Information Theory and Coding
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Lecture - 38
Variable Length Coding and Problem In Quantizer Design

Design of a quantizer mainly involves three steps; first selection of decision boundaries, second selection of reconstruction levels and third selection of code words. Two performance measures which are associated with the design of a quantizer are minimization of distortion that is minimization of mean square quantization error, and minimization of the rate of the quantizer. So far, in our study of quantizer we had assumed fixed number of quantization levels and fixed length coding. And therefore the selection of code word was not an important issue, because the rate of the quantizer was decided by the quantizer alphabet size.

But now if we relax the restriction of fixed length coding and move on to variable length coding, then selection of code word will be an important issue. It will decide the rate of the quantizer which in turn is lower bounded by the entropy of the output of the quantizer. Now, there are two different approaches for variable length coding, the first approach is to directly incorporate the entropy of the output of the quantizer in the design of quantizer itself which is a difficult approach. A simpler approach would be to design the Lloyd max quantizer, which considers only the minimization of distortion as a criterion and then entropy code the output of this quantizer.

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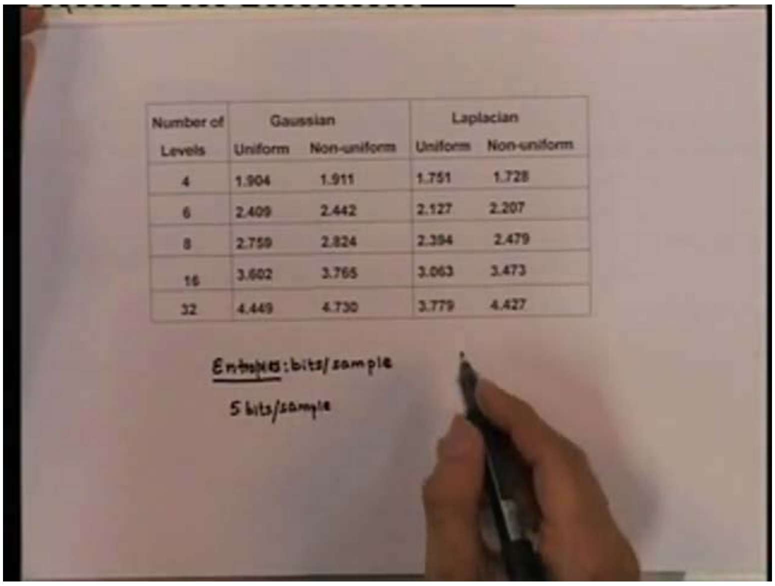

Now, let us study the second approach; first this table depicts the entropies in bits per sample for the quantizers designed, based on the criterion of minimization of distortion, that is minimization of mean square quantization error. So, there are two inputs; one is Gaussian P D F another is Laplacian P D F. For both these inputs, we design quantizer based on uniform P D F optimized and non uniform P D F optimized principles. Now, there are certain observations to be made from this table; first we find that for the lower levels that is low number of quantization levels, the difference in rate is relatively small between the fixed rate and entropy coded cases, but this is not the case for higher levels.

For example, for 32 levels, a fixed rate quantizer would require 5 bits per sample, however the entropy of a 32 level uniform quantizer for the Laplacian case is 3.779 bits per sample, which is more than 1 bit less. Another observation is that difference between the fixed rate and the uniform quantizer entropy is generally larger than the difference between the fixed rate and the non uniform quantizer. This is true for both Gaussian and Laplacian P D F.

So, for example, if we take number of level to be 2, the fixed rate quantizer which will require 2 bits per sample whereas, uniform quantizer would require 1.904. So, difference between 2 and 1.904 is more than the difference between 2 and 1.911, this is the case given for the Laplacian P D F. Now, the reason for this is as follows; non uniform quantizers have similar step size in high probability regions and larger steps in low probability regions. Therefore, the net effect is that the probability of input falling into a low probability region and the probability of input falling in a high probability region are approximately close together. Because of this, the output entropy of the non uniform quantizer is higher with respect to the uniform quantizer.

Another observation is, closer the P D F is to being uniform, the less difference in this rates, therefore it is observed that the difference in rate is much less for the quantizer for the Gaussian source than the quantizer for the source having Laplacian P D F. Now, as said earlier that entropy coding the output of the Lloyd max quantizer is a simpler approach, but it would be more beneficial to incorporate the entropy as a major of rate instead of alphabet size in the design of the quantizer itself. So, let us do this for a specific case where we want to minimize the distortion, given the entropy of the quantizer.

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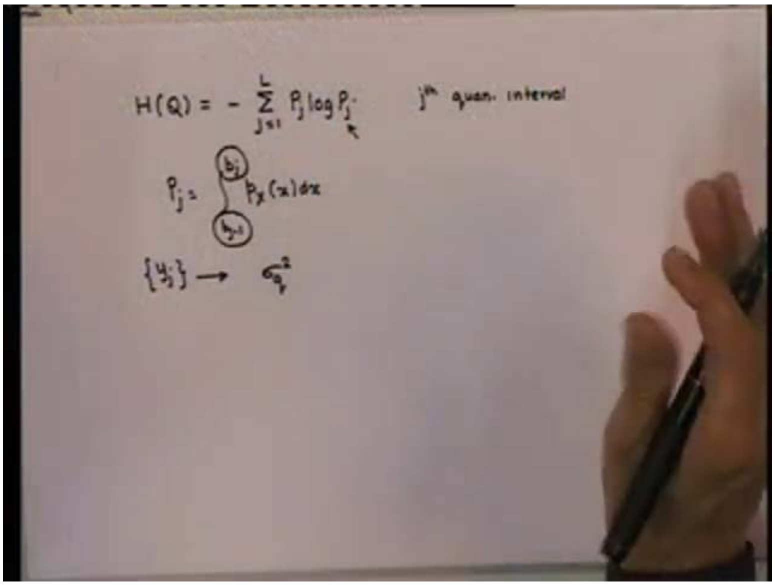

Now, the entropy of the quantizer output is given by H Q is equal to minus summation j equal to one up to l where we assume that the number of quantization levels are l P j log P j, where P j is the probability of the input to the quantizer falling in the j th quantization interval. This probability P j is given by integral of P D F over the interval b j minus 1 up to b j. Now, it is to be noted that the selection of the representation levels that is y j do not affect the calculation of the rate of the quantizer which is lower bounded by the entropy. So, what this implies is that that y j can be selected solely to minimize the distortion that is the mean squared quantization error.

However, the selection of the decision boundaries do affect the rate and the distortion. Earlier in our design of the quantizer, we had assumed that the rate is fixed by the quantizer alphabet size that is the number of levels. We also assumed fixed rate coding and then obtained reconstruction levels and decision boundaries based on the criterion of minimizing the distortion. Now, we can fix the entropy and try to minimize the distortion and the problem can be posed as follows.

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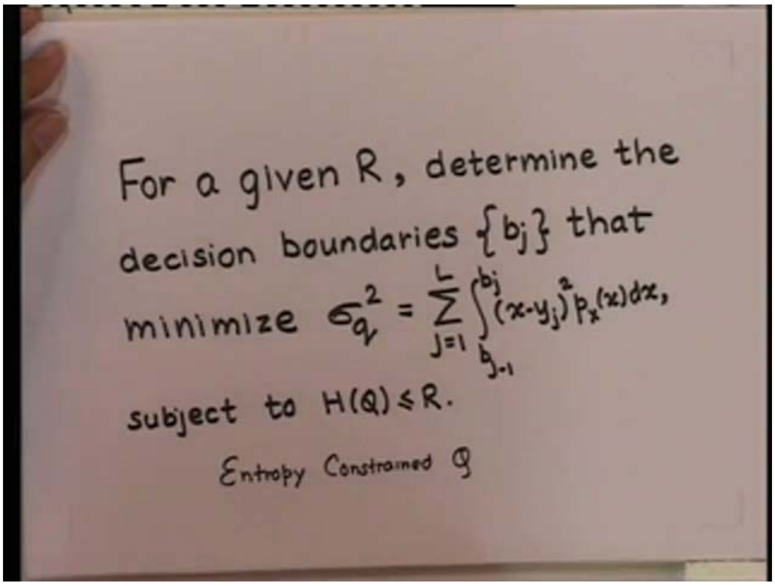

For a given rate, determine the decision boundaries b j that minimize, the mean squared quantization error which is given by this expression subject to the constraint that the entropy is less than equal to R. Now, the process of finding this optimum entropy constraint quantizer is quite complex and the details are provided in this reference.

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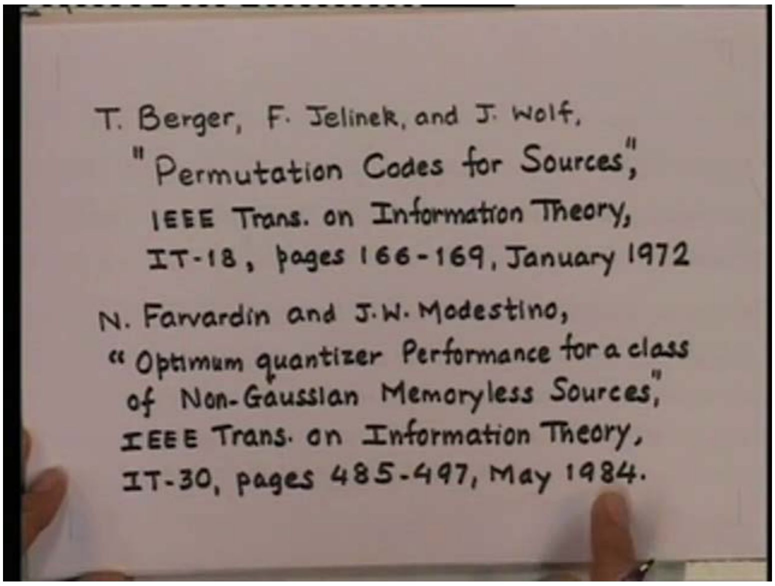

So, detailed derivation for this case for the entropy constraint quantizer is provided here, now fortunately at higher rates, we can show that the optimal quantizer is a uniform quantizer. It is also important to state that this results which bare valid for the high rate valid are also valid for lower rates. Now, for details one can refer to the following paper by Farvardin and Modestino which appeared in I E E E transaction on information theory in the year 1984. So, in general design of entropy constraint quantizer is little more complex, now we have also seen that any non uniform quantizer can be represented by a compressor followed by a uniform quantizer and followed by an expander.

Now, it is also possible to design a quantizer wherein we try to minimize the entropy given the constraint of satisfying particular level of distortion. Now, this is a very difficult problem, but it is possible to derive the characteristic of the optimum compressor at high rates which minimize the entropy for a given distortion. So, let us look at this problem.

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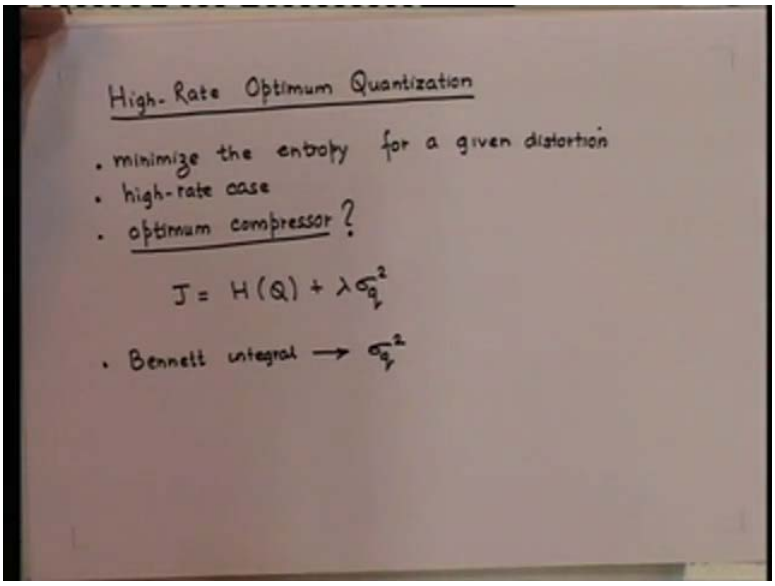

So, we will consider the design of high rate optimum quantization which takes into account both the entropy and the distortion. So, we will assume that we want to minimize the entropy for a given distortion, we will also assume it is high rate case and for this case we want to find out what is the optimum compressor. So, in order to do this let us exploit the calculus of variation approach. Let us construct the functional J equal to entropy of the output of the quantizer plus a constant lambda times distortion in the form mean squared quantization error. Then obtain the compressor characteristic to minimize this functional J, we will use the Bennet integral, which we studied in the last class to characterize the distortion.

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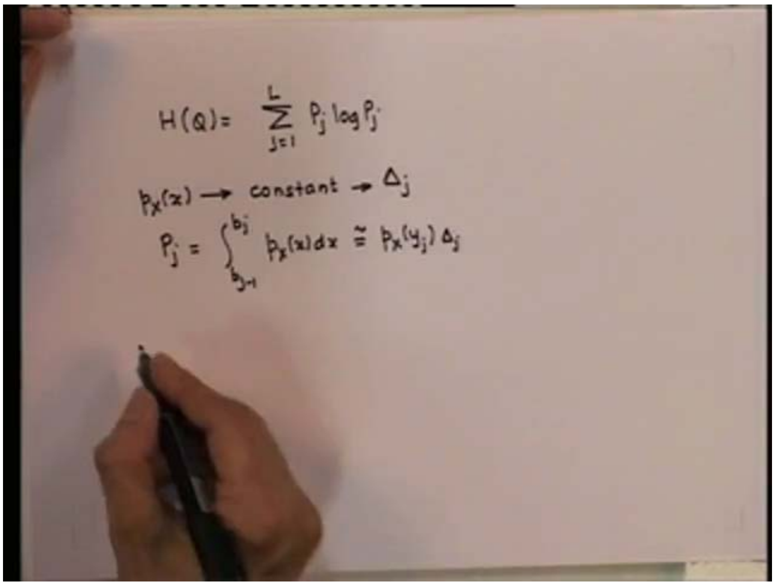

Now, the quantizer entropy is given by H Q is equal to summation of P j log P j. Now, for high rates we can make a reasonable assumption that the P D F P x is constant over each quantization interval delta j. If P j denotes the probability of the input of the quantizer falling the j th quantization interval, it can be written as integral of P x d x over the interval b j minus 1 up to b j and this is approximately equal to P x y j, y j denotes the reconstruction level for the interval b j minus 1 to b j. And delta j is the quantization interval. So, using this we can rewrite the expression for entropy as follows.

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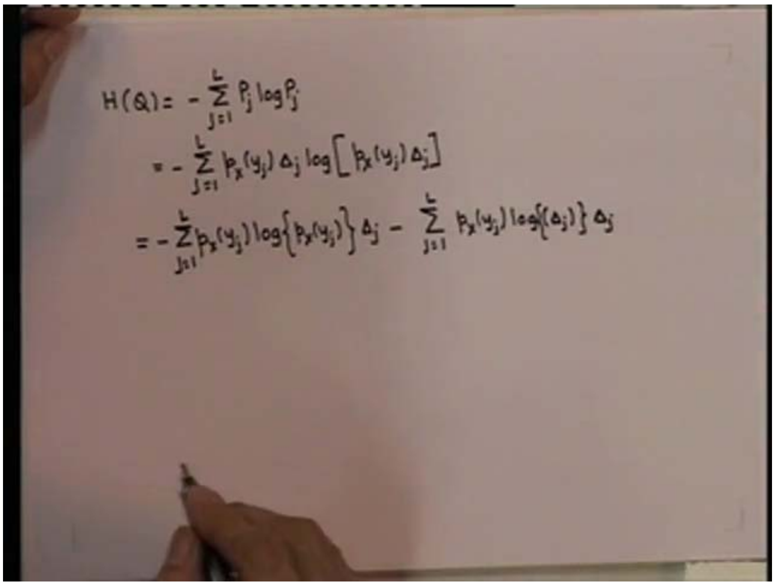

H Q is equal to minus summation P j log P j equal to 1 to L and this can be rewritten as log of P y j delta j. This can be rewritten as log of delta j.

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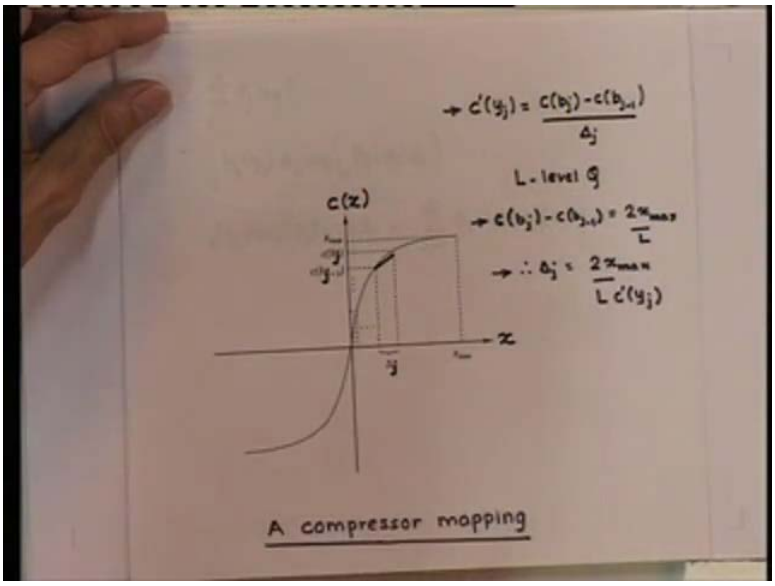

Now, in the earlier class we had seen that a compressor mapping in general has the following characteristics and if the interval is very small for very high rates. Then the derivative at reconstruction level y j can be approximated by this expression and then c of b j minus c b j minus 1 itself is equal to 2 x max upon L in which case, the delta j is equal to this expression out here. So, this we had seen in the earlier class we will use this result and substitute in this expression out here.

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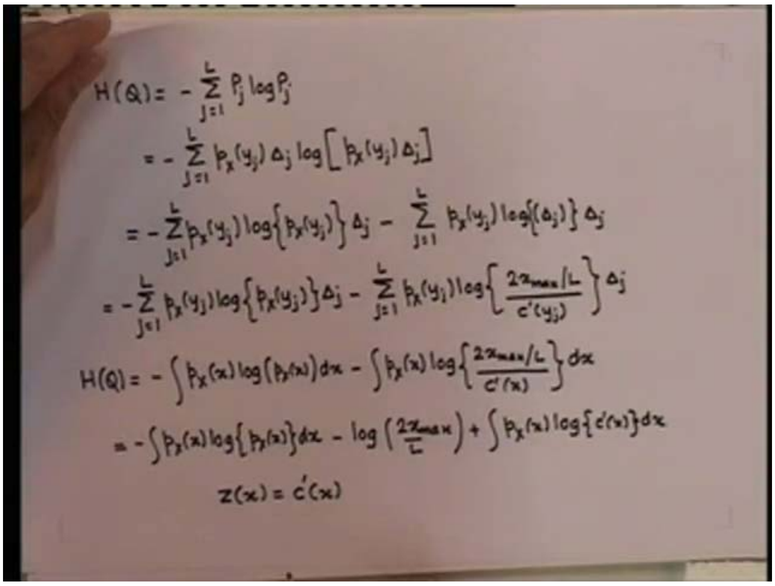

To get the following expression in place of delta j, we substitute 2 x max divide by L the whole thing divided by derivative evaluated as y j delta j. Now, if we assume the rate to be high enough in that case, the quantization interval delta j will be very small and in that case we can rewrite this expression, where summations are succeeded by the integrals. We get the expression for H Q as follows, now this integral can be broken up into two parts and finally, we get the expression as follows. Now, let us make a small substitution as Z x is equal to the derivative of c x and if we do that and rewrite the expression and also write the Bennett integral for distortion in the expression for J.

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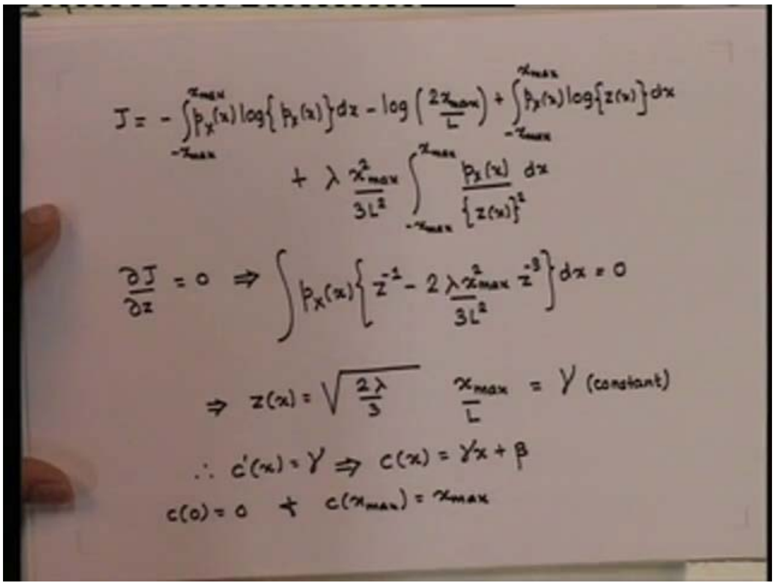

We will get as follows, J is equal to this is H Q which we just derived plus lambda times distortion which can be written based on what we did last time as 3 L squared x squared x max minus x max where we have substituted Z x for derivative of c x. All this integrals are over minus x max to plus x max. Now, to minimize J we can differentiate J with respect to Z and equate the result to 0, this will give us the following expression. So, this implies that Z x is equal to root of 2 lambda by 3 times x max by L which itself is a constant. Let us call it as gamma and therefore, c dash x is equal to gamma which implies c x is equal to gamma x plus beta, where beta is another constant and since we are given the initial conditions as c 0 is equal to 0 and c x max is equal to x max.

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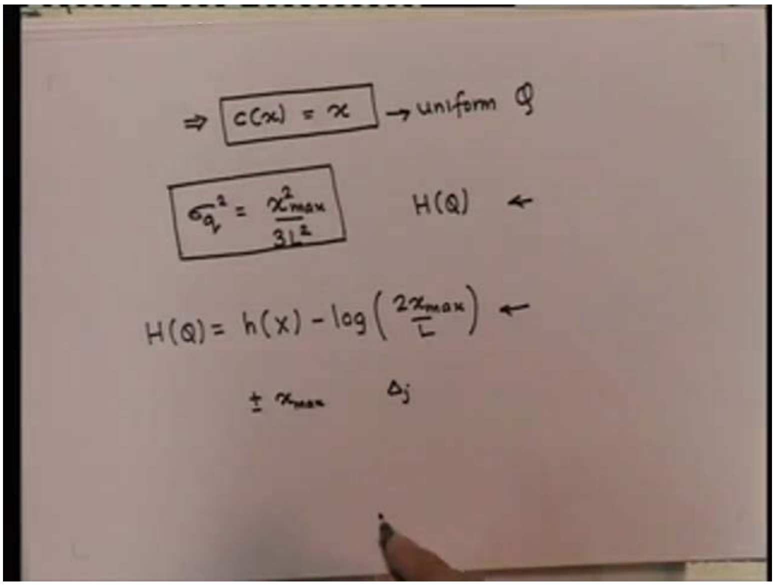

This implies that the compressor characteristic is equal to c x equals x, so the compressor characteristics is that for a uniform quantizer. Therefore, at high rates the optimum quantizer is a uniform quantizer, so if we substitute c x equal to x in the Bennet integral, we obtain an expression for the distortion as and also substituting c x equal to x in the expression for entropy. We obtain the expression for the entropy of this high rate optimum quantizer as follows; H Q is equal to h x minus log of 2 x max by L where h x is the differential entropy.

So, this is the differential entropy, now it is important to note that this derivation is valid only if the source P D F is bounded by plus minus x, max and if the step size is small enough so that we can reasonably assume the P D F to be constant a quantization interval. So, having studied the different types of quantizer, now let us solve a few more problems related to the design of quantizer.

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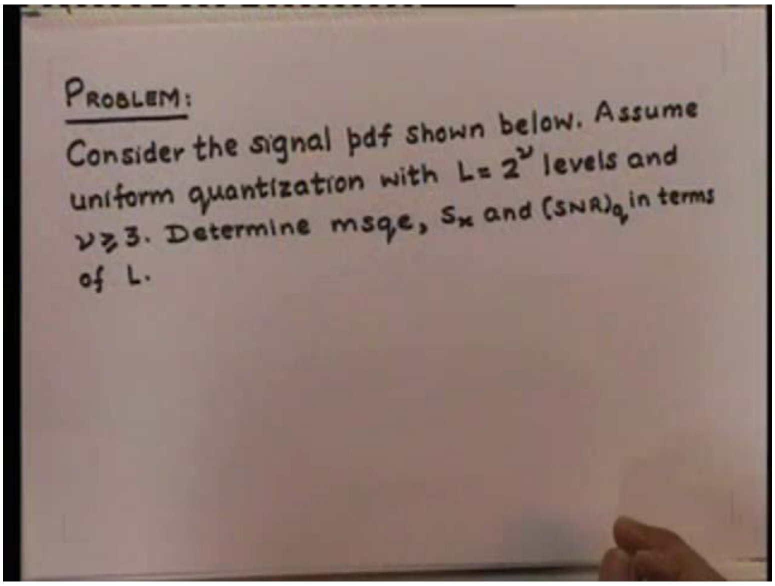

So, let us take problem number one which says that consider the signal P D F shown below assume uniform quantization within the number of levels equal to 2 raised to nu, where nu is greater than equal to 3. The problem is to determine the mean squared quantization error, the variance of the signal and signal to quantization noise ratio in terms of the quantization levels n and the signal P D F is as shown here.

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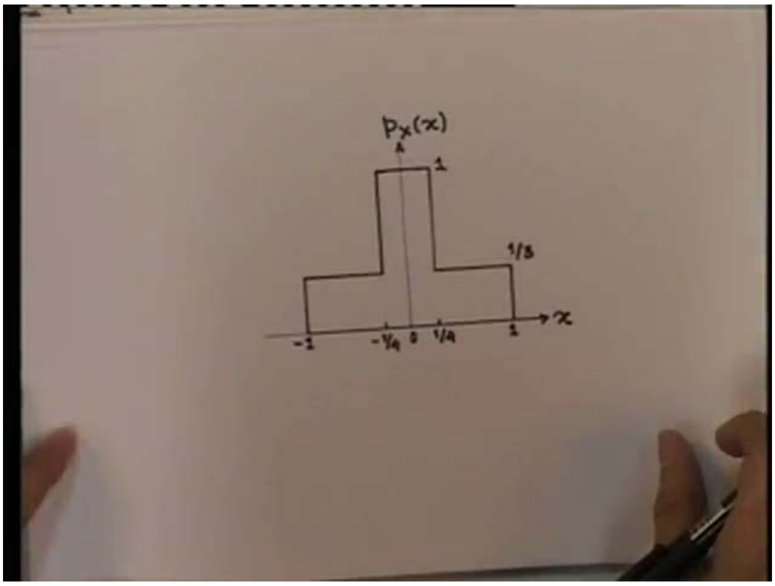

So, the value is between minus 1 to plus 1 where it is uniform between minus one fourth to 1 by 4 with the value equal to 1 and other interval is 1 by 3. So, let us try to find the solution for this problem.

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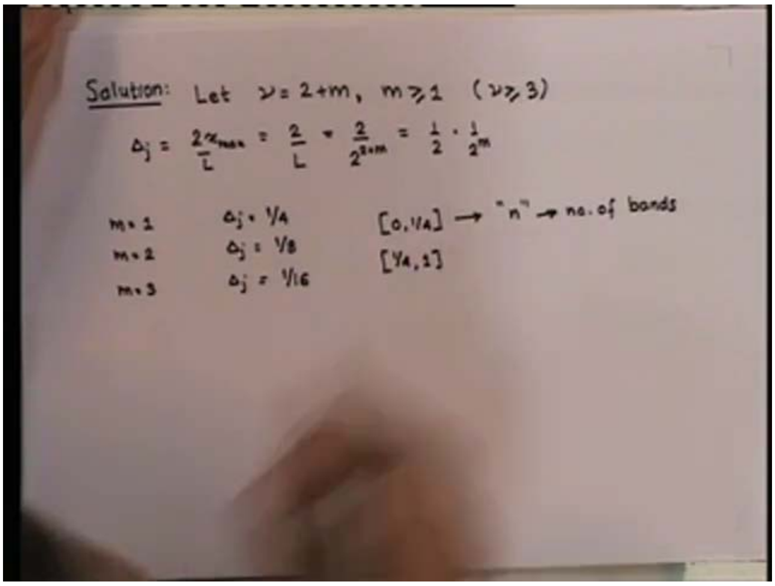

So, let us assume nu equal to 2 plus m where m is greater than equal to 1. Since we have been given nu is greater than equal to 3, and if we restrict our design to uniform quantization, the delta j is equal to twice x max by L which in our case is equal to 2 by L because x max is equal to 1. This is equal to 2 by 2 plus m is equal to half times 1 by 2 raise to m. So, for m equal to 1, delta j is equal to one fourth, for m equal 2 delta j is equal to one-eighth and for m equal to 3 delta j is equal to 1 by 16.

Now, because of symmetry of this P D F and the symmetry of our quantizer design, we will restrict our discussion to the interval between 0 and plus 1. Now, what this results show is that for any m, the interval between 0 to one-fourth is divided into integer number of bands. Let us call those bands as n and so is the case for the interval from one-fourth to 1.

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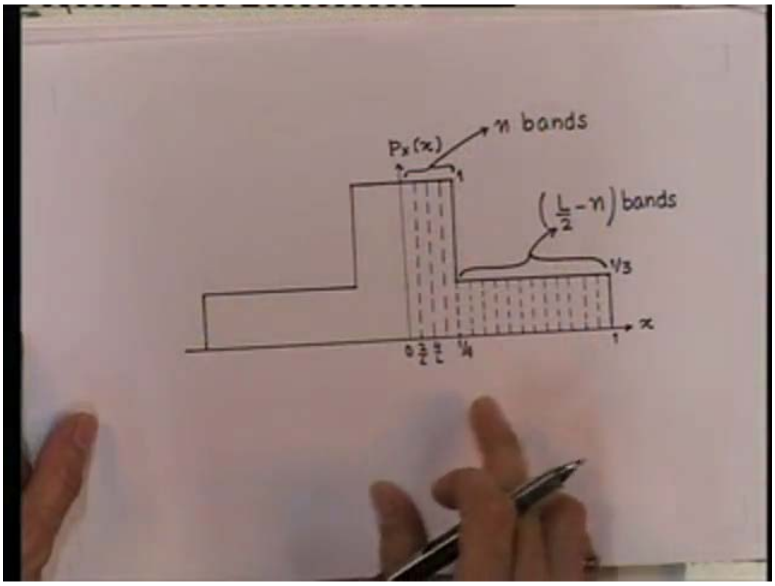

So, this P D F will be divided as follows; depending on value of m, there will be n number of bands between 0 to one-fourth. The number of bands between one fourth to 1 will be L by 2 minus n because L denotes the number of bands between minus 1 to plus 1.

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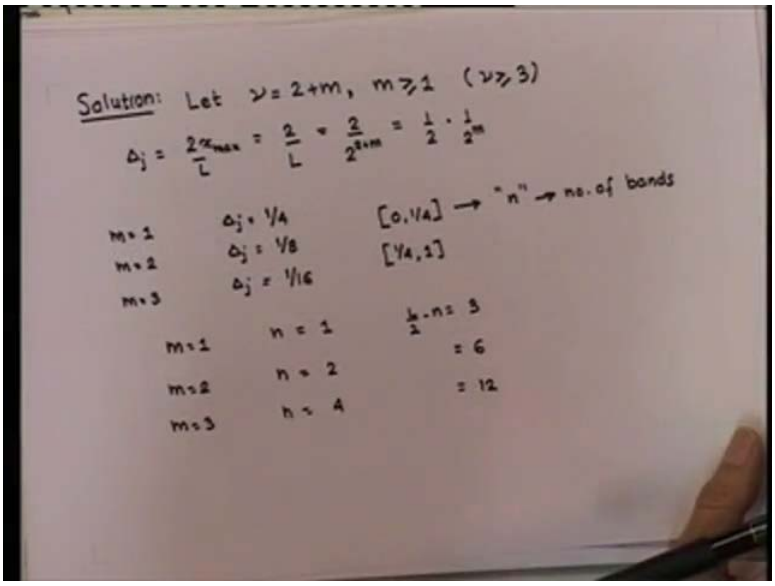

So, m equal to 1, the number of bands lying between 0 and one fourth interval would be equal to 1, and the number bands lying between one fourth and 1 would be l by 2 minus n is equal to 3. Similarly, for m equal to 2, n would be equal to 2 and this quantity would be equal to 6, m equal to 3 n is equal to 4 and the number of bands in the interval between one-fourth to 1, would be equal to 12. Now, what this result show that P D F is constant over each band and all boundary values and reconstruction are also equally spaced.

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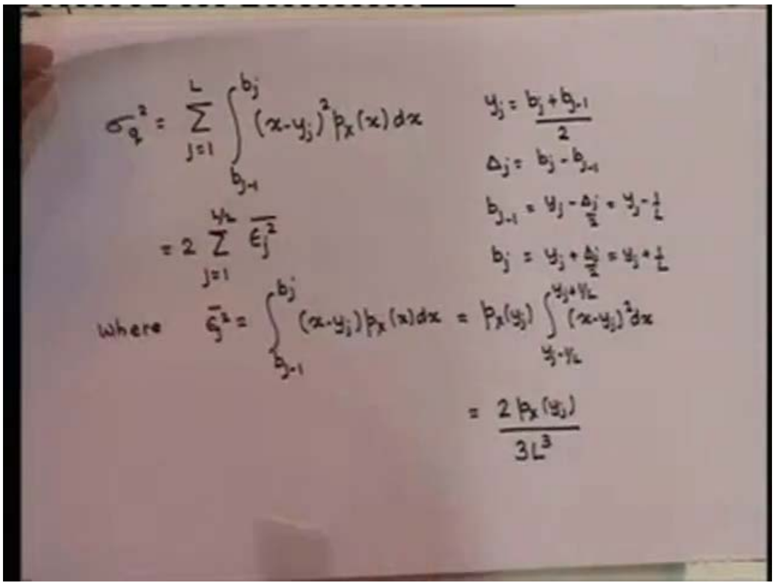

Now, we know that distortion is given by the following expression; now in our case y j is equal to b j plus b j minus 1 that is by 2. That is the mean value of the decision boundaries; delta j is equal to b j minus b j minus 1, b j minus 1 is equal to y j minus delta j by 2 is equal to y j minus 1 by L b j is equal to y j plus delta j by 2 is equal to y j plus one by L. Now, using this values out here and knowing that the quantization design is symmetric, we get this equal to twice 1 by 2 j is equal to 1 of where squared is equal to and this is equal to 2 P y j 3 L cube. Therefore, finally we can write the expression for the distortion as follows.

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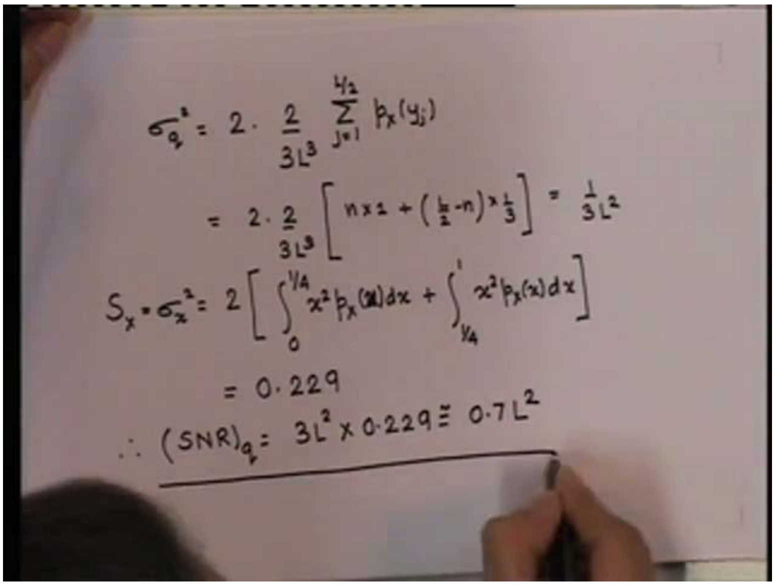

This is equal to all the bands in the interval between 0 to 1 by 4 have P y j equal to one-fourth and all the bands in the intervals between one-fourth to 1 have P y j equal to one-third. Therefore, this reduces to n x 1 plus l by 2 minus n times one-third is equal to 1 by 3 L Squares and the variance for the signal is equal to and this can be shown equal to 0.229. Therefore, signal to quantization noise ratio is equal to 3 L squared by 0.229 which is equal to approximately 0.7 L squared.

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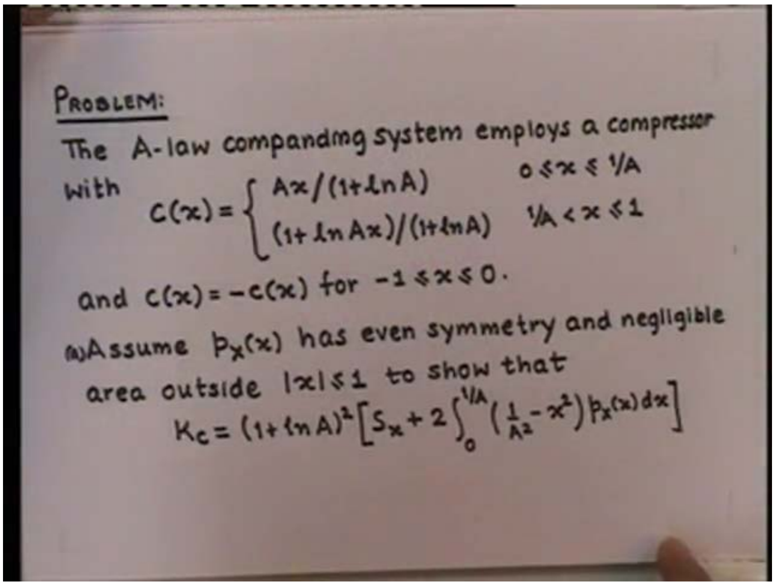

Now, let us take another example, this example resembles the earlier example which we did for nu law companding. Now, this is for the A law companding system which employs a compressor of this form and we assume that the input P D F has even symmetry and negligible area outside k mod x less than equal to one. We are required to show that the value of K c, which we had discussed in the last class is given by this.

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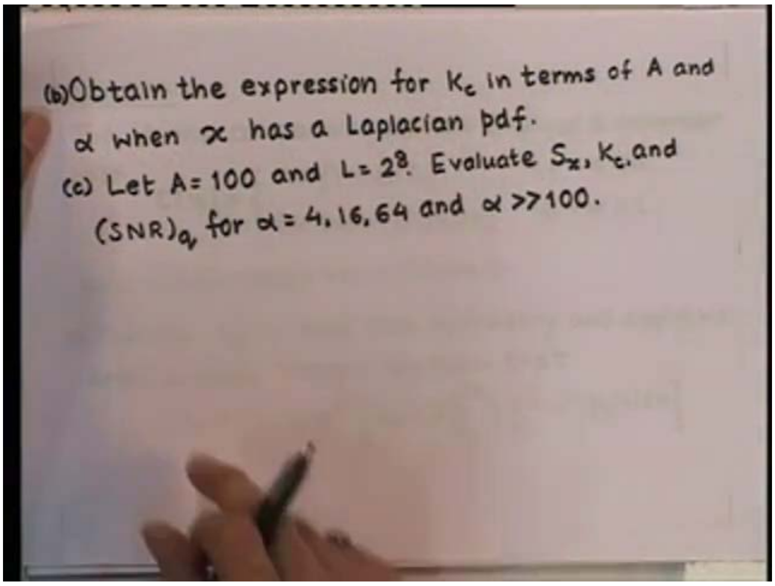

For a particular input P D F that is Laplacian P D F, we are expected to obtain the expression for K c in terms of A and alpha which is a parameter for Laplacian P D F. Then finally, for a specific a value equal to 100 and L equal to 256, we are required to evaluate the variance of the input. The constant K c and signal to quantization noise ratio for different values of alpha, so let us try to provide a solution to this problem.

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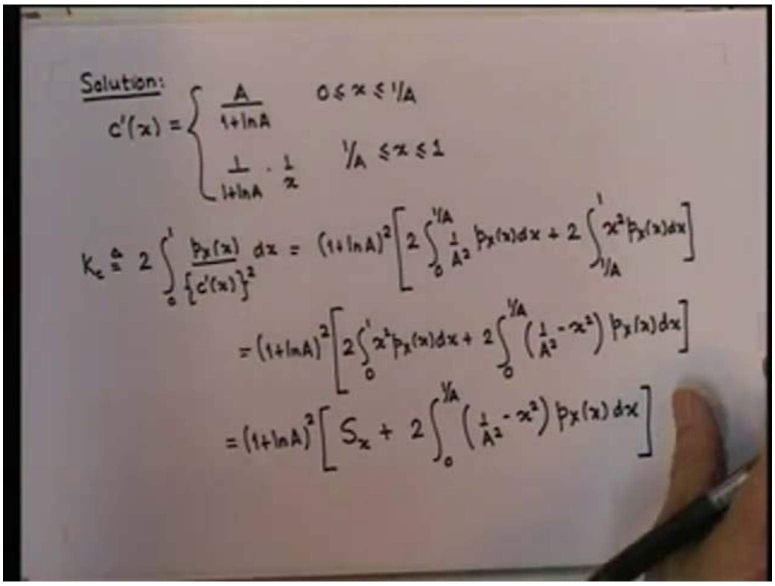

So, the solution would be as follows for the given A law companding system compressor character is given by this. So, the first step is to obtain the derivative of this compressor characteristic in the form c dash x, which is equal to 1 plus log A for x lying between 0 and 1 by k and e is equal to 1 plus log k times 1 by x for x lying between 1 by k and 1. Now, we have seen the definition for K c in the last class that is equal to twice integral from 0 to 1 of the quantity P x divided by c dash x squared d x. Now, if we plug in this values, we get the following expressions which can be rewritten as and which is equal to the desired expression. This is equal to S x, that is the input variance plus twice 0 to 1 by A square minus x square p x d x.

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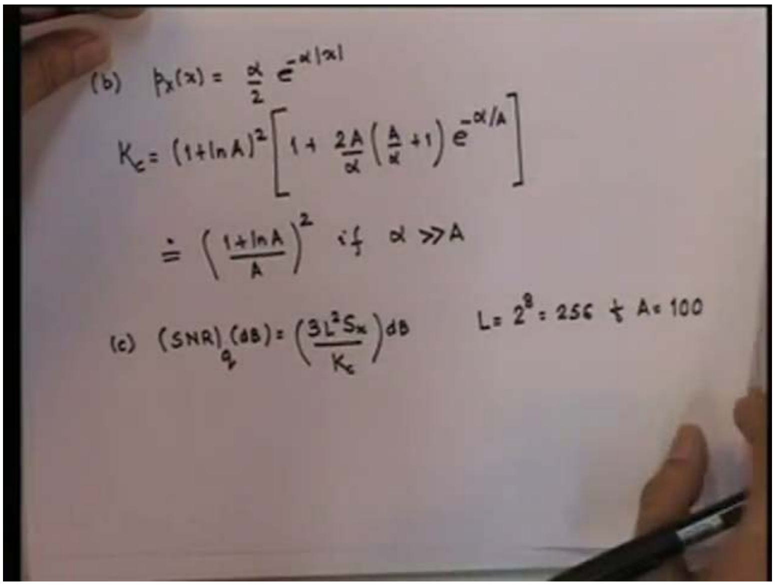

Now, for a specific case of P x equal to Laplacian P D F that is alpha by 2 e raise to alpha times mod x, it is easy to show that K c is equal to 1 plus log A squared times 1 plus 2 A, alpha times A by alpha plus 1 e raise to minus alpha times A. This is approximately equal to 1 plus log A by A squares A alpha is much, much larger than A. So, let us try to evaluate the signal to quantization noise ratio which is given by in dB is equal to for L is equal to 2 raise to 8, that is 256 and A is equal to hundred. So, if we plug in these values in the value for K c and this expression.

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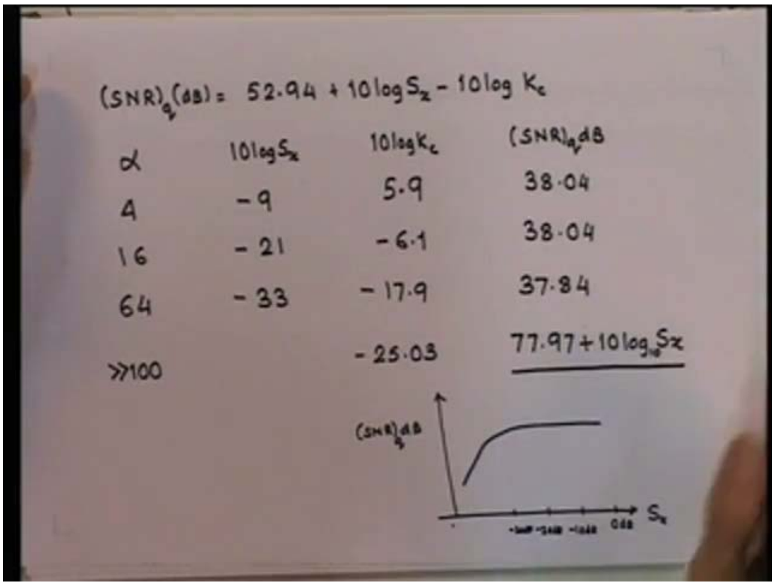

Then, we get the signal to quantization noise ratio as follows plus m log S x minus 10 log c and this is shown as follows for different values of alpha. For alpha equal to 16 this is equal to minus 21 minus 6.1 38.04, 12 pi equal to 64 minus 33 minus 17.9 37.84 and for alpha much larger than A that is 100. The general expression for signal to quantization noise ration in terms of input variance is as follows. So, if you plot this, we would get a result something like this, so what this shows that with the companding the signal to quantization noise ratio remains more, or less fixed and this result is similar to what we had obtained earlier for nu law companding.

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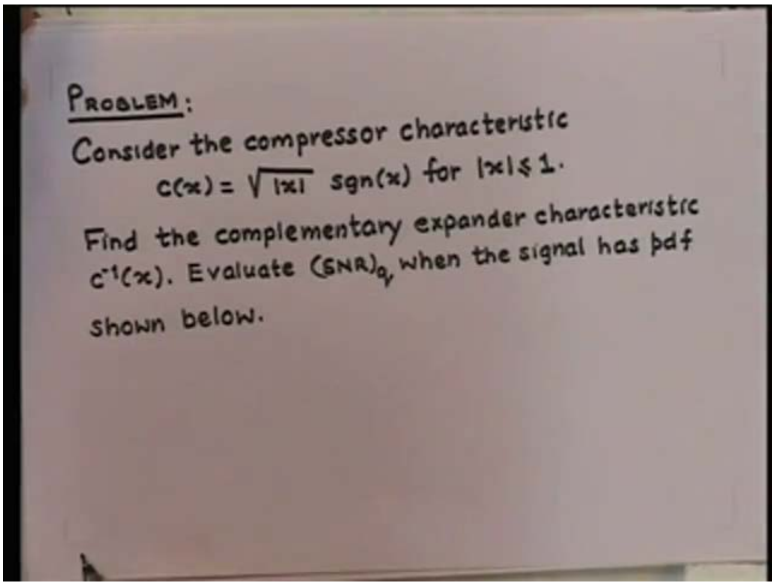

Now, let us take one more example, suppose we had the compression characteristics given by this expression, and it is desired to find the complementary expander characteristics that is c inverse x. We are required to evaluate signal to quantization noise ratio when the signal has P D F shown below it is of this form. So, let us try to find the solution to this again.

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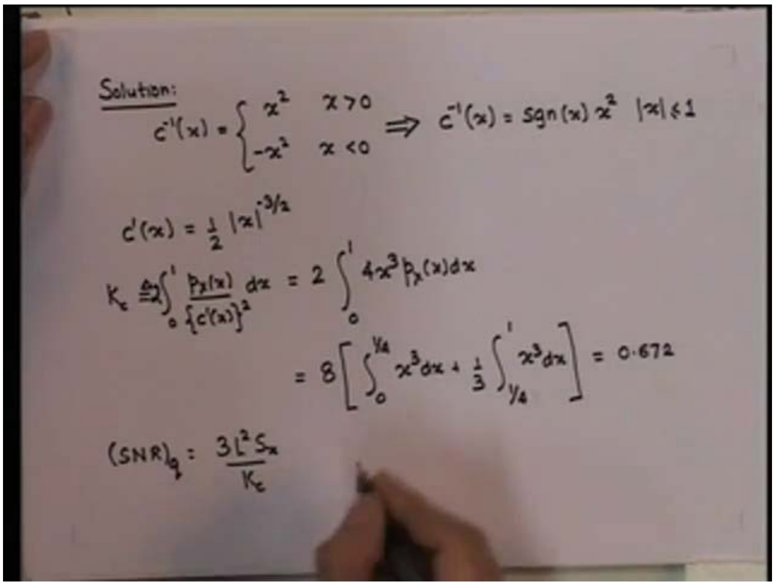

The first step would be to obtain the expanded characteristic that is c inverse x can be shown is equal to x squared, where x greater than 0 and minus x square for x less than 0 which implies that c inverse x is equal to sign x squared mod x less than equal to 1. Calculate the signal to quantization noise ratio; the first step would be to calculate c dash x which in this case is equal to half mod x minus 3 by 2 and K c can be evaluated as follows. This is equal to twice 4 x cube p x d x and this can be rewritten for a given P D F as follows, and it can be shown that for this P D F this expression reduces to 0.672.

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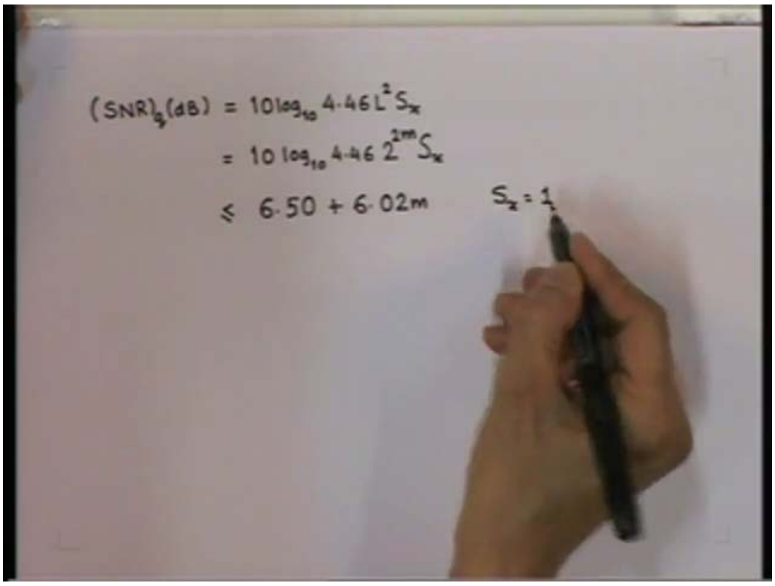

Finally, signal to quantization noise ratio which is equal to 3 L squared S x by K c in terms of dB is equal to 10 log to the base 10 of 4.46 times L squared times the input source variance is equal to 10 log base 10 4.46 2 raise to 2 n times S x and less than equal to 6.50 plus 6.02 m and we will have an upper bound for S x is equal to 1. So, we have studied different quantization schemes for a case where the output of the source is quantized individually.

Earlier in our study of lost less compression, we had studied two cases; one where we encode the source output individually and the second case, where we encode source output in terms of block. We reach to the conclusion that by grouping the source output together, we could achieve better compression ratios. So, in the similar manner we can extend this idea to the case of quantization instead of quantizing the individual sample values we could group the sample values together and quantize them in blocks. We will explore this block quantization in the next class.