Information Theory and Coding
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Lecture - 37
Compounded Quantization

We have looked into the design and performance of both uniform and non uniform P D F optimized quantizer that is Lloyd max quantizer. In the case of uniform quantizer, we have seen that the decision boundaries, or the boundary values and the reconstruction levels are equally spaced. The size of the quantization interval is the same. However, in the case of non uniform P D F optimized quantizer that is Lloyd max quantizer, both the decision boundaries and the reconstruction levels are unequally spaced. The quantization interval is smaller in those regions that have more probability.

Mass implementation of a non uniform P D F optimized quantizer is much more difficult than a uniform quantizer, but mean square quantization error is lesser in the case of a non uniform P D F optimized quantizer compared to uniform quantizer. There is another approach to achieve the similar beneficial region of lesser mean squared quantization error of a non-uniform quantizer.

Instead of making the quantization interval small, we could make the region in which the input lies with high probability large. We could expand or stretch the region in which the input lands with high probability in proportion to the probability with which the input lands in the high probability region. This is the main idea behind compounded quantization. The compounded quantization can be represented by the following block diagram.

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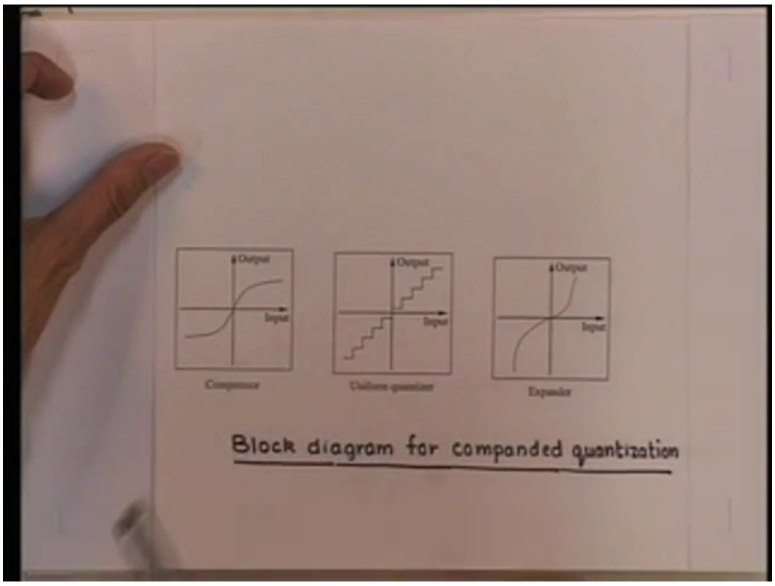

First the input is mapped through a compressor function. The compressor function task is to expand or stretch the high probability regions closed to the origin, and compress the low probability regions away from the origin. The net effect of this is that regions which are close to the origin in the input to the compressor occupy a greater fraction of the total region in the output of the compressor.

Now, the output of the compressor is uniformly quantized. The quantized value is transformed via an expander function, which is the inverse of the compressor function. The net effect of this whole process is the same as using a non-uniform quantizer. This is known as compounded quantization. Now, before we look into the mathematics of compounded quantization, let us try to get the feel of this process with the help of a simple example.

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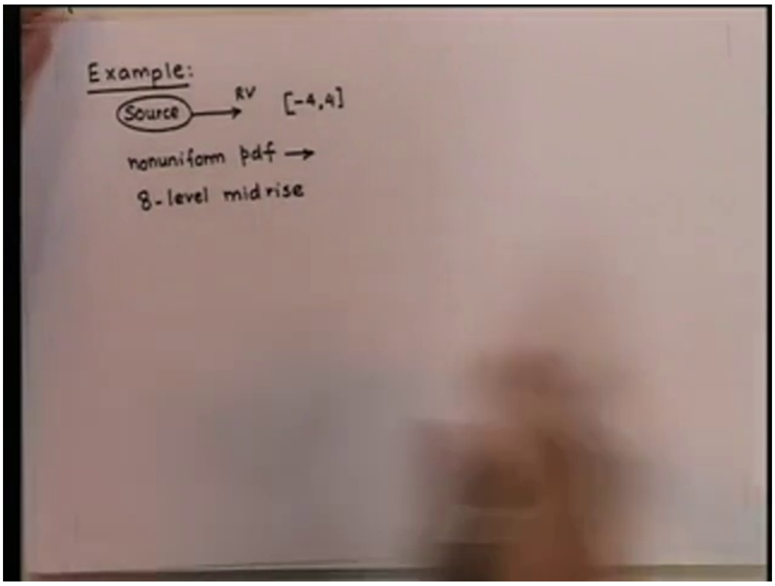

Let us assume a source whose output can be modeled as a random variable. This random variable takes values in the interval minus 4 to plus 4. Let us assume that this random variable has non uniform P D F with higher probability mass near the origin than away from it. Let us use an 8 level midrise quantizer of the form shown in the figure here.

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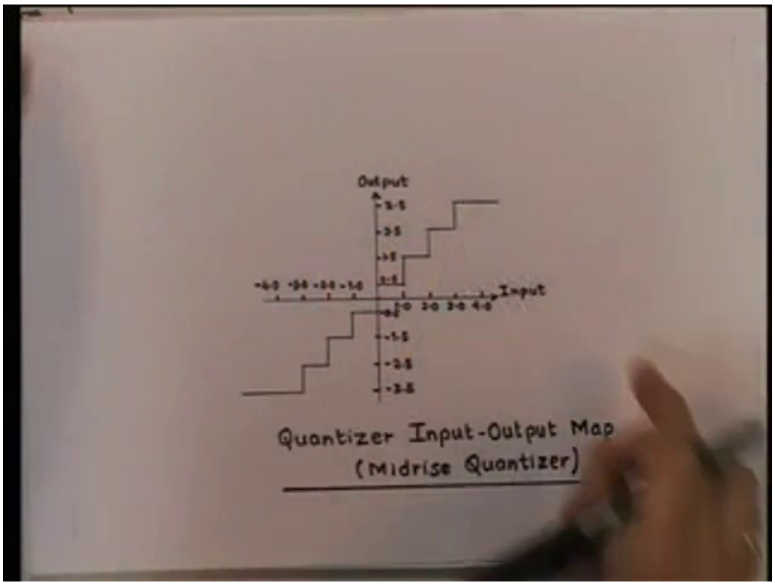

So, this is the quantizer input output mapping for a midrise quantizer. Quantization step size is 1 and reconstruction levels are the mid points of this interval.

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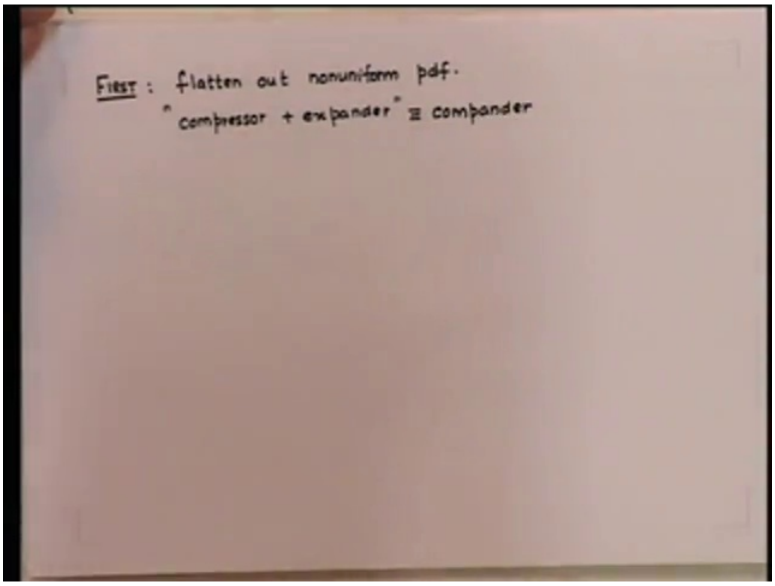

So, first let us try to flatten out the non uniform P D F using the following compressor and expander, which together is known as compander.

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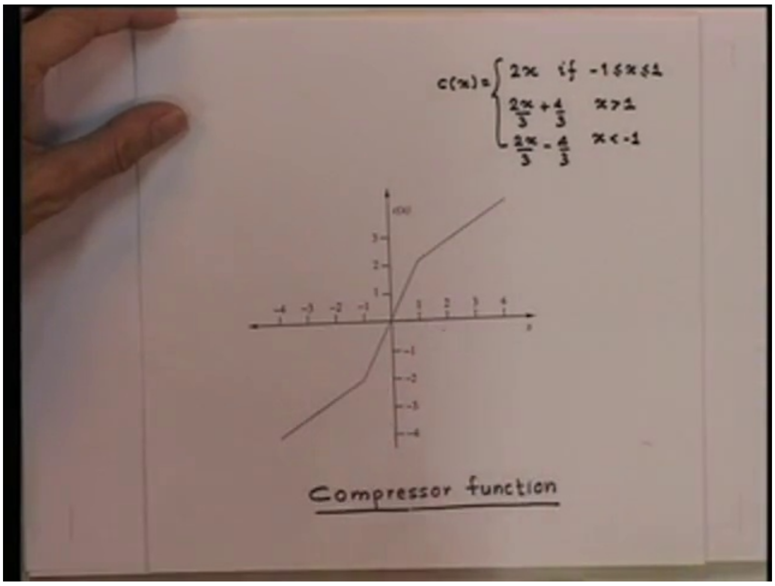

The compressor function is shown here graphically and mathematically. It is c x is equal to 2 x if x lies between minus 1 and plus 1. It is equal to 2 x by 3 plus 4 by 3 if x is greater than 1. It is equal to 2 x by 3 minus 4 by 3 if x is less than minus 1. It is this. So, this is the compressor function or compressor mapping and the expander mapping.

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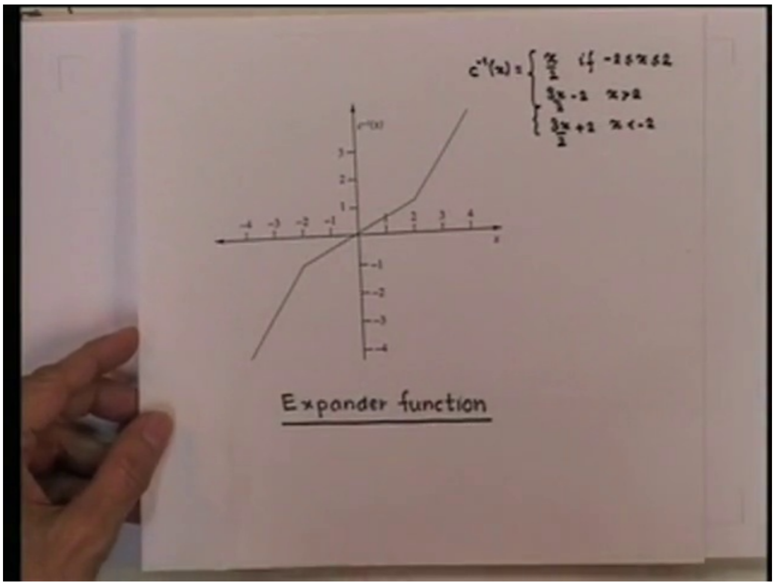

The expander function is shown here in terms of c inverse x. The characteristic can be written as c inverse x is equal to x by 2, if x lies between minus 2 and plus 2 that is this region. It is equal to 3 x by 2 minus 2, if x is greater than 2 that is this line. Finally, it is equal to 3 x by 2 plus 2, if x is less than minus 2 that is this line. So, between compressor and the expander, we have the uniform quantizer and that is shown here. We have the range in minus 4 plus 4 with 8 levels.

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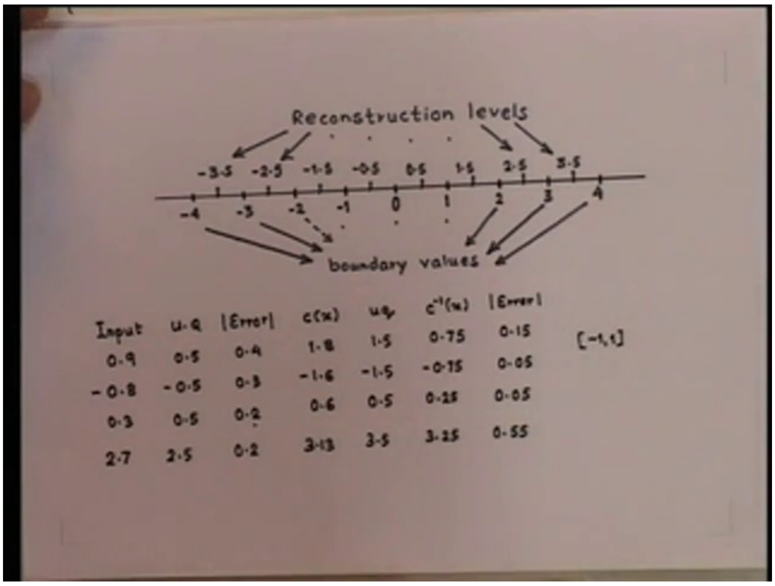

So, we denote minus 4 minus 3 minus 2 and so on as boundary values and 3.5, 2.5, 1.5 as the reconstruction levels. Now, let us study how this compressor and expander functions affect the quantization error, both near and far from the origin. So, let us say we have input as 0.9. Now, if we directly quantize this without the compander, the output of the uniform quantizer would be 0.5 because 0.9 lies between 0 and 1. The error in this case would be 0.4. Now, if the same input was passed through a compander, then let us see the output. So, it is 0.9 when it passes through the compander characteristic.

We just discussed the output would be 1.8. The output of the compressor will be fed to the uniform quantizer whose output will be 1.5. So, the apparent error is 0.3. Finally, the quantized value passes through the expander to give the value 0.75. In this case, the error is difference between 0.9 and 0.75 that is 0.15. So, we see that without compander, the error is 0.4, whereas with the compander the error is 0.15. Similarly, if we take minus 0.8, this would be quantized to minus 0.5 without the compander. So, the absolute error would be 0.3. Now, if we pass this through a compander, the output would be minus 1.6, which when uniformly quantized will give the value of minus 1.5.

When this passes through the expander, it will give minus 0.75. So, the error in this case is again less. It is equal to 0.05. Similarly, we could extend this 0.3. This would give to 0.5. The error is 0.2. When you pass through compressor, it will give 0.6, which will be quantized to 0.5. When passed through expander, it will give 0.25. So, the error is 0.05. Again, the error here with the compander is less than without the compander. Finally, let us take a value of 2.7, which without the compander would be quantized to 2.5. So, the absolute error is 0.2.

This is passed through the compressor. This will give us 3.13, which will be quantized to 3.5 uniformly. When passed through the expander, it will give back 3.25. In this case, the error is 0.55. So, the companded quantizer effectively works like a non-uniform quantizer with small quantization intervals in the interval between minus 1 to plus 1 and larger quantization intervals outside this interval. So, the next question is what is the effective input output mapping for this quantizer?

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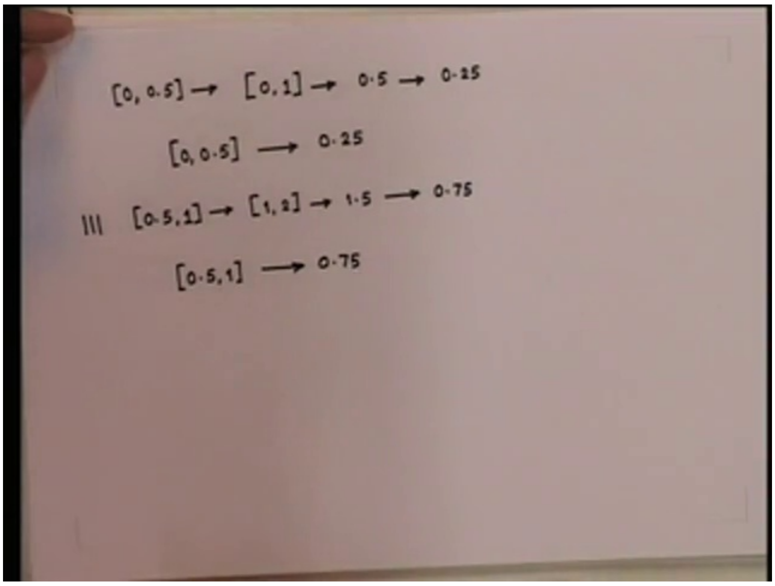

Now, it will be easy to see that when the input lies in the interval between 0 and 0.5, then using the compressor, it gets mapped into the interval between 0 and 1. When this output of the compressor is passed through a uniform quantizer, it is quantized to output value of 0.5. This when it passes through the expander, it gives us the reconstruction level of 0.25.

So, effectively the input between the ranges 0 to 0.5 gets mapped to 0.25. Similarly, the input between the intervals 0.5 to 1 on passing through the compressor get mapped to the region between 1 and 2. This gets quantized to 1.5. The output of the expander is 0.75. So, again effectively, the interval between 0.5 and 1 gets mapped to 0.75. So, the effective quantizer input output map is shown in the figure here.

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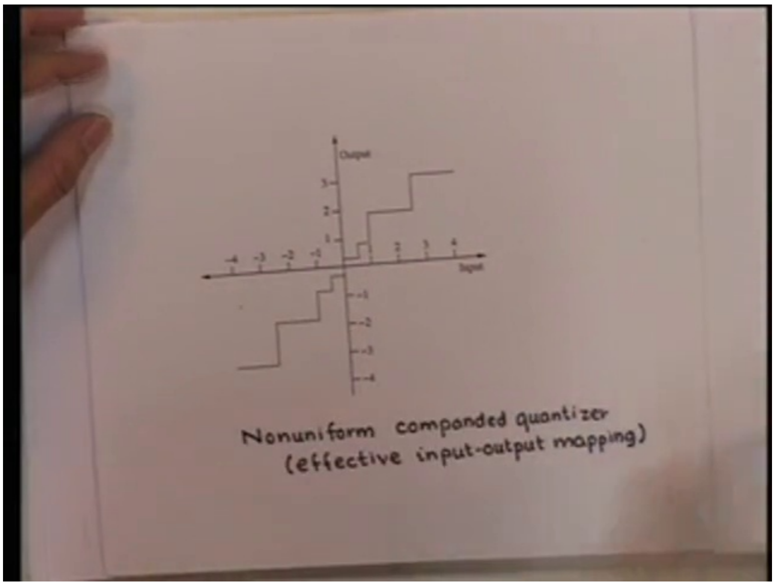

The values between 0 and 0.5 get mapped to 0.25; between 0.5 and 1 get mapped to 0.75 and so on. Now, having studied this simple example, let us try to extend this concept to a more generic input. For this, let us assume that the source output is bounded by some maximum value. Let us call it as x max. Let us also assume that a number of quantization levels are high that is we deal with high rate quantizers. Now, with these assumptions, let us denote the distance between 2 boundaries of a quantizer as delta j.

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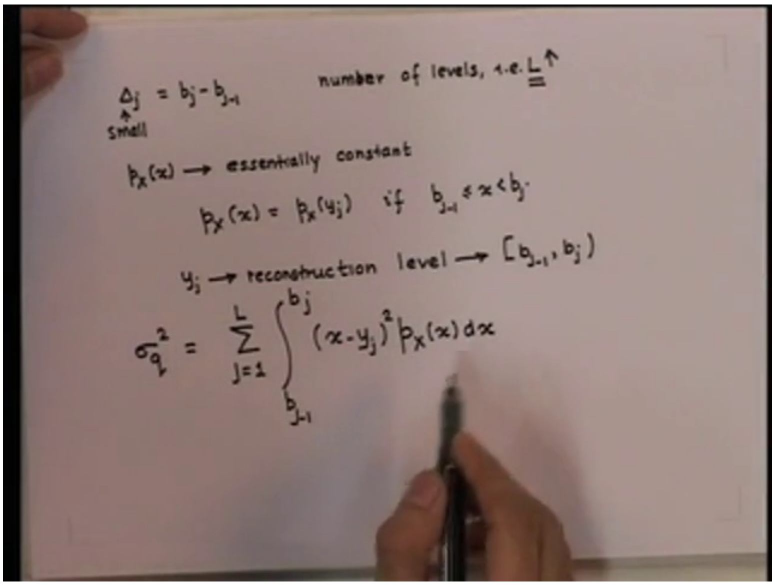

It is equal to b j minus b j minus 1. Now, if we assume that the number of quantization levels that is l is high enough, then the size of each quantization interval will be small. Therefore, it would be quite reasonable to assume that the P D F of the input that is given by p X will be essentially constant in each quantization interval. We can approximate p X is equal to p X y j, if x lies in the interval between b j minus 1 and b j. y j is the reconstruction level of the interval b j minus 1 b j. Now, we know that mean squared quantization error is given by the following expression. Now, this expression can be rewritten based on the above assumptions as follows.

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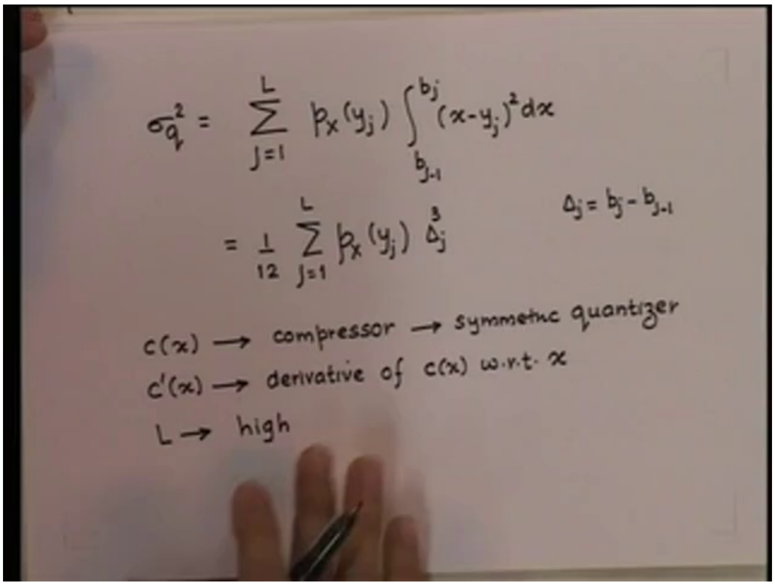

This comes out of the integral because of the constant. This can be simplified as follows. The value for this expression is equal to where delta j is equal to b j minus 1. Now, for the companded quantization, if we let c x denote the compressor characteristic for a symmetric quantizer and let c dash x denote the derivative of c x with respect to x. Let us assume that the number of quantization level that is l is high. Now, given this, any general compressor mapping will look as follows.

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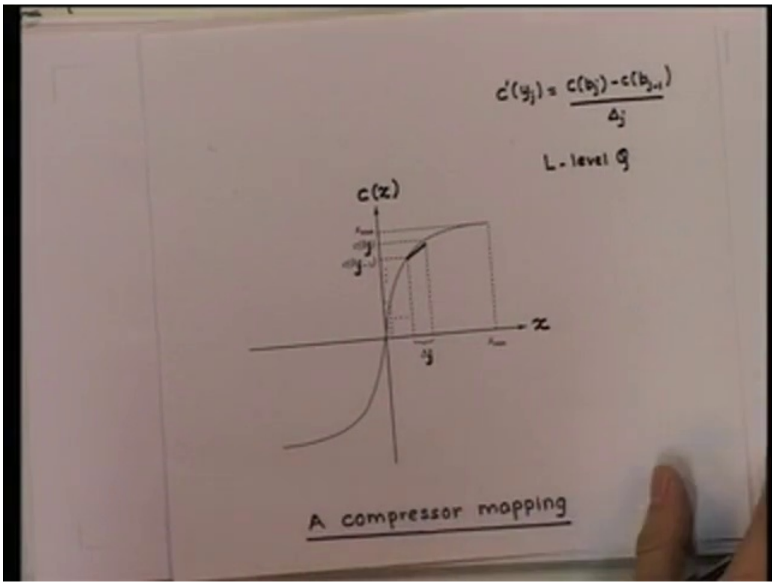

So, within this interval, the compressor characteristics can be approximated by a straight line. Therefore, the derivative can be approximated as c dash y j is equal to c of b j minus c of b j minus 1 divided by delta j. This is an approximation of the derivative. From this figure, we also see that c b j minus c b j minus 1; this difference is the step size of a uniform l level quantizer. Therefore, we can write c b j minus c b j minus 1 is equal to 2 x max by l. This is the step size of a uniform quantizer, where the input is bounded by x max. There are L numbers of levels.

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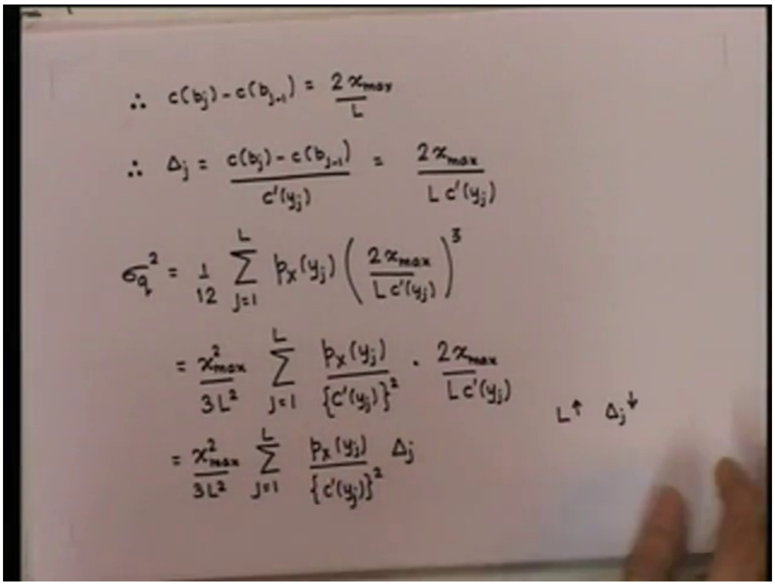

Therefore, with the given approximation delta j is equal to c b j minus c b j minus 1 divided by derivative as y j is equal to 2 x max divided by L times c dash y j. Using this relationship, we can rewrite the mean square quantization error as follows. 1 by 12 summation over j equal to 1 to L delta j can be approximated by 2 x max upon L times c times y j cube. This essentially comes from this expression, which we saw some time back. This can be rearranged as follows and again simplified as x squared max upon 3 L squared summation over j equal to 1 to L p x y j delta j is equal to this quantity. Now, if we assume that L is high, then delta j will be small. In that case, we can replace this summation by integration as follows.

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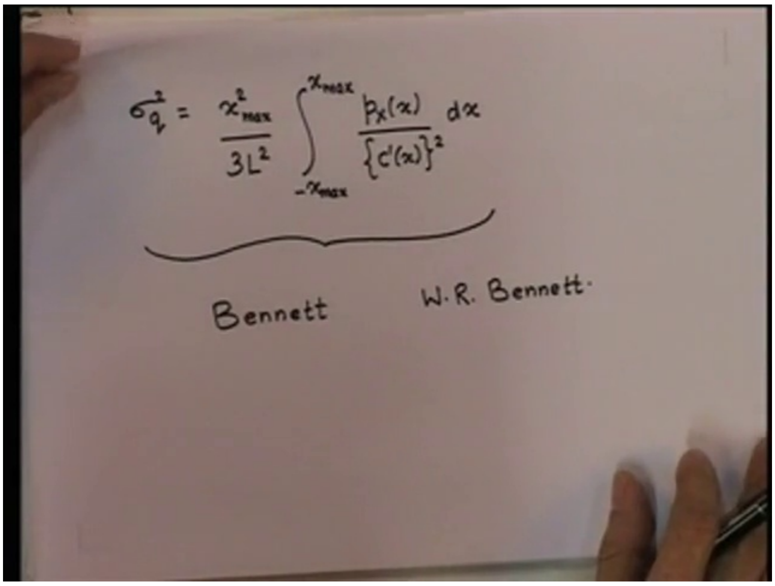

The mean squared quantization error expression reduces to this integral is known as Bennett integral. This is discovered after W.R. Bennett. This famous result is widely used to analyze quantizers. Now, if you observe mean square quantization error, it is dependent on P D F of the input to the quantizer.

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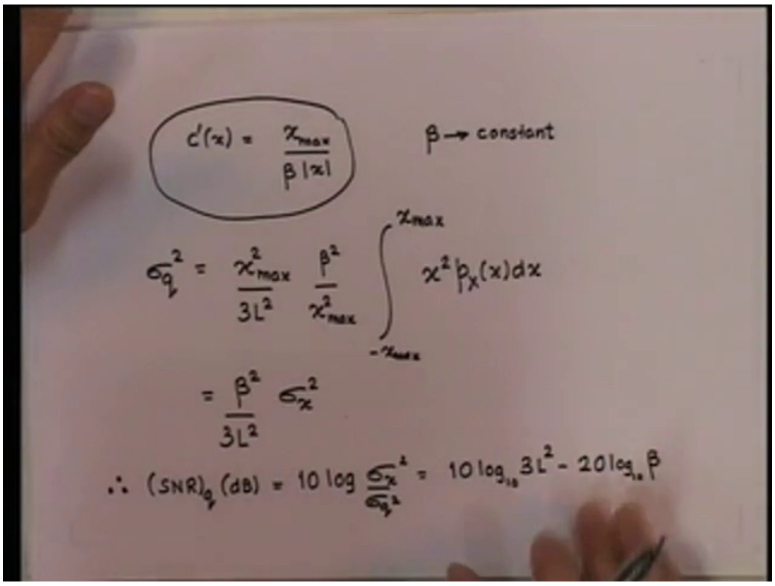

So, let us define c dash x as x max by beta times mode x, where beta is a constant. Now, using the Bennett integral and using this relationship, it is easy to show that mean square quantization error reduces to the following expression. It is equal to beta square that is a constant divide 3 L squared times sigma x squared. This is the variance of the input. Therefore, signal to quantization noise ratio in dB will be equal to 10 log sigma x squared by sigma q squared is equal to 10 log 3 L squared minus 20 times log beta. This expression shows signal to noise quantization ratio is independent of the input media. So, this implies that if we use compressor characteristics, which satisfies this relationship, then signal to quantization noise ratio will remain constant regardless of the input variance. Now, it is important to note that this impressive result is valid as long as underlying assumptions used to derive these results are valid.

For an instant, if input variance is very small, then our assumption that the input P D F remains constant over the quantization interval will be no longer valid. If the input variance is very large, then our assumption of the input being bounded by x max will be no longer valid. So, assuming that these assumptions are valid, we can derive the compressor characteristics as follows.

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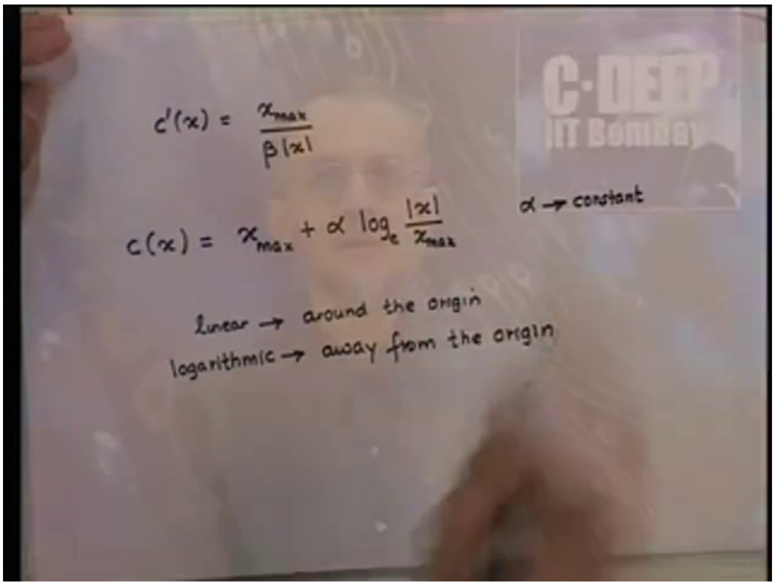

We saw that c dash x is equal to x max by beta times mod x. Therefore, this on integration results in to compressor characteristic given by c x is equal to x max plus alpha times log mod x by x max, where alpha is the constant. Now, for small x, c x assumed very large value. Therefore, in a practical situation to avoid this technical difficulty, the compressor characteristic is approximated with the function which is linear around the origin. It is logarithmic away from the origin. Now, 2 popular companding characteristics are as follows.

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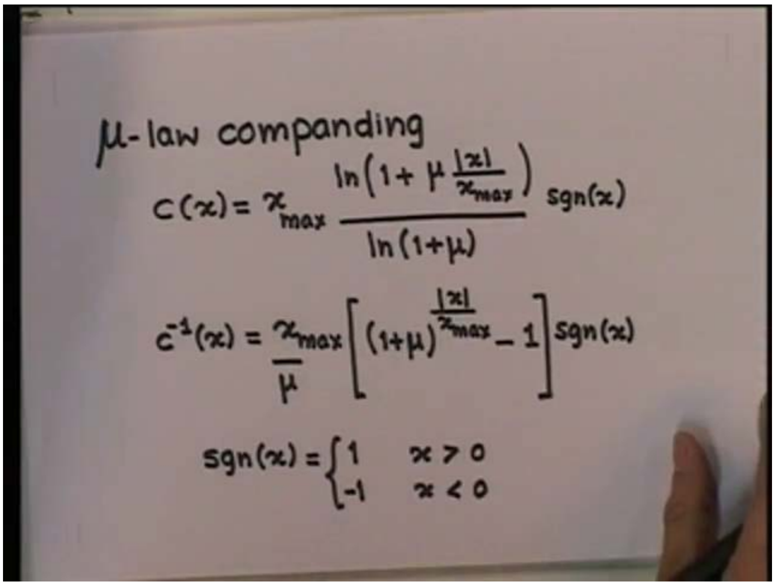

One characteristic is known as mu law companding, because the characteristic is a function of a parameter mu. It is given here. This is the inverse characteristic that is the expander characteristic where signum x is equal to 1 for x greater than 0 and minus 1 for x less than 0.

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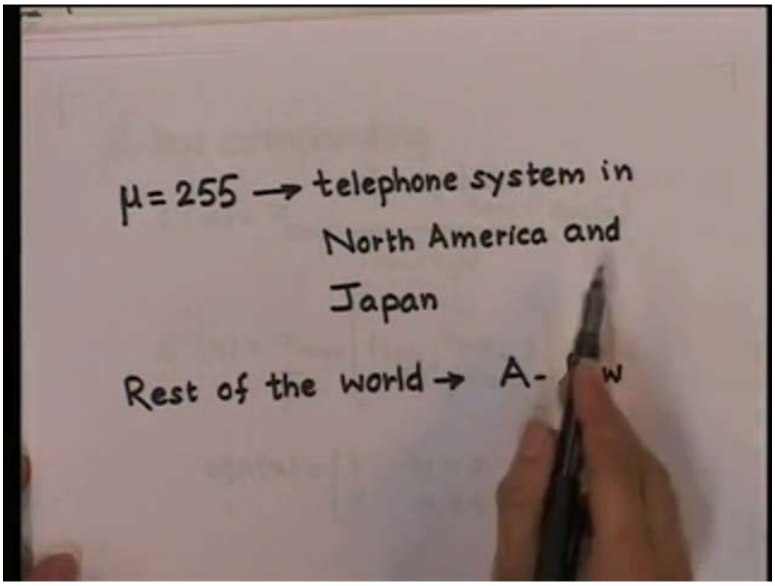

Mu equal to 255 is the value which is used in telephone system in North America and Japan, whereas the rest of the world uses a law characteristic for companding. That is given as follows.

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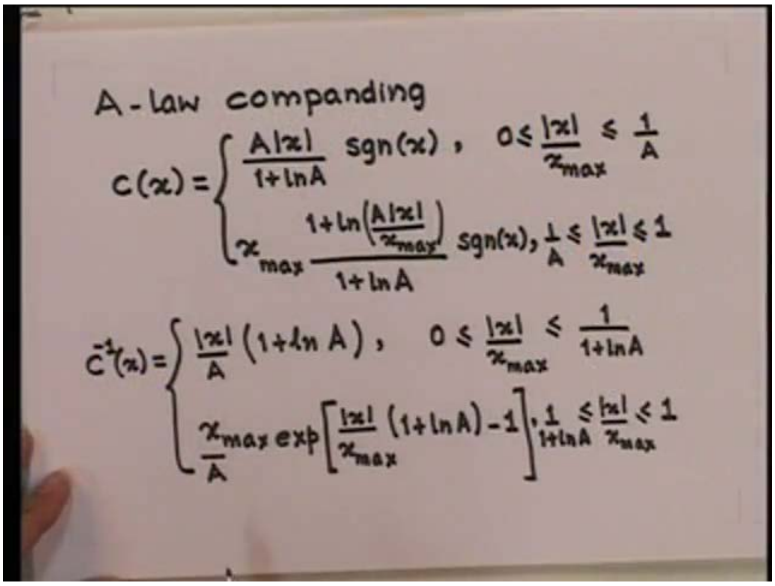

So, this is A law companding, where the function or characteristic is a function of the parameter. The typical value for A is equal to 100. For the same A law companding, expander characteristic is specified as shown here. Now, let us take few examples to understand this concept in a better way.

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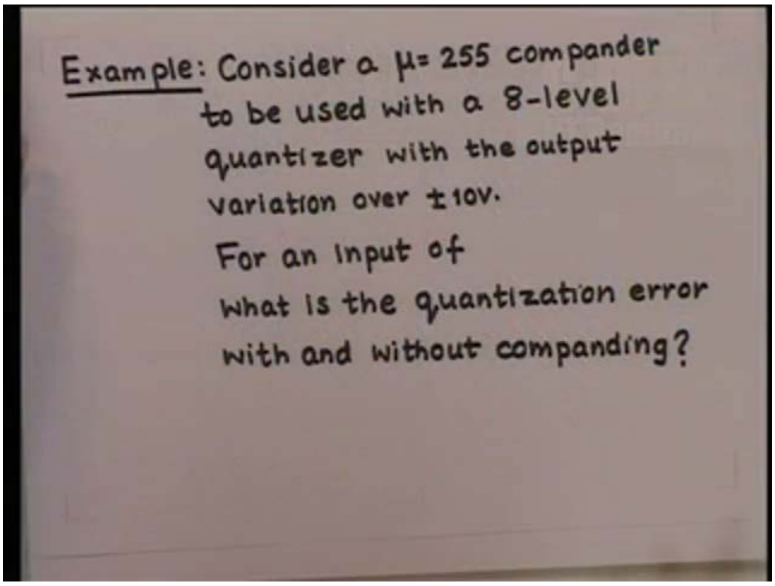

Let us take the first example as follows. Consider a mu equal to 255 compander to be used with an 8 level quantizer with the output variation over plus minus 10 volt. For an input of say 0.5 volt, it is desired to find the quantization error with and without companding. So, let us try to find to provide the solution for this.

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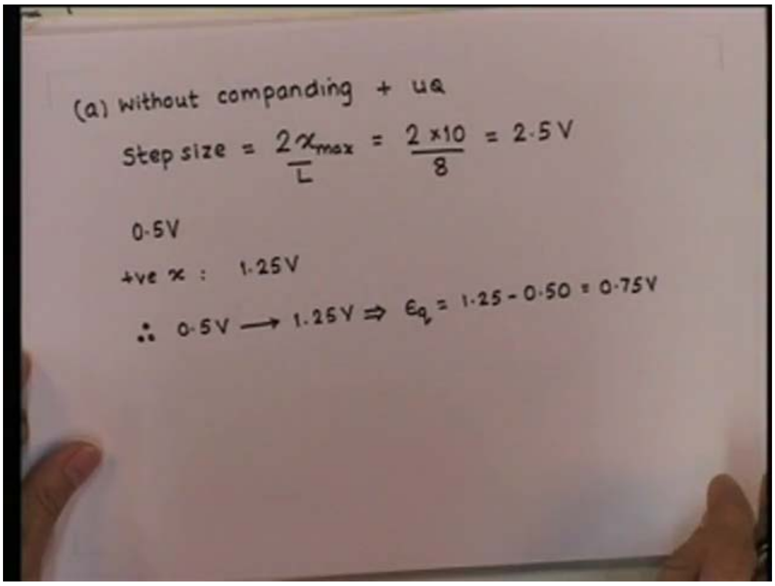

So, let us first consider without companding and using uniform quantization. So, in this case, the step size is given by 2 x max by L. In this case, we have 2 into 10 by 8t level quantizer. So, the step size is 2.5 volts. Now, the given 0.5 volt lies between 0 and 2.5 volt. Therefore, the first reconstruction level for the positive x is 1.25 volt. Therefore, 0.5 volt will get quantized to 1.25 volt, which implies that the quantization error will be equal to 1.25 minus 0.50 is equal to 0.75. Now, let us look at the quantization of 0.5 volt with the help of mu law compander.

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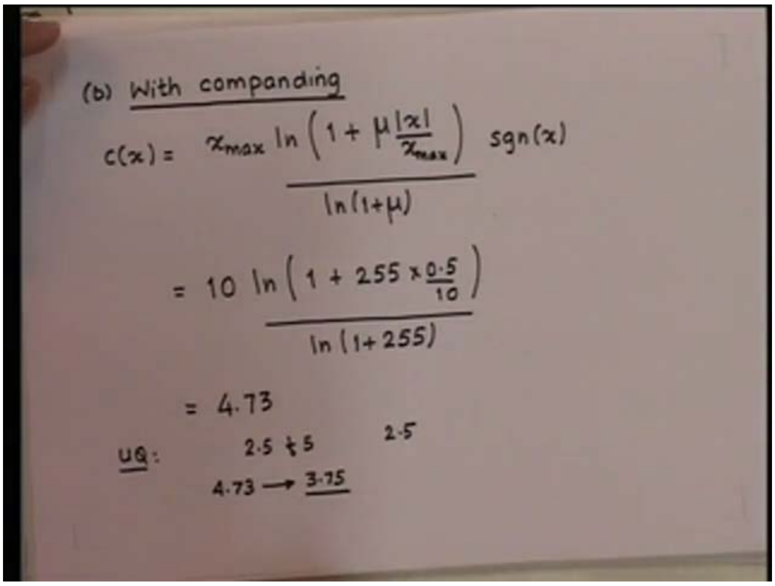

So, with companding, the first step is to pass it through compressor, which is equal to c x max log of 1 plus mu times. So, we substitute the given values as this reduces to 4.73. So, this is the first step. The next step is to take the output of the compressor and feed it to the uniform quantizer. So, next step is uniform quantization. Now, 4.73 lies between 2.5 and 5. We know that the quantization step size is 2.5. Therefore, 4.73 will get quantized to 3.75, which is the reconstruction level for the interval between 2.5 and 5. That is the average of these 2 values. The final step is to take this 3.75 and pass it through the expander characteristics.

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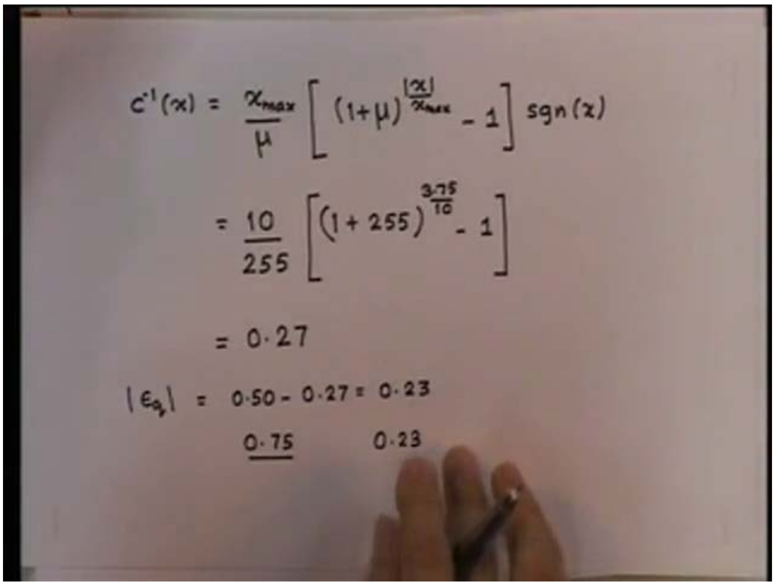

It is given as c inverse x is equal to x max by mu times 1 plus mu raise to mod x by x max minus 1. So, this is equal to and this can be evaluated as 0.27. So, in this case, the quantization error with companding absolute value is equal to 0.50, which is the input minus the quantized value 0.27, is equal to 0.23. So, without companding, we have an error of 0.75, whereas with companding, we have an error of 0.23. So, this shows an advantage of companding. Now, let us look into the performance of mu law companding little more in depth. In order to simplify our discussion, we will assume that the input to the mu law compander is normalized by x max. So, in this case, we have the input which lies in the interval between minus 1 to plus 1.

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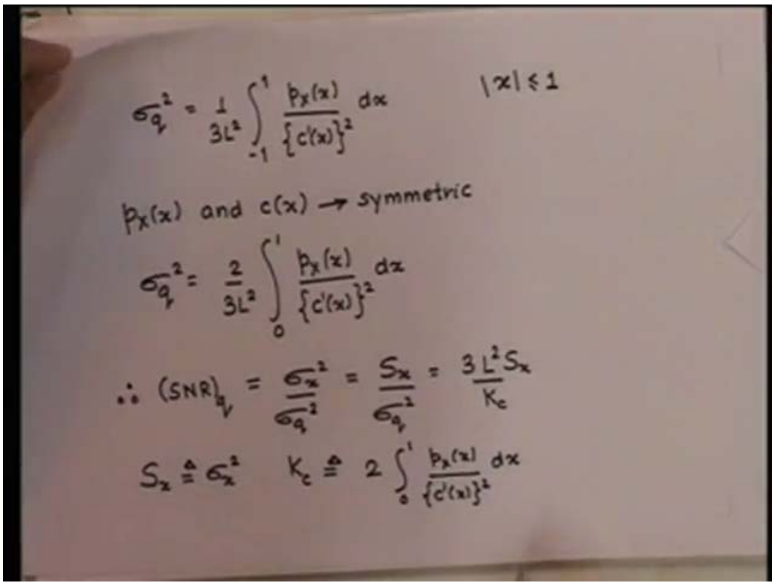

So, if we assume that, then in this case the quantization noise expression that is this reduces to where x is a normalized random variable. This x is lies between the interval between minus 1 and plus 1. Now, if we assume the symmetry both for input P D F and compressor characteristic, then this expression reduce to 2 by 3 L squared. Therefore, signal to quantization noise ratio is equal to input variance divided by the quantization noise, which is equal to S x divide by sigma q squared is equal to 3 times L squared S x by K c. S x is by definition input variance. K c is by definition equal to twice the integration of input P D F divided by the squared of c dash x. So, this is the constant which is dependent on the compressor characteristic and the input P D F. So, in this case, the popular mu law companding for voice telephone employs the following compressor characteristic.

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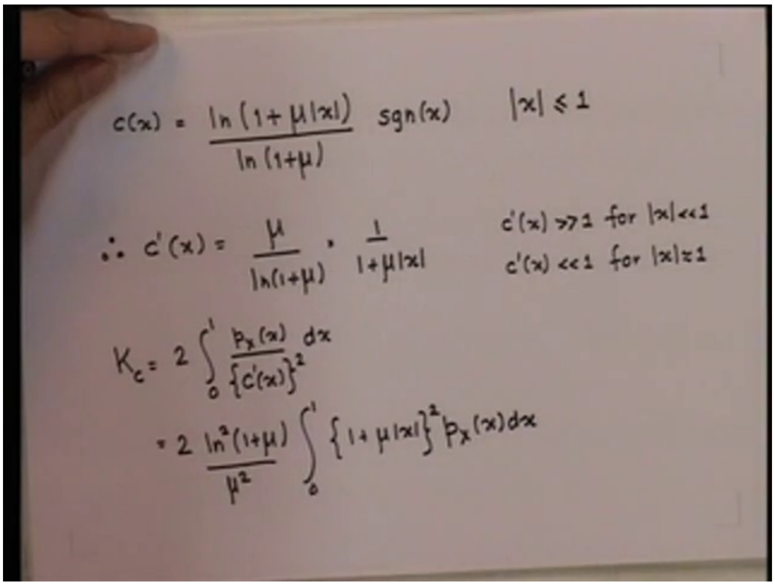

Therefore, c dash x is equal to mu by log 1 plus mu times 1 by 1 plus mu times mod x. The parameter mu is a large number. So, c dash x is much larger than 1, for x much smaller than 1 c dash x is less than 1 for mod x approximately equal to 1. Therefore, in this case, K c which is equal to twice integral for 0 to 1 of p X c dash x squared is equal to twice log squared 1 plus mu by mu squared. It can be very easily shown that this expression reduces to K c.

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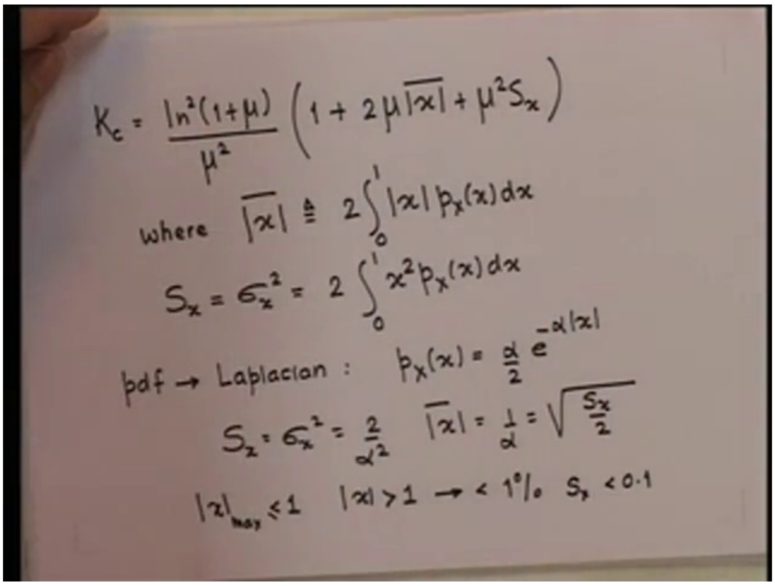

K c is equal to log squared 1 plus mu by mu squared 1 plus 2 mu mod x plus mu squared S x. Average of mod x is equal to by definition twice S x and S x is input variance which is equal to sigma x squared. Now, we need the values for average of mod x and S x to test the efficacy of the mu law companding. Now, empirical results have shown that the P D F of a voice signal is reasonably modeled by Laplacian P D F of the form equal to alpha by 2 e raise to minus alpha times mod x. Alpha is a constant and in this case, S x is equal to 2 by alpha squared.

Average of mod x is equal to 1 by alpha is equal to square root of S x by 2. Now, unfortunately, this P D F cannot be normalized for x max less than equal to 1. But the probability of mod x being greater than 1 is less than 1 percent if S x is less than 0.1. Now, otherwise P D F models feel about the same relationship between mod x and S x, which is the critical factor for evaluating K c.

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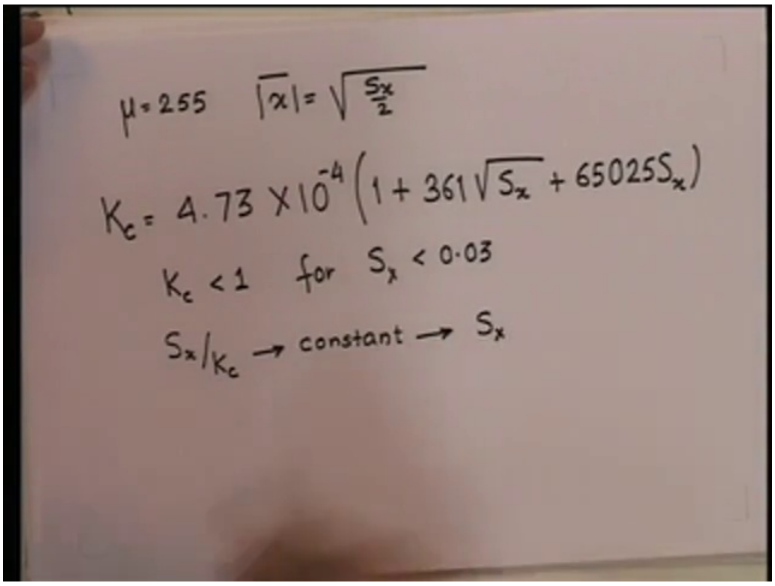

So, taking the standard value for mu equal to 255 and mod x average equal to square root of S x by 2 gives the value for K c equal to 4.73 multiplied by 10 raise to minus 4 times 1 plus 361 square root S x plus 65025 S x. Now, numerical evaluation shows that K c is less than 1 for S x less than 0.03. But, more significant is the fact that S x by k c stays nearly constant over a wide range of S x. So, consequently mu law companding for voice signal provides an essential fixed value of signal to quantization noise ratio despite the variations of S x, the input variance among the individual talkers.

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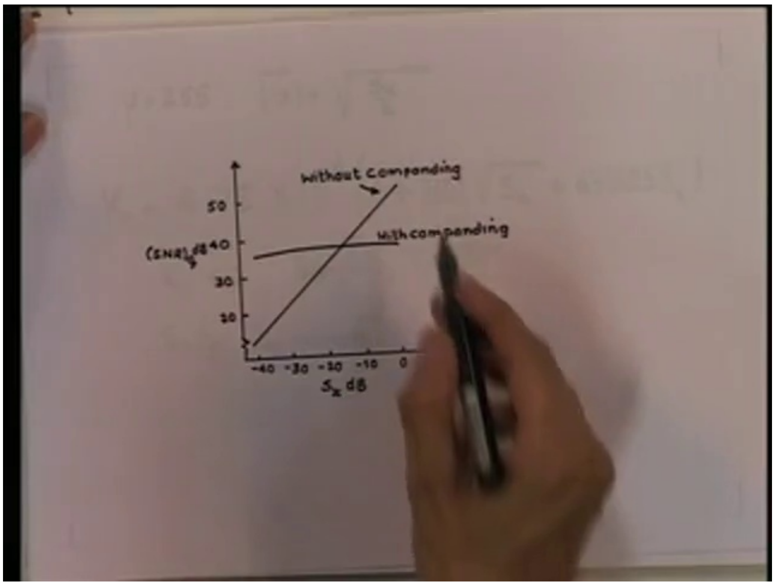

Now, the figure here brings out this desirable feature by plotting signal to quantization ratio in dB versus input variance in dB with and without the companding when L is equal to 256. From this figure, it is clear that there is a considerable companding improvement for S x less than minus 20 dB. So, in a design of a quantizer, there are 3 important issues. These are first selection of decision boundaries or boundary values, second selection of pre construction levels and third selection of code words.

In our study so far, we have assumed fixed length coding for a given number of quantization levels of a quantizer. Now, if this is the case, selection of code word is not an important issue. Therefore, we have only considered the selection of decision boundaries and reconstruction levels with consent of minimizing the mean square quantization error.

But, now if we relax this constraint of fixed length coding and deploy variable length coding, then the selection of code word becomes an important issue. It will be beneficial to design the quantizer, wherein we can incorporate the entropy of the output of the quantizer. This will decide the rate of the quantizer. So, in the next class, we will look into the entropy constraint quantization.