understanding that I have now the electric field which is oriented in the x direction and is the function of z only. So, let us say now the field which we are investigating that is my electric field is now equal to some function of z, E to the power j omega t is implicit and this field is oriented in the x direction.

Once we do that then I can go back to and substitute this E in the wave equation and that is your del square E that is equal to minus omega square mu epsilon into E. And since I have only one component which is the x component, it is the, take the substitute the x component into this. So, your this equation should be satisfied by the x component. So, essentially I have here del square the x component is equal to minus omega square mu epsilon into x component. I can expand this del square in the Cartesian system and that is d 2 by dx square plus d 2 by dy square plus d 2 by dz square that is operated on E x that is equal to minus omega square mu epsilon into E x.

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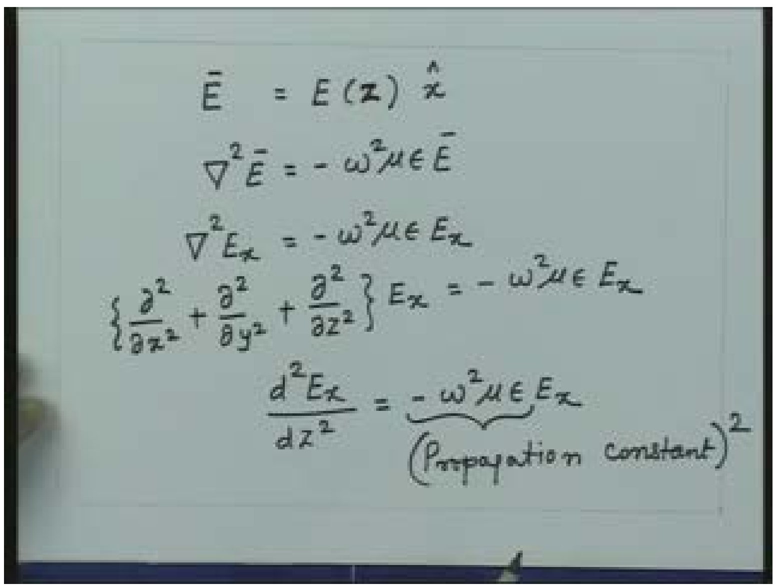

Since, E x is the function of z only now the second derivative of this E x is identically 0, dy is identically 0. So, I now get equation which is d 2 E x by dz square that is equal to minus omega square mu epsilon into E x. However, since now E is not varying as a function of x and y, we can as well convert this partial derivative in space as the full derivative because now E is, E x is a function of z only.

So, we can write down this as d 2 E x by dz square that is equal to minus omega square mu epsilon into E x. This equation now is identical to the equation which we had got for transmission line because this is now a one dimensional case. We are simply saying this quantity E x now is a scalar quantity is a second derivative in space thus for E is to get for transmission lines. So, this equation if I replace E x by v, I get the equation for transmission line. So, this case the simplest solution which we get in a unbound medium is the solution which is identical to the solution which we use to get for transmission lines and since we have investigated transmission lines in details most of the concepts which we are developed for transmission lines would be applicable in this case also.

So, in transmission line when we are investigating, the first thing we had done for other we had defined this quantity minus omega square mu epsilon which was in their transmission line case for related to r l g and c. So, we have called this quantity as what is called the propagation constant with the same kind of understating we can call this quantity as the propagation constant for this case. So, we can call this as a propagation constant square or square of the propagation constant as we defined there. So, following the notation and convention which we had for transmission line I can call this quantity propagation constant we said gamma, so we can define this quantity gamma square that is equal to minus omega square mu epsilon or gamma is equal to square root of minus omega square mu epsilon that is equal to j omega square root of mu epsilon.

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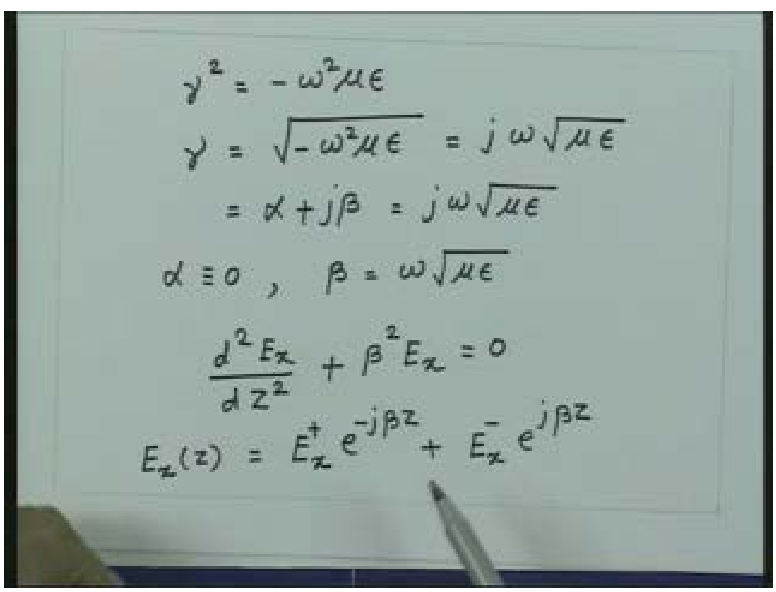

If I go by the way the parameters for gamma were defined which was gamma was nothing but alpha plus j beta where alpha was the attenuation constant and beta was the phase constant, I can follow the same notation here. So, this propagation constant gamma in this case now is equal to j omega square root mu epsilon. What that means is that for this case alpha which is the attenuation constant is identically 0 and your phase constant beta is equal to omega square root mu epsilon. And if you recall when we are discussing transmission lines alpha equal to zero case represented the loss less transmission line and what that physically meant was that when the wave travels on transmission line, the amplitude of wave does not decrease it has a constant amplitude.

Precisely the same thing should, this thing should represent here that in this case the solution of this equation will be identical to the solution like wave kind of solution for E x and the amplitude of this wave should not vary as the function of distance that is in the direction z. So, this represent now in terms of transmission line and terminology which we have developed, this would represent now a loss less propagation on transmission line and the solution as we have got for the transmission line case, these are second order differential equations once we defined this parameter as phase constant beta as we have done here then the solution of this can be written very easily.

So, we can write down this equation on here that is d 2 E x by dz square plus beta square E x that is equal to 0. And the solution of this can be easily written as E x as a function of z that will be some arbitrary constant E, we can call that as plus so we say E plus x component only E to the power minus j beta z plus some other arbitrary constant E minus E to the power j beta z.

So, on the lines similar to that use for transmission lines for this simplest case of wave propagation in an unbound medium, we get a electric field which will be combination of this two terms and we know from our earlier knowledge that this essentially represents your traveling wave traveling in z direction, positive z direction. This term essentially represents a traveling wave traveling in negative z direction. So, again we get now the solution for the electric field which is combination of two traveling waves of amplitude E x plus and E x minus traveling in opposite directions that is in positive z and negative z direction.