Transmission Lines & E. M. Waves
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Lecture – 17
Vector Calculus

In the last lecture we studied the basics of vectors, we defined the basic operations on vector like cross and dot product and then we saw the differential operations on vector.

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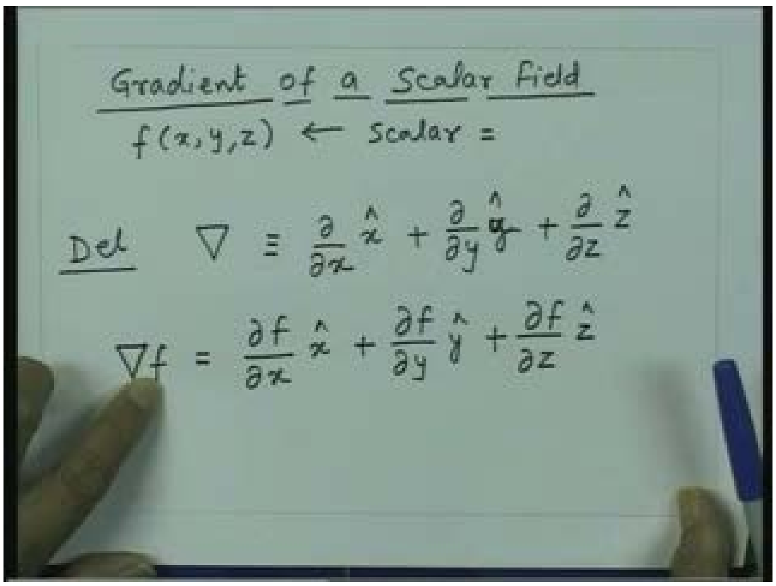

So we introduce this differential parameters which is vector operator called del which is defined as this and by using this operator then we defined three operations on vectors the del of f where f is a scalar of quantity and del of f gives you what is called gradient of the scalar quantity f so this is del of f is a vector quantity and physically we saw it tells you the maximum rate of change of this function f in the three dimensional space.

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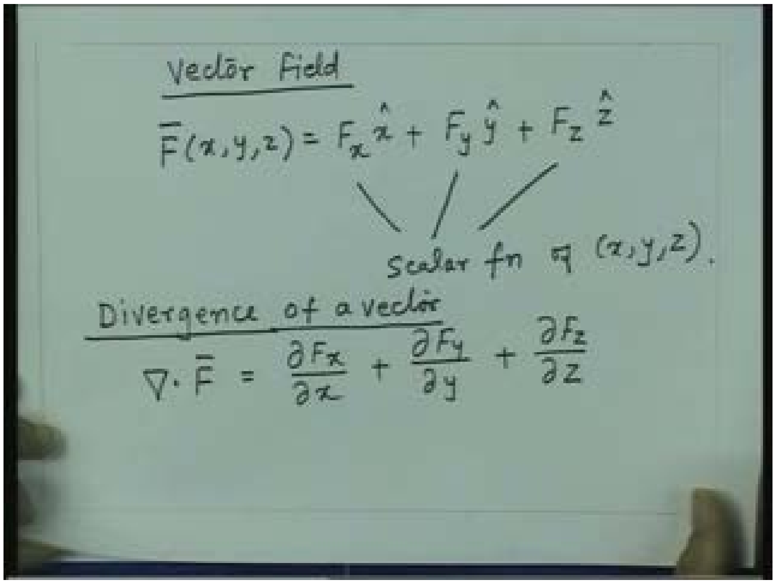

Then we defined for the vector fields the operations which was the operation del dot f and this quantity we call the divergence of vector f, this quantity is a scalar quantity. Thirdly, we had defined the vector operations (Refer Slide Time: 02:33) which was the cross operation and we called that as the curl of vector f so del cross f is given as the determinant of the matrix.

In this lecture today we will try to get a physical feel for this operation: the divergence and the curl operations and we define then the basic theorems which are used in the vector operation. So just to get a feel for the dot product; essentially if to you consider the vector field like a fluid consider a small box or small volume and ask how much is the net flow of fluid from that box per unit volume that quantity is nothing but the divergence of vector.

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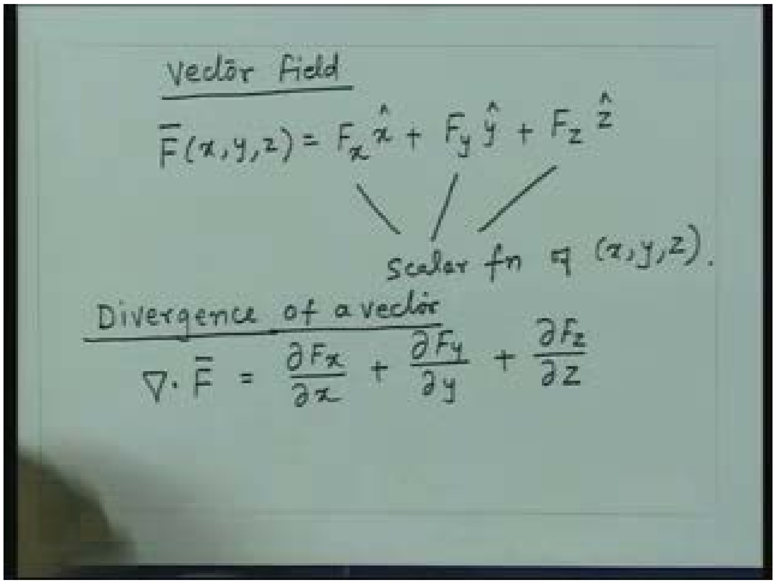

So if you imagine the vector field something like a fluid then the net flow which is coming out of the volume per unit volume will be essentially represented by the this which is the divergence of the vectors. Similarly when we look at the curl of the vector as the name suggests it is something curling or there is some rotation involved here.

If you consider the vector field and keep an object in treat the vector field like let us say again like a fluid flow or some surface of a river and try to keep some object on the surface of a river because of the differential flow of the layers of the water there will be some kind of a rotational effect which will be created on the surface of the river that is the effect which is captured by this operator what is called the curl operator.

So if you define net rotation created on an object per unit area of the object then that quantity essentially is the curl of that vector. So just to get a little better feel let us ask what kind of fields would give me divergence and what kind of field will give me the curl.

So if I draw let us say a vector field which is given by that again we are writing the arrows which are representing the vectors, so if I consider a vector field like this; if I consider the magnitude of the vector here here here (Refer Slide Time: 5:15) they are same, it is here here

here is same, here here here is same but if move in this direction this value this value is same but this value is different, this value different and so on.

Now imagine if I keep a small area here then depending upon the value of the vector it will create some kind of a shear on this object so this vector will try to rotate the object this way whereas this vector will try to rotate the object in the anti-clockwise direction so there is a net rotation effect which is created on this object. So if I have a vector field something like this then I will have a rotation created and this vector field will have a curl in this region. If I consider a field which is like that (Refer Slide Time: 6:14) that means now the field magnitude is changing in this direction so here certain value of the vector the vector increases, increases and so on.

If I consider now a small volume in this region and if I treat this vector like a fluid it will represent... the fluid which is going inside is having the value which is this whereas the field which is coming out of this will be having a magnitude which is much larger so then it will be give me a net flow of fluid it form this volume if I keep some volume at this point.

So, if I have field something like this then it will have a divergence; if I have something like this then it will have a curl. So whenever we have some kind of a flux coming out of a volume then the concept of divergence is the correct concept it can capture that effect of something oozing out of that volume or getting inside that volume.

If I have a physical phenomena where some rotational kind of effects are involved then the phenomena which can captured that effect is a curl.

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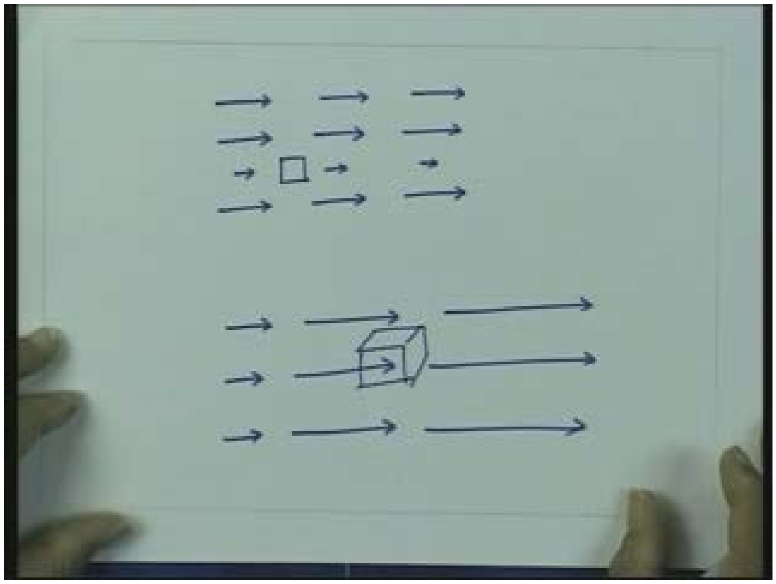

So divergence curl essentially are the mathematical operations which capture this phenomena. So you will see later on when we go to the general analysis electric and magnetic fields. Essentially the concept of physics are mathematically captured by the curl and divergence operations. Another operation which is defined in terms of the del or this differential operator is what is called Laplacian operator.

This is actually a second-order differential operator and this is denoted by del square which is equivalent to del dot del. So if I treat the del like a vector then the dot product of this del vector is the operator del square. So this operator the Laplacian operator is a scalar operator and therefore this is... if I take a dot product of the del operator (Refer Slide Time: 9:00) and if I treat this like a vector I will get the dot product of del with itself so this will become d2 by dx square so this is d2 by dx square plus d2 by dy square plus d2 by dz square.

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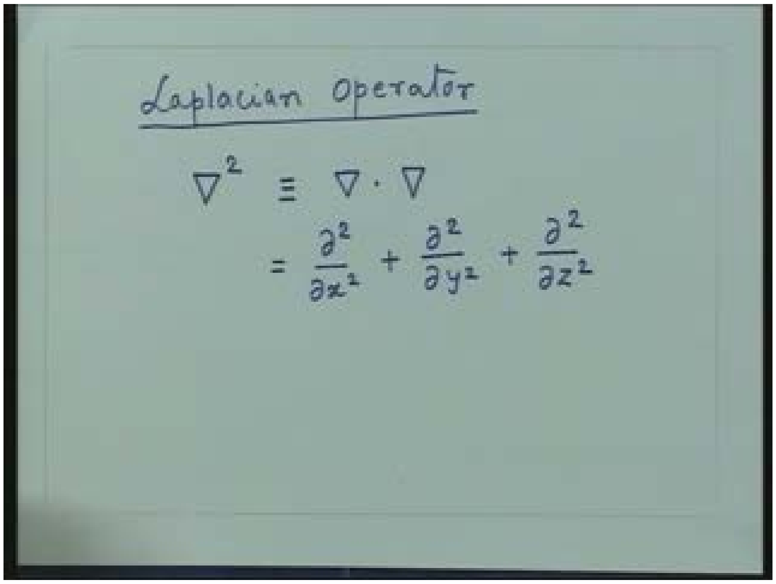

So the Laplacian operator is a second-order differential operator and this operator is a scalar operator. If I operate this on a scalar quantity the Laplacian operator will be del dot del of the scalar quantity. So, for a scalar function f function f the del square operator del square of f will be equal to del dot del of f. Since we have defined this quantity del of f as the gradient of the scalar function f and this dot represents the dot product or the divergence the Laplacian of the scalar quantity is nothing but a divergence of the gradient of the scalar function. So this Laplacian operator in this case is divergence of gradient of the scalar function.

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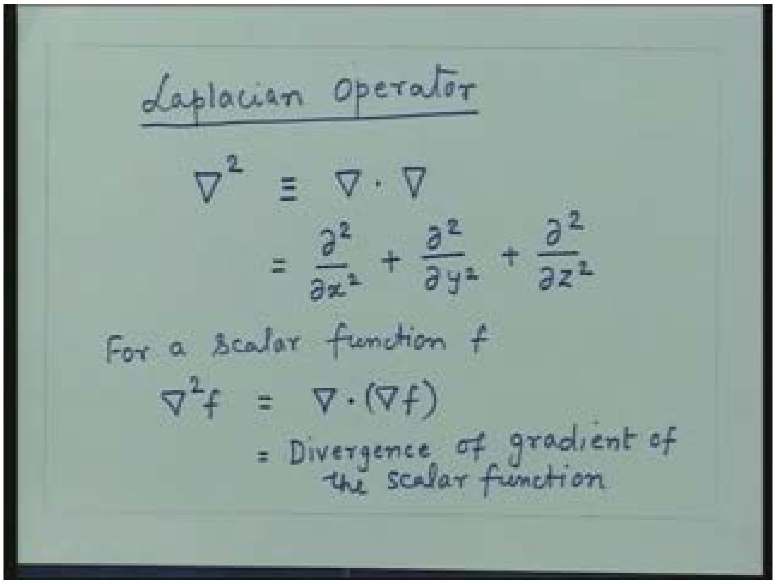

The del square operator however is not restricted to the scalar function only. The del square operator can operate also on the vector. And if I follow the same thing as you have done here that del dot del and if I put a vector in front of it I will get del dot del of the vector f capital f but that will not have any meaning because you know it defines the operation del of the vector quantity. When the del operates on a vector quantity it can operate either a divergence or the curl. So in this case when you operate this one on the vector quantity we directly take this and operate this directly on the vector quantities.

So, when you have a Laplacian of a vector f then del square of f where f is a vector quantity that is equal. Operating this (Refer Slide Time: 12:13) on all the three components of vectors so essentially you take the second derivative d2 by dx square of the entire vector which is three components; you take second derivative with respect to to y of the entire vector and you take second derivative with respect to z for the entire vector.

So in this case although the del square operator is a scalar quantity that operator is a scalar operator when it operates on a vector you get a quantity which will be a vector quantity. So, when del square operates on the scalar quantity you get a scalar function. When you operate this

this on a vector quantity you get a vector quantity. So in this case you will get d2 by dx square f plus d2 by dy square f plus d2 by dz square f.

What it means is if I expand f in its components f x f y f z you have d2 by dx square for all the three components, d2 by dy square for all three components and d2 by dz square for all the three components. So if I expand this and if I say the f is f of x into x plus f of y y plus f of z z and I can substitute into this so del square of f will be equal to d2 by dx square into f of x x plus f of y y plus f of z z plus d2 by dy square same thing plus and so on. Then I can combine the next component in all the components. So, that will give the d2 f x by d x square d2 f x by dy square d2 f x by dz square that will be the ex-component of this vector. So this will be d2 f x by dx square plus d2 f x by dy square plus d2 f x by dz square that will be the ex-component plus similarly for y component and so on.

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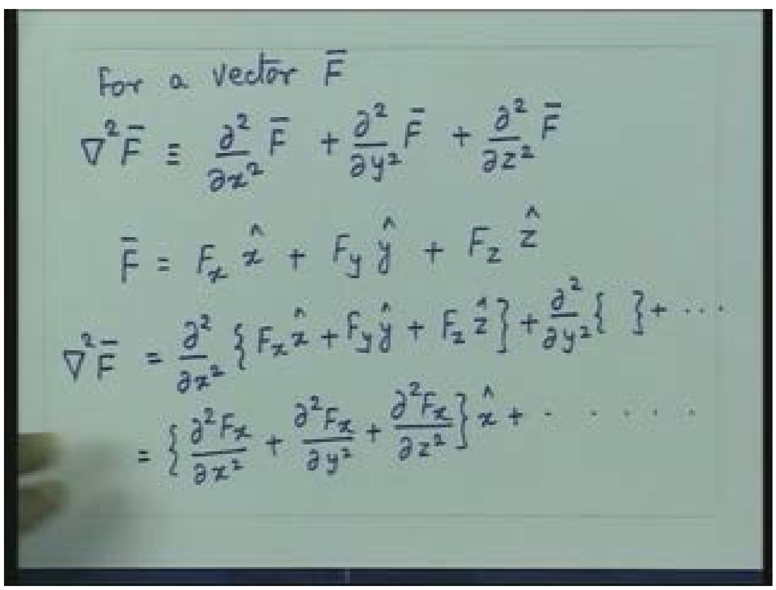

So essentially I take this Laplacian operator which is operating on each of the components. So this is I can say this is equivalent to saying this this is del square of f x into x plus del square f y into y plus del square f z into z.

So if I take each of the components of the vector make the scalar Laplacian operation on each of the components and write the final vector that will be the quantity which will be del square f where f is the vector quantity. So later on when we do the analysis of the vector fields we will see this operation will be needed for solving the electromagnetic problems.

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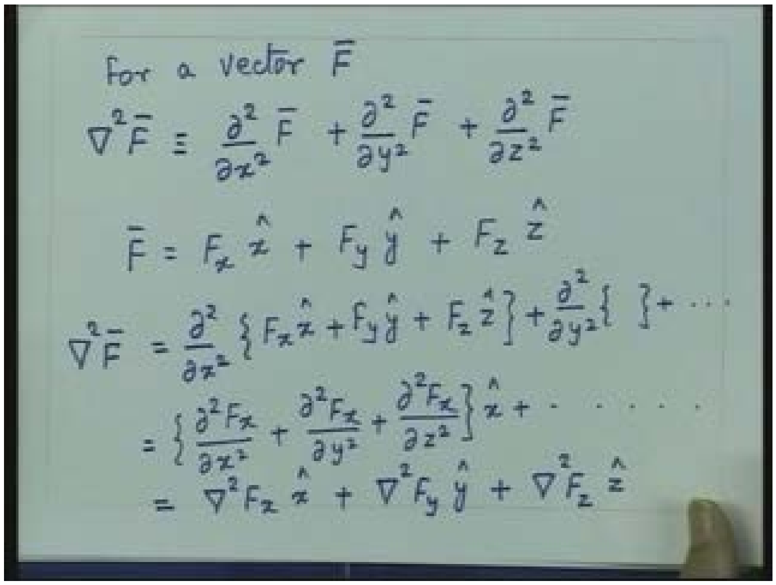

These are the differential operators and then you have to do the operations which are the integration of the vector fields so we require certain operations which we can do in the integration kind of operation. So if I have a vector field there is a possibility that I can take the integral of this vector field in a plane along a contour or along a line so if possible I have certain vector fields and I can take the integration around a path a contour. If possible I can take the integration of this vector on a surface which could be a closed surface or open surface so I can have something like that is a surface; I will require the vector field which is something like this, here again vector field could be like that so I will require the integration of this vector along this path, I may get the integration of this vector on this surface or I can take the integration over the volume if is the surface is a closed surface.

So now when we do the integration if the integration is done along this path we call this as the contour integration; if I take the integration over a surface we call that as surface integration and since surface is a two dimensional thing essentially we will have a double integral of the surface integration; the contour integration is a single integration and if I go to a volume integration then it will be a triple integral in the three dimensional space.

Then we have an important theorem which connects the line integral to the surface integral and surface integral to the volume integral. So if I have a volume here I will have integration over the volume, so I have here line integral, I have surface integral and I have volume integral and invariably we need to change from one integration to another; we may have to convert from the line integral to the surface integral or from surface integral to the volume integral. It should be kept in mind, however, that the integral is a scalar quantity. So when we do the line integration the final answer will be a scalar quantity, when we do the surface integration the final answer will be a scalar quantity.

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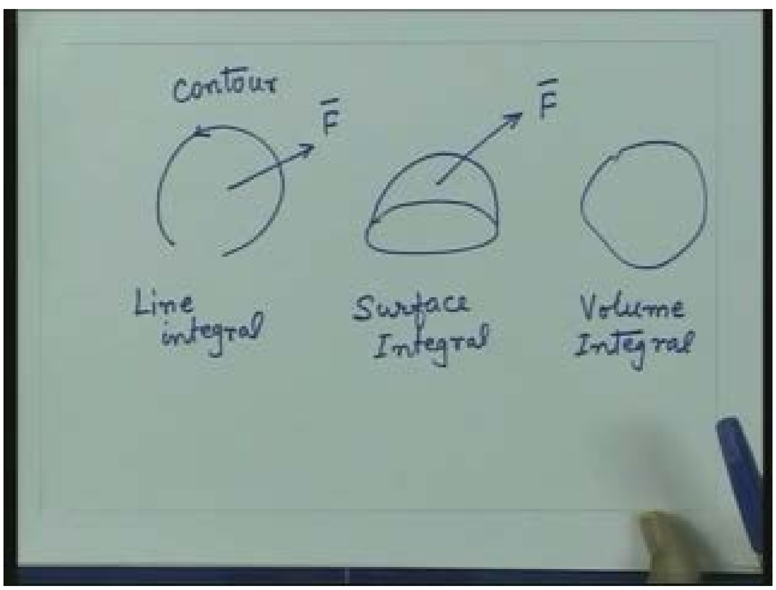

So now we can define the surface integration and the volume integration and the line integration that is if I take some vector A some vector field and say denote it by A is my vector field then the line integration around a path of contour some contour c is integral A dot dℓ where dℓ is the segment along the length. So this quantity is dℓ and this is the direction of segment; if I am integrating along the contour like this so dℓ is a vector quantity because it has a length and it has a direction. If the contour is closed contour then I will have the integral which will be a closed integral. So here this is open integral, open contour. If the contour is closed then integral will be denoted by this: A dot dℓ this is closed contour.

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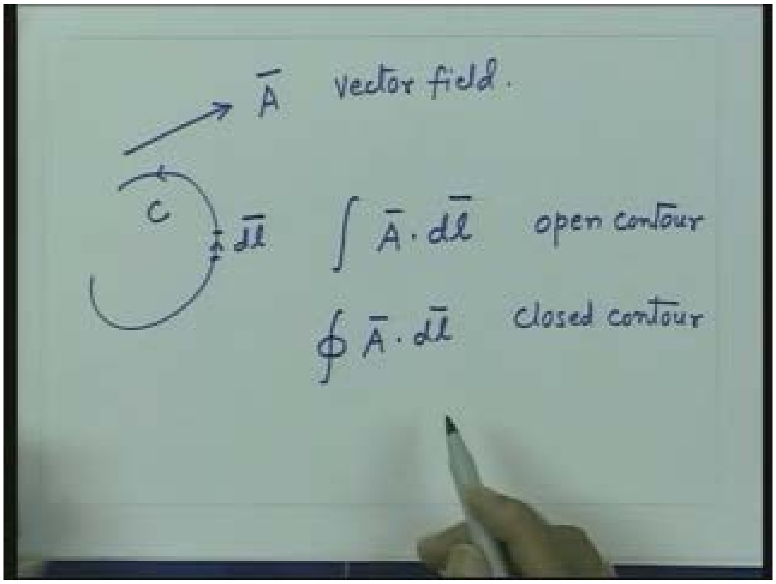

So, at every location essentially what we are doing is we have this vector A, we find out the dot product of A and dℓ at every location, you add up that along the contour and that is this line integral. So this is the line integral (Refer Slide Time: 21:39). We can do the similar operation for the surface integration.

So let us say we have some surface like this and there is an infinitesimal area on this which is given by da, this is the direction of the area (Refer Slide Time: 22:04) this is the unit vector in the direction perpendicular to that infinitesimally small area and the vector field is again A so I can define the surface integration; again the surface may be closed, it may be open so here this vector and the area which is electrical quantity so this is infinitesimally small area and the direction of the area is the direction which is perpendicular to this area so da vector area is the infinitesimally small area and then multiplied by the unit vector which is in the direction perpendicular to this area. So, if I take the dot product of these two A and da and if I sum up over the entire surface, that gives me what is called the surface integral.

So we have here now surface integral. So, if I have a surface which is an open surface then I have this double integral A dot da this is for open surface and A dot da for the closed surface. So the integration now is a double integration because we are talking about a surface and if you write the integration like this then it represents the open surface, if you write loop around the integration then that represents integration around the closed surface. Again you can note here we are taking a dot product of the vector A and the area at that location. So this quantity is a scalar quantity (Refer Slide Time: 24:29) so this integration is going to be a scalar quantity.

Now, direction of the normal which we take for A if it is a closed surface then the direction of normal generally is taken as the outward's normal from that volume that direction is taken as the positive unite direction. However, if you consider a surface which is an open surface then of course there is no preference for defining the unit normal so we can define the unit normal in either direction.

If I consider a surface like this the unit normal can be defined as this way what can be defined either this way. If I am having a surface which is a closed surface like this then the normal which is coming out from this surface that normal is taken as the positive direction. So, while defining the unit normal for the surface there is no specific notation. However, later on when we will try to connect the contours to the surface for an open surface that time we will follow again the convention of the right hand rule and then we will specifically choose the direction for the unit normal for the surface in the contour.

So in this case let us say we can define if your volume was closed, surface was closed then we define the unit normal coming outwards if the surface is open then any direction can be taken as the direction of the unit normal.

The third possibility we said is the volume integral. So since we are now defining this quantity over the volume we have some function which is a scalar quantity which is filling this volume. See if I integrate the total scalar quantity over this volume that gives me the volume integral. So I have some volume here and I have some scalar quantity filling this volume which is given by f. if I take an infinitesimally small volume in this that volume let us say is given by dv where v is the volume then integration over this will be a triple integration so we have here volume integral which is triple integral; this function f dv.

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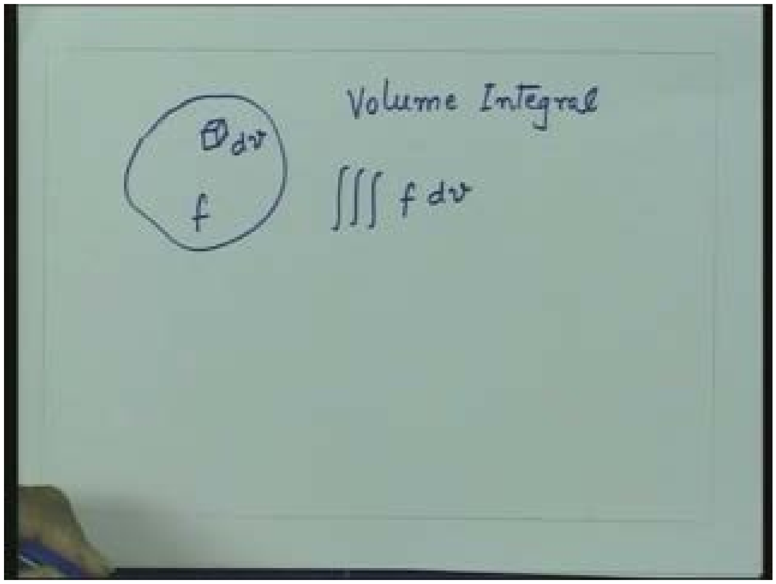

So, essentially when we try to now relate the different integrations: the line integration to the surface integration to the volume integration essentially the vector fields first are operated by the operator del and then there is the relationship between these del operated fields in the integration domain. So there are two important theorems which essentially relate this and they are called divergence theorem and the Stokes' theorem. So we have an important theorem what is called a divergence theorem which converts a surface integral for a vector field to a volume integral.

So if I have a vector field A vector field A then the divergence theorem states that for a closed surface the integral A dot da that is equal to the volume integral of divergence of A. So if I consider a surface it has a total surface area let us say is given by s and the total volume of this which is v. If I take a surface integral of the vector A on the total surface S then that is equal to the volume integral of the divergence of A over this volume.

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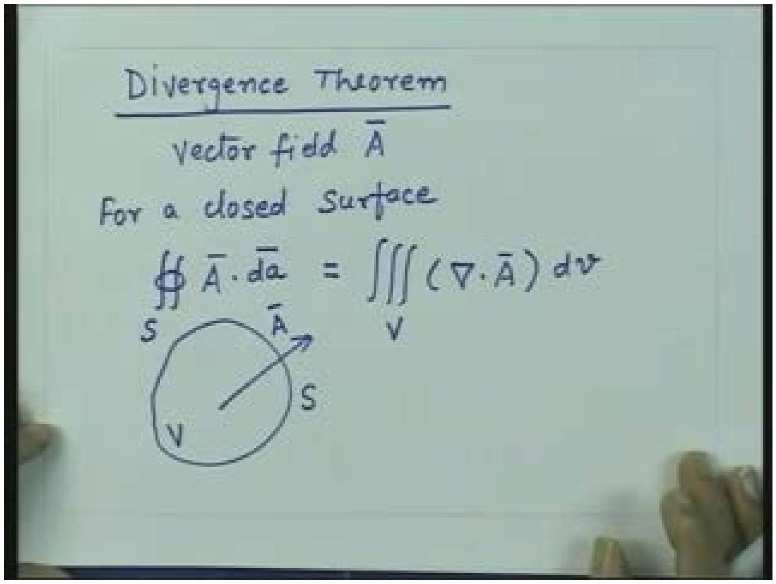

So if you are having a vector and if you know the surface integral of this vector the surface integral can be converted into a volume integral by operating this vector A with a del operator with a dot product that means it makes a divergence of vector A and take the volume integral of that that is same and the surface integral of the vector A. This theorem is called the divergence theorem. So whenever we have a need to convert from volume integral to surface integral or vice versa this theorem comes very handy.

Another theorem which is again an important theorem and that converts the surface integral into the line integral and that theorem is called the Stokes' theorem.

If I have an open surface now something like this I have a contour of this surface so this is the contour c and I have this surface so I have some surface area s. So the surface is open and the boundary of this surface is the contour c. Now I can have the unit area here and that is the direction of let us say n which is the unit vector for the area, then I have this contour for which I have to define the direction of ℓ; when I can do integration I will require ℓ so essentially first you have to say how should I define this ℓ with respect to n and the convention is that if I again follow the right hand rule if my contour goes like this then will be like this so the n will be coming inward. If I take contour which will be like this (Refer Slide Time: 32:16) then the n will be going outwards.

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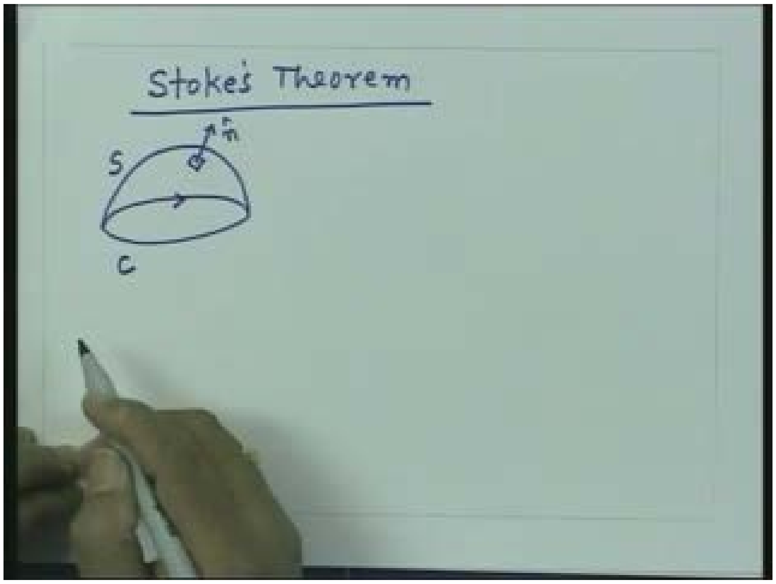

So, if I take the direction of the contour like this that means the dℓ vector if I define that way so if I take the contour positive direction like this then the direction of the unit vector for the surface will be like that. If I change the direction of the contour integration in the opposite direction the normal direction of the surface integral will be inwards. So, for this then we can write the contour is a closed contour in this case so the stokes theorem states that over this contour c the line integral of this vector A you have a vector field again here which is A that is equal to the surface integral and in this case it is open surface curl of A dot d.

So the line integral of this vector a on this contour is equal to the surface integral of the curl of this vector A. So this theorem can be used now for converting the line integral to surface integral and vice versa. So again summarizing this (Refer Slide Time: 34:04) if I have a closed surface then the surface integral and the volume integral can be related by the divergence theorem and if I have an open surface then the line integral and the surface integral can be related to the Stokes' theorem. So, later on for solving the problem of electromagnetics especially when we write the Maxwell's equation for the integral and differential form these two theorems become very important, become very handy in converting from the differential to the integral form.

Having understood these basics of the vectors now we can go to the basic quantities of the electromagnetics which we are going to make use of in further analysis that is the quantity like electric and magnetic fields when the origin of the entire electromagnetic phenomena is the basic charge. See if I consider charge you know from our basic Coulomb's law this charge has the effect around it so if I put another charge in the vicinity of that it experiences a force. This effect is measured by a quantity what is called the electric field.

So, if I take a charge and go into the vicinity of that I experience a force which is characterized by a quantity called electrical field. However, if I keep this charge in motion then it constitutes a current because current is nothing but the rate of change of charge so if the charge starts moving you have a sustained flow of charges that simply gives you current and then we have the magnetic fields. So the same charge well it is static stationary it gives you the electric field and when it starts moving then it gives you current and that gives you the magnetic field.

The charge gets accelerated also. So if you accelerate the charges then it gives you the electric and magnetic fields both. So essentially we are dealing with the quantities here: the charges, currents, electric and magnetic fields and try to establish the relationship between these quantities. The relationship which we have between these quantities is given by what is called Maxwell's equation.

So essentially now starting with the basics of these quantities we will try to establish the relationship between them from the laws of physics and when we write the laws of physics in the mathematical form using the vector notation and the vector theorems which you establish we get what is called the Maxwell's equation. So the quantity which we have now is what is called first quantity is the electric field E which is nothing but the force experienced by unit charge.

So if I have some charge here by Coulomb's law I will experience a force at the location in the vicinity of this charge. If I measure this force per unit charge that quantity is what is called the electric field at that location. Since here we are talking about the force per unit charge, the electric field, the vector quantity so it has the directions and it has the magnitude. The unit of electric field is volts per meter. So electric field is given by volts per meter. then we have a medium property that if I measure the electric field in let us say vacuum I will get certain force, if I measure the same thing if I change the medium parameter to some other direction then the force will change so the quantity which does not depend upon the medium parameters is the electric displacement vector.

So the electric field is a quantity which is related to a charge which is producing this field. But it also is related to the medium parameter which is what is called the permittivity of the medium. So we have a medium parameter called permittivity which is denoted by epsilon and it has a unit: Farad or meter. So if the medium parameter changes the permittivity of the medium changes and because of that the electric field measured at that point changes. the permittivity of the vacuumed or the free space is denoted by epsilon 0, so for the free space we have epsilon is equal to epsilon 0 and its value is approximately 1 over 36 pi into 10 to the power minus 9 Farad per meter.

So, in the space which is nothing which is vacuum; even for that the permittivity has a value and which is... many times this is also written as 8.86 10 to power of the minus 12 and so on. However, it is easy to remember 1 upon 36 pi into 10 to the power minus 9 rather than remembering it as 0.8 something. Generally we prefer to write the permittivity of the free space as 1 over 36 pi into 10 to the power minus 9 Farad per meter.

Then if you take another media whose permittivity is some epsilon, the ratio of epsilon to the epsilon 0 is what is called the dielectric constant or the relative permittivity of the media. So we have the dielectric constant epsilon r also called as relative permittivity that is equal to the permittivity of the medium divided by the permittivity of the free space that is epsilon 0. Then the quantity which is independent of the medium parameters that is what is called the electric displacement vector that is the product of electric field and the permittivity of the media.

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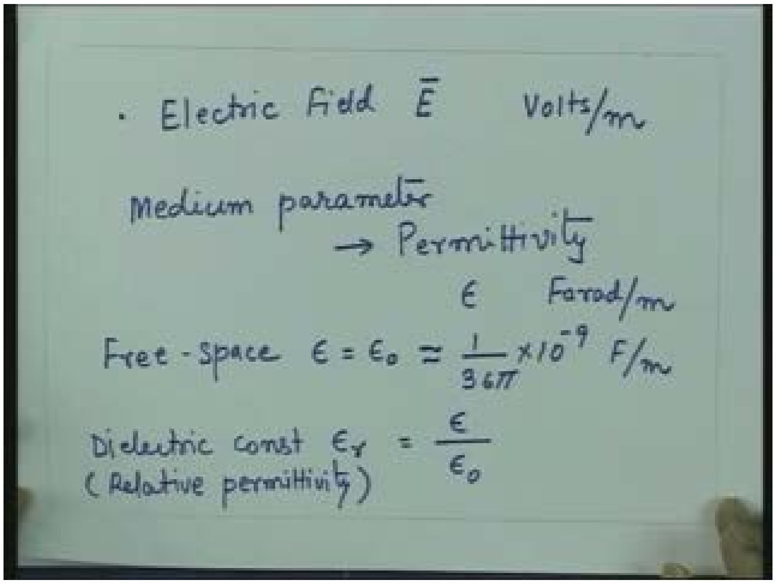

So we have a quantity electric displacement vector denoted by d which is equal to permittivity of the media multiplied by the electric. So, when the medium property change the epsilon changes but the d remain same; d does not depend upon the medium properties it depends upon the actually charge which is creating this field irrespective of which media it is creating this field. So d remains same, it does not change from media to media, so the quantity which changes is the electric field depending upon what is the permittivity of the medium.

Now this quantity epsilon in a general media can be constant everywhere, can be uniform or it can vary as a function of the space in three dimensional space. It can also depend upon the direction. What that means is if I measure the permittivity in certain direction it has certain value, if I measure the permittivity in some other direction it will have another value, so in general this quantity epsilon may be direction dependent, it may be space dependent so if epsilon varies as a function of space then we call the medium as a inhomogeneous medium.

So epsilon varies with the space, this gives you the inhomogeneous medium. If epsilon is a function of direction, if epsilon is direction dependent then the media is called anisotropic media. So if epsilon varies as a function of space we call the medium inhomogeneous, if the epsilon is direction dependent then the media is called anisotropic, if the media is not varying as a function of space then we call that medium homogeneous and if epsilon is not varying as a function of direction then we call that media as anisotropic media.

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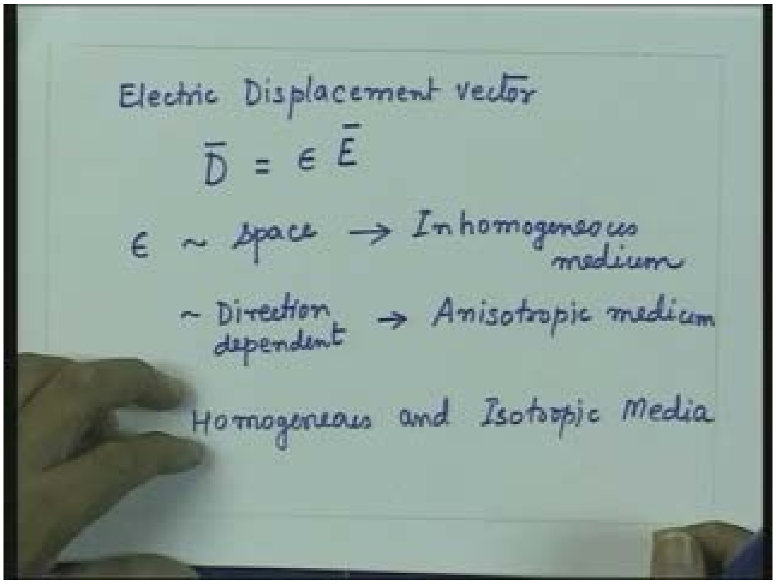

In this course we deal with the media which are homogeneous and isotropic media. So here we consider only homogeneous and isotropic media. What that means is that the dielectric constant of the medium or permittivity of the media is neither direction dependent nor it is varying as a function of space. In general, however, if the media was an isotropic then this quantity epsilon is not a scalar quantity. In fact it becomes a 3 by 3 matrix and D is now equal to this 3 by 3 matrix epsilon multiplied by this E which is the vector.

So, for anisotropic case the epsilon is a 3 by 3 matrix whereas if I take a medium as isotropic then epsilon is a scalar quantity. So, in this course we essentially deal with the media for which epsilon or the dielectric constant is a scalar quantity. What that means is that in a homogeneous medium if this epsilon is a scalar quantity that D is nothing but a scaled version of E. So if I compare D and E they have different magnitude these two vectors but the direction of E and D are same that means in anisotropic medium the displacement vector and the electric field they are in the same direction. However, if I take the medium which is isotropic that in general this is a 3 by 4 matrix so essentially it rotates this vector E and in general the displacement vector and the electric field vector are not in the same direction.

So we say that this is E and this D (Refer Slide Time: 48:11) this is E, this will be the case if the medium is isotropic whereas this will be the case if the medium is anisotropic because we said that the displacement vector D is given in this case by 3 by 3 matrix so let me just write that as epsilon x x epsilon x y epsilon x z epsilon y x epsilon y y epsilon y z epsilon z x epsilon z y epsilon z z multiplied by this vector which is Ex Ey Ez. So this is equal to three components of the vectors D x D y D z.

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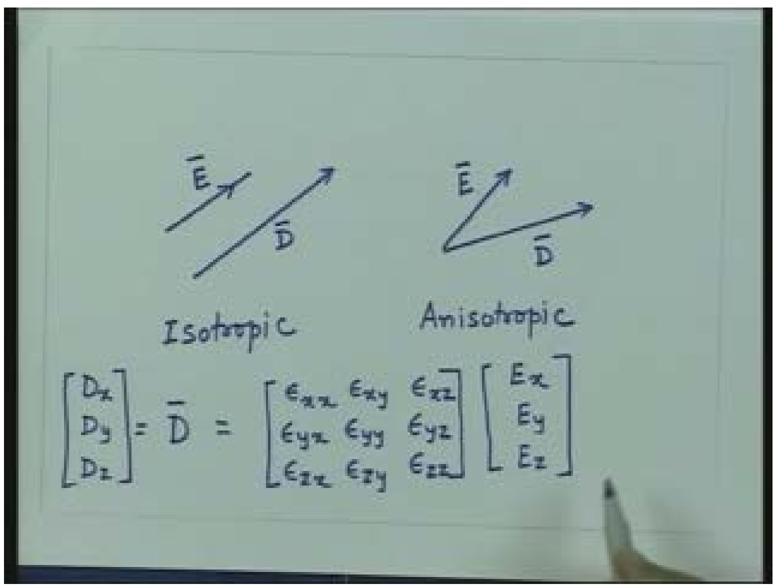

So you can see here; if I had a vector whose components are E x E y E z after transforming to this matrix which is the permittivity of the anisotropic medium that will give me displacement vector which will not have the same direction of the electric field so I will have a situation as something like that. Then we have a quantity which is useful which we define and that is the electric potential at a point in the field and that is nothing but the negative gradient of the potential is the electric field. So the electric field is related to the electric potential; electric potential is the scalar quantity V and that is related to the electric field as the electric field is minus gradient of the voltage.

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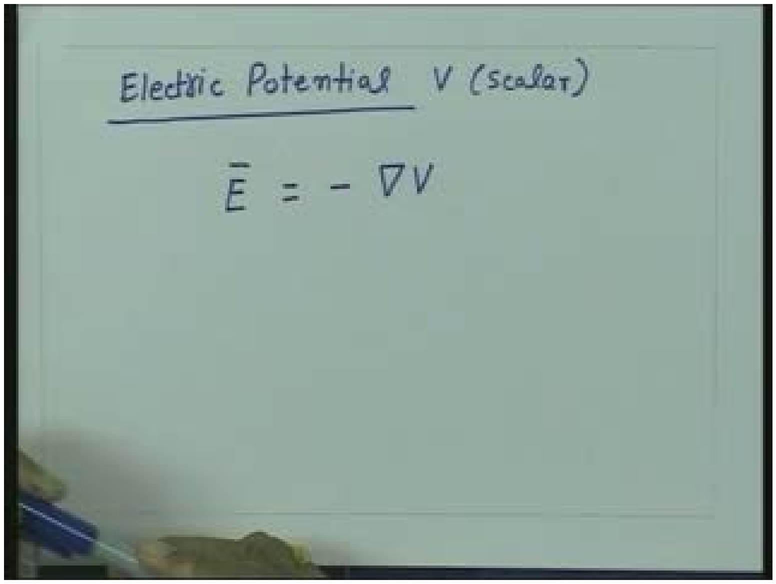

So if I know the potential at a point then I can take the gradient of that and that gives me the electric field at that location. From here the unit which we have got for the electric field essentially comes from this definition. But here we have defined the potential which is which is like that, the unit for the voltage is volts, the del operator is a differential special differential operator so it is d by dx so its units are 1 by length or per meter so that gives me the unit of the electric field which is volts per meters.

So this relation for converting or for finding out the electric field from the potential it turns out very handy; whenever we find try to find the electric field in a general complex distribution of the charges, if you calculate the electric field at a particular location and if I find the electric field because of each component of the charges which are distributed in space you have to carry out the vector additions at that point for the electric field so generally it turns out to be easier to find out the potential at that point due to all the different charges; I can add those potentials because these are scalar quantities and then you find out the gradient of that that gives me the electric field.

So these are the very basic qualities for representing the electrostatic parameters: the electric field and the electric potential. When you meet next time then we will try to now establish the basic laws which connect the electric displacement and the charges which are responsible for creating the electric displacement. And then subsequently a similar analysis we will carry out for the magnetic fields for the magnetic flux densities.

So let us summarize what we did today. Today we saw saw some of the more vector operations, we saw two very important theorems for the vector integration what is called divergence and Stokes' theorems which converts the surface integral to volume integral and line integral to surface integrals and then we also saw other basic relationships between the displacement vector and the electric field and a parameter what is called permittivity of the medium and how it changes for isotropic medium to anisotropic medium.