Digital Communication

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Lecture - 23
Calculation of Probability of Error (Part - 2)

Hello every one, in the last class we started discussion on probability of error calculation for different digital modulations schemes. And we actually calculated the probability of error for pulse amplitude modulation, for binary pulse amplitude modulation. And in this class we will continue our discussion, on probability of error calculation and specially try to see, how to compute probability of error for some other modulation techniques.

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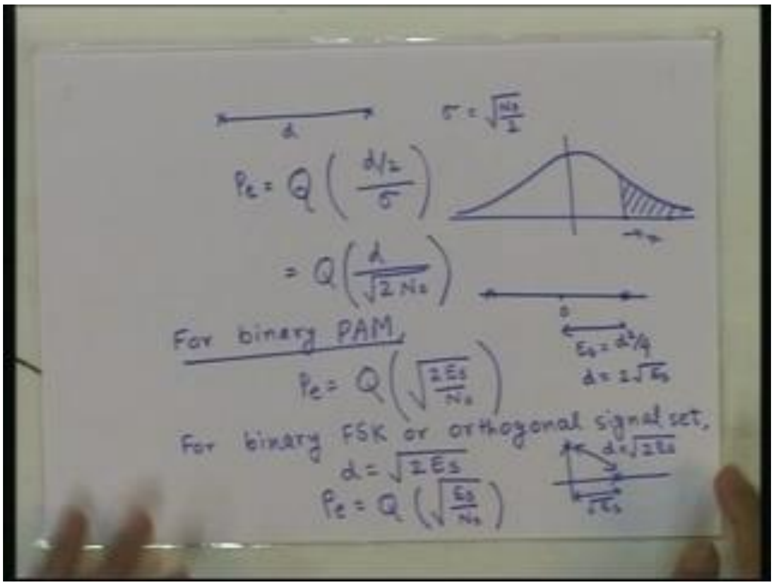

So, in the last class we saw that, for a binary signal set if we have 2 points at distance d from each other and if the noise variance is sigma and we know that that the sigma is nothing, but root over N naught by 2. Because, sigma is the standard deviation of the noise component, in 1 dimension or dimension noise.

So, the variance sigma squared is a N naught by 2 and then, the probability of error for this modulation will be Q of d by 2 by sigma, while Q function is the area under the Gaussian pdf probability density function of unit variance; 0 mean unit variance. If we take Gaussian pdf of 0 mean and unit variance and take a point here then, this area is the value of the Q function at this point.

Then we have seen that, this is the probability of error for the binary modulation scheme. And we can replace sigma by this and we can write this as Q of d by root over 2N naught. For binary PAM, we can see that the 0 is here and these towards the points. So, if this distance is d, then this is d by 2, this is d by 2 and so the energy of the signal, each of the signal is d square by 4.

So, d is 2 times root Es and so for binary PM, we have Pe equal to Q root over 2 Es by N naught. And for binary FSK or any other orthogonal modulation scheme; binary orthogonal modulation scheme, binary FSK or binary orthogonal signal set, what do we know? We know that d is root over 2 Es because, the signal set is like this; there 2 points here and the energy of each of these points is Es. So, this distance is root over Es. So, this distance d is root 2 times root over Es that is root over 2 Es. So, PE in this case will be Q of root over Es by N naught.

So, for any binary signal set, we can compute the probability of error, if we are able to compute the Q function value at any point and only thing we want we need to know, is the distance between the 2 points; the signal points and the noise variance. So, this is actually a fundamental component in computing the probability of error of a any signal set. This, the method of computation of probability of error for binary signal set, is a very standard technique. This will be used; the similar technique will be used in computing the other probability of error, for other modulation technique as well.
So, let us try to generalize these 2 other type of signal sets.

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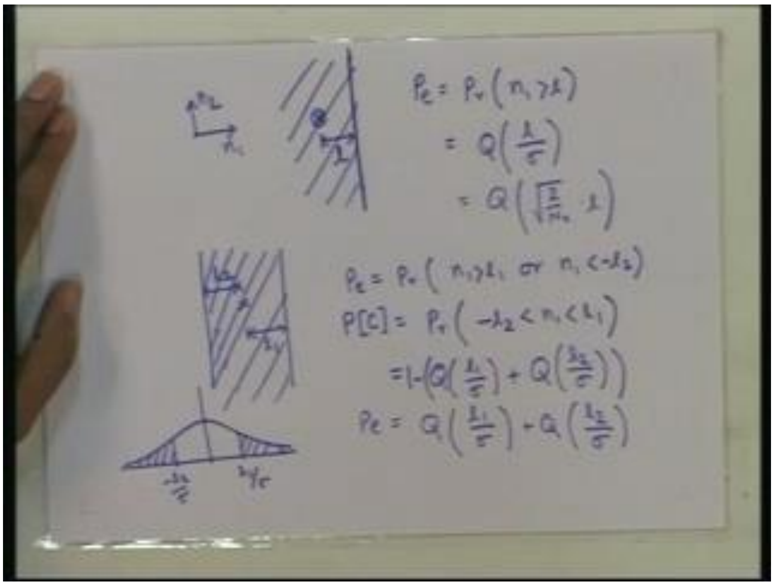

Let us consider 1 point in a signal set. Suppose, we have a signal set and we are considering 1 particular point and we want to find out, what is probability of error if this point is transmitted? So, there will be some other points and because of that, there will be decision region for this point that is, there will be an area of the signal space; in the signal space in which if the received vector lies, then we will detect this point.

So, if we transmit that point and we the received point is not in that decision region of this point then, there will be an error. So, suppose that the signal set is such that, the point has the decision region of this type; half plane on the left hand side of this line, this is the decision boundary for this point. This is our point and this is the decision boundary that is, if the received point is on this side of this line then, we detect this point otherwise, we detect some other point.

Then, what will be the probability of error for this point? So, we are transmitting this point and we have do find out, what is the probability that the received point will be on the right hand side of this line. So, note that the received point depends, only on the noise component in this direction. This is 2 dimensional plane and there are 2 components of the noise n1 and n2. The n2 does not matter; if we transmit this point, whatever is the value of n2 it will still lie on in this decision region.

So, whether it is in the decision region of this point or not, will be determined by the value of n1 alone. And for the received point to be in the decision region; this decision region, n1 should be less than this value. So, suppose this distance is l; l is the distance of this point from the decision boundary then, what is the probability of error? The probability of error is, the probability that n1 is greater than l.

If, n1 is greater than l the received point will be on the right hand side of this line. And what is this? This is nothing, but Q of l by sigma that is, Q of root over 2 by N naught times l. Now, let us consider another type of decision region; another type of decision region is this; this is a strip. So, on either size suppose there is, this distance is l1 and this distance; distance of the point from the decision boundary on the left is l2. Then, what is the probability of error? The probability of error all again, the error probability error depends on the value of N1 alone, N2 does not matter.

So, this probability is nothing, but the probability of error, is the probability that n1 is greater than l1 or n1 is less than minus l2. In other words the probability of correct decision, we denote it by this notation; probability of correct decision is the probability that, n1 is in this range. If the noise is in this range then, the decision will be correct because, the point received point will be in this region. And what is this? This is the area under the Gaussian PDF of unit variance; the sum of these areas. This is the probability that, n1 will be greater than l1 and this is the probability that, n1 will be less than minus l2. Of course these points need to be computed because, there will be normalization.

This is the Q function is so that, this is the area of the Gaussian PDF with unit variance whereas, the noise may not have unit variance to start with. So, we need to normalize the, this point. We have seen these details in the last class. So, this will be l1 by sigma and this will be minus l2 by sigma. So, note that this area is Q of l1 by sigma and this area is Q of l2 by sigma. This is minus, but, this area is same as, on this side, if we take this point on this side and take Q function the same it is symmetry; Gaussian PDF is symmetric.

So, these 2 areas together is Q of l1 by sigma plus Q of l2 of l2 by sigma. This is the probability of error. So, this central of this area is 1 minus that. So, probability of error is Q of l1 by sigma plus Q of l2 by sigma. Similarly, we can compute probability of error for decision regions of other some other types, which we will do in a moment. Let us now consider the decision region of this type. Now, it is not only no 1 along 1 dimension, but let us say we have a decision region of this type.

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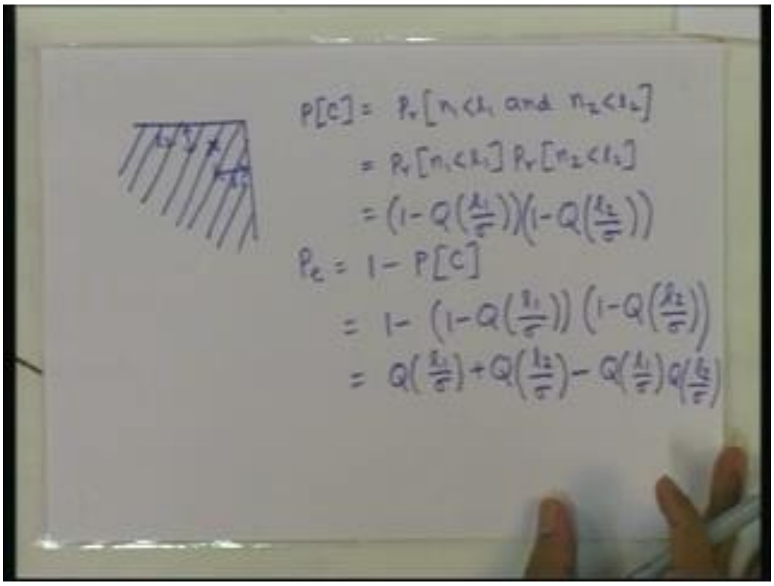

So, this is our point and suppose, this distance is l1 and this distance is l2. Then what is the probability of correct for decision for this point? Probability of correct this point is the probability that, N1 the noise component in this direction is less than l1 this value, then it will be on this side of the line. So, n1 should be less l1 and the noise component in this direction, should be less than l2. So, n2 should be less than l2.

Since, n1 have and n2 are independent of each other, this can be written as fact factor of the 2 probabilities. So, probability of n1 less than l1 times probability of n2 less than l2. And what is this probability? This probability is nothing, but 1 minus the probability that n is greater than l1 and that is, Q of l1 by sigma. And this is similarly 1 minus Q of l2 by sigma.

So, the probability of correct decision is expressed in terms of l1, l2 and sigma this way. Now, from here we can compute the probability of error that is, 1 minus probability of the correct decision that is, 1 minus these 2 factors times 1 minus Q of l2 by sigma. This can be of course, simplified and this product, from the product 1 will cancel and then, we will have Q of l1 by sigma plus Q of l2 by sigma times minus Q of l1 sigma times Q of l2 by sigma.

Now, let us consider another type of decision region. So, these are all similar in nature and can be computed in similar manner. So, this type of decision region.

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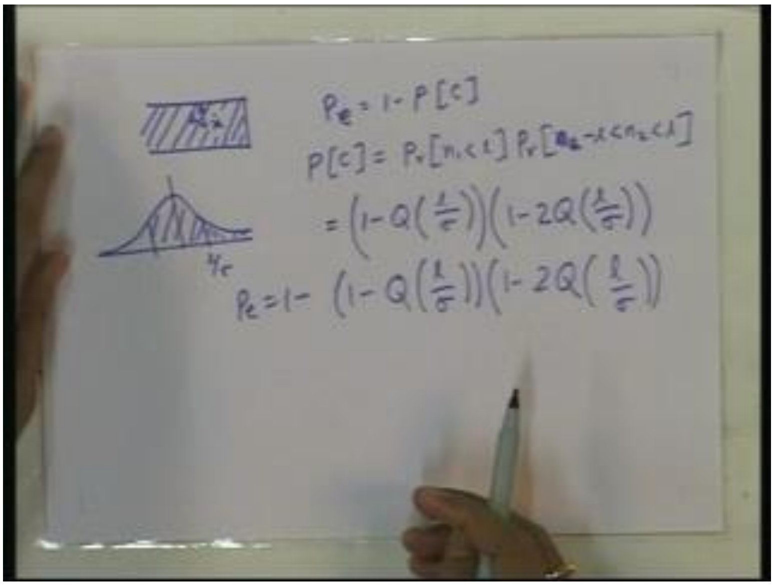

So, this is the decision region and let us not, considers any arbitrary distance here, here and here of course, that also can be computed simply, but we will let us just simply assume that all these distances are same. If they are not same, the expression will be little more complicated, but the method is as simple as this case; just like the previous 3 examples.

So let us suppose that, all these distances are l that is, this is distance l away from all these lines; this line, this line and this line. Then, what is the probability of error? Probability of error is 1 minus probability of the correct decision and what is probability of correct decision for this? This is the probability that, n1 is less than this l and n2. So, and we can now factorize because, n1 and n2 are independent of each other and probability that n2 is, n2 is in this range minus l to l. Then, it will be in this range. So, this is the probability of correct decision. And what is this; probability of n1 less than l is 1 minus probability of n1 greater than l and that is, 1 minus Q of l by sigma. And then this is, remember we have already computed probability of this type in 1 of the slides. This is basically the area of this type; probability of n2 being in this range. So, this is 1 minus the 1 minus 2 times this area. So, 1 minus 2 times Q of, this is l by sigma then 2 times this area that is, 2 times Q of l by sigma.

So, this is the probability of correct decision. So, probability of error is 1 minus 1 minus Q l by sigma 1 minus 2 Q l by sigma. And we can expand the product and simplify the expression. Similarly, we can similarly and finally, we let us compute the probability of error, for a point with this type of decision region.

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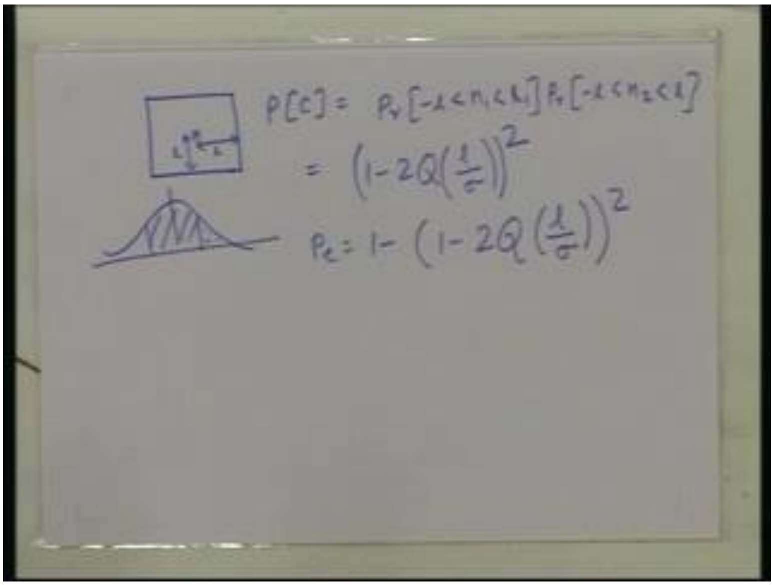

The point is at the centre and this distance is l from all sides; l this is l from all sides. Then what is the probability of correct decision? It is probability that, n1 is in this range and also n2 is in this range. And we know that n1 and n2 are independent, but identically distributed, both are Gaussian with same variance. So, these 2 are same. So, this is, 1 of this square and what is this probability? This is the again the area here; this area and that is 1 minus 2 times this area. So, this is 1 minus 2 times Q of l by sigma and Whole Square.

So, probability of error for this type of decision region is, 1 minus 1 minus 2 Q l by sigma whole square. So, we have seen for different type of decision region, what will be the probability of error. Now, in a particular signal constellation different points will have different shape of decision region. And there are some, there is a whole family of signal sets, for which the points have decision regions of these types, that we have just now considered. Either it is a box or it is 1 side open, but 1 side closed or all these types that we have considered.

So, there is a whole family of signal sets for which the decision regions of different points are of these types. So, for those signal sets, we will be able to use this probability of error calculation techniques to compute the average probability of error for the signal set. Let us take 1 example and try to compute the probability of error for that signal set. So, let us take 16 QAM constellation for example.

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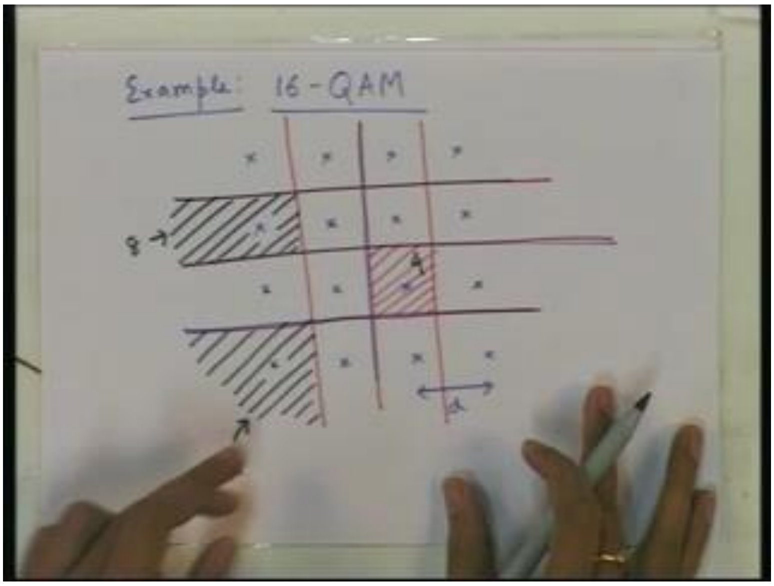

So, we have a 16 QAM constellation and suppose that, this distance is d. And we know that, this d actually related to the average energy of this signal set. If we know d, we can compute the average energy of the signal set and vice-versa. So, let us just take d as the parameter and compute the probability of error in terms of d and the noise variance sigma square.

So, let us try to find out first what of the types of decision regions that these points have? So, decision regions are like this; these are the decision boundaries and this is also there. So, let us plot; let us draw with red ink. So, this is 1 decision 1 boundary, this is another decision boundary, this is a decision boundary then, this is a decision boundary, this is a decision boundary and this is also a decision boundary. So, what are types of what are shapes of decision regions we have got?

We have got, 1 this close rectangular decision region. We have 4 such points, for which decision region has this shape, this point, this point, this point and this point and then, we have decision region of; say this type and there are 4 such points for which the decision region has this shape this 1, this 1, this 1 and this 1. The orientation of this shape does not matter for the probability of error because, it is just rotation or we know that, rotation of decision region does not change the probability of error rotation or translation.

So and there is another type of decision region in this and that is, like this and there are 8 such points for which, the decision region has this shape; this point, this point, this point, this point, this, this, this and this. So, there are 8 such points. So, this decision this type of decision region is there for 8 points, this type of decision region is there for 4 times and this type decision region is there for 4 times.

Now, let us; so this is important because, we would like to compute the average probability of error. Now average has to be taken over all the 16 points. So, we will compute the probability of error for this point that is if, this point is transmitted, what is the probability that, there will be an error in detection. We will compute that and we will assume that, that is the probability of error for each of these 4 points. And then; we will compute the probability of error for this point, we will compute probability of error for this point and then this probability of error will be the same for all these points and this will be same for all these points.

So, let us compute the probability of error for 3 of those points. First let us take this point and we have already computed, we have seen how to compute if this distance in this case is d by 2 because, this is the d and this is d by 2. So, l equal to d by 2. We have already computed this probability of error for such decision region and for we have do just put l equal to d by 2 here.

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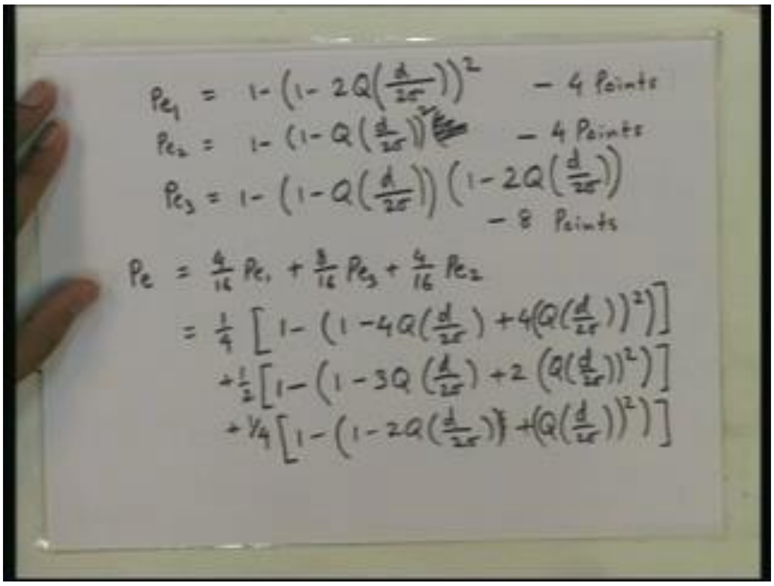

We will just copy this and so we have here; let us denote this probability of error by Pe1. This is 1 minus 1 minus 2 times Q d by 2 sigma whole square. So, now let us take another point; let us take this point and find the probability of error. And this distance is also d by 2, this distance is also d 2. So, we have also computed the probability of error for this type decision region and that is; here.

The probability of error is; let us denoted by Pe2. This is 1 minus this product. So, 1 minus 1 minus Q and both l1 and l2 in this case is d by 2. So, d by 2 sigma times 1 minus Q, again the same quantity. So, we will have square of this. And there is another type decision region that is, now this is left. So, for this also we have computed the probability of error before and that is; this.

So, let us denote it by Pe3. So, this Pe3 is 1 minus 1 minus Q d by 2 sigma times 1 minus 2Q d by 2 sigma. So, we have computed the probability of error for these 3 types of decision regions that come in this 16 QAM constellation. Now this Pe1; this probability this same probability of error is valid for these 4 points. So, there are 4 points for which, this is the probability of error and there are; this Pe2 is for this point and there are 4 such points also.

So, there are 4 points for which, this is the probability of error and there are 8 points for which, this is the probability of error. So, what will be the average of the average probability of error of the 16 QAM constellation? We have to take 4 times this plus 4 times this plus 8 times this by 16. So; that means, 4 by 16 times this plus 4 by 16 time this plus 8 by 16 this. So, average probability of error Pe is: 4 by 16 times Pe1 plus 8 by 16 times Pe3 plus 4 by 16 times Pe2.

So, let us write this down; this is the one-fourth times 1 minus let us take the whole square. Let us expand the square; this will be 1 minus 4 times Q d by 2 sigma plus 4 times Q d by 2 sigma whole square. Then, plus half Pe3 this let us expand this product also; we have 1 minus 3 times Q of d by 2 sigma plus 2 times Q d by 2 sigma square plus one-fourth times Pe2 that is, 1 minus let us again expand this square we get, 1 minus 2 Q d by 2 sigma and plus Q d by 2 sigma square.

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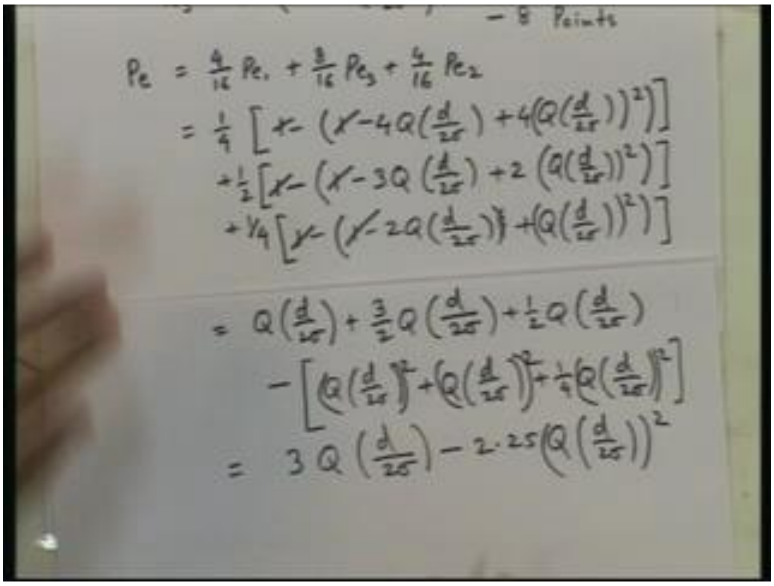

So, this can be simplified; this 1 all, this 1 cancels with this 1, this 1 cancels with this 1 cancels with this 1. So, what we have now is; this minus goes inside and this signs change. So this is plus, this is plus, this is plus and these signs are minus. So, let us add all these terms and we get after multiplying by these factors; so one-fourth times 4. So, this is Q d by 2 sigma plus half times 3. So, that is 3 by 2 Q d by 2 sigma. Then, one-fourth times 2 that is, half Q d by sigma and minus all these terms multiplied by the corresponding factors.

So, this is the one-fourth times 4 that is Q d by 2 sigma whole square then half times 2 that is, again Q d by 2 sigma whole square, it is one-fourth d by 2 sigma whole square. The square is on the whole Q function. So we get: this is 3 by 2 plus half is 2 to 2 plus 1 is 3. So, 3 Q d by 2 sigma minus, this is 2.25 Q d by 2 sigma whole square. So, this is the probability; average probability of error of 16 QAM constellation and this is in terms of d and sigma. And d can be expressed in terms of the average energy if wanted.

For example, let us just see; what will be the average energy of the signal set in terms of d. We have the signal set like this. Now, what is the…? This is the origin. So, what is the energy of this point? This point has energy; this d by 2. So, d by 2 whole square plus d by 2 whole square, this is also d by 2. So, 2 times d by 2 whole square that is, d square by 2. So, the average energy Es is; there are 4 such points.

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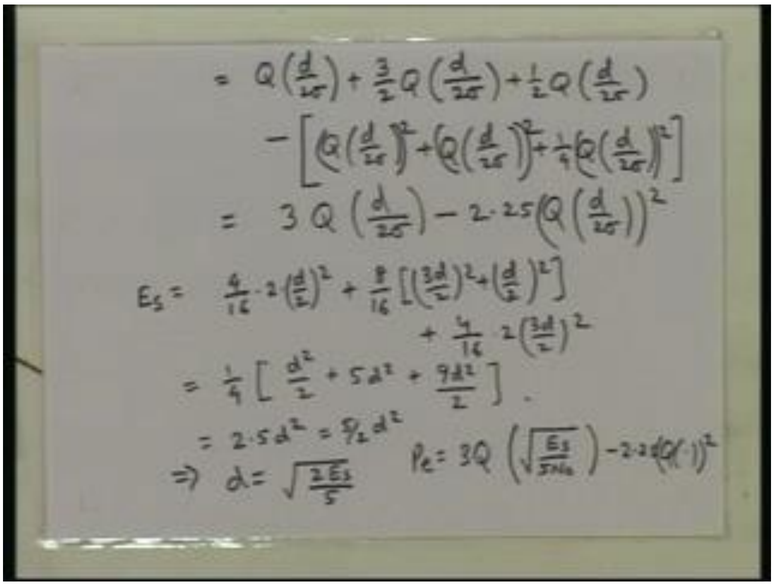

So, 4 by 16 times 2 times d by 2 whole square; 2 times d by 2 whole square is the energy of that point and there are 4 such points so, 4 by 16. Total number of points is 16. So, you are averaging the energy of all the points. Then, there are 8 points of same energy of this type; this this this this this this this this this.

So, energy of this point is 3d by 2, this distance is 3d by 2 whole square plus d by 2 whole square. This is the energy of this point and that is there 8 times. So, 8 by 16 times 3d by 2 whole square plus d by 2 whole square plus again; there are 4 points of this type and energy of this is; 3d by 2 whole square plus 3d by whole square that is, 2 times 3d by 2 whole square. And this becomes one-fourth d square by 2 plus 5d square plus 9 d square by 2. that is, we get 9d square by 2 plus d square by 2 is 10d square by 2 that is, 5d square plus 5d square plus 5d square 10d square 10square by 4 That is, 2 point 5d square.

So, that means, d is, this is 5 by 2d square d is root over 2 Es by 5. So, this d by 2 sigma term; now the sigma is root over N naught by 2. So, this term becomes; so, the probability of error becomes 3Q root over Es by 5N naught minus 2.25 then the same quantity whole square Q same thing, then square. So, this is instead of expressing the probability of error in terms of d, we expressed in terms of the average energy of the signal set so that, we know as we change the average energy of transmitted energy; how the probability of error changes. And it will be good exercise for you to actually verify this by simulation.

Take a 16 QAM constellation, add white Gaussian noise in the channel and do usual deduction and find out what probability of error you get by simulating, by transmitting the large number of symbols. And then, compare it with this value; this value can be computed in terms of the Erfc function that is available in mat lab. So, 1 can verify this by simulation. Let us now, try to see how to compute probability of error for other types of signals sets. For QAM type of signal sets this is very simple because, we have the decision regions of those shapes. Now, let us take some other modulation techniques and find out how to compute the probability of error. We next consider orthogonal signal sets.

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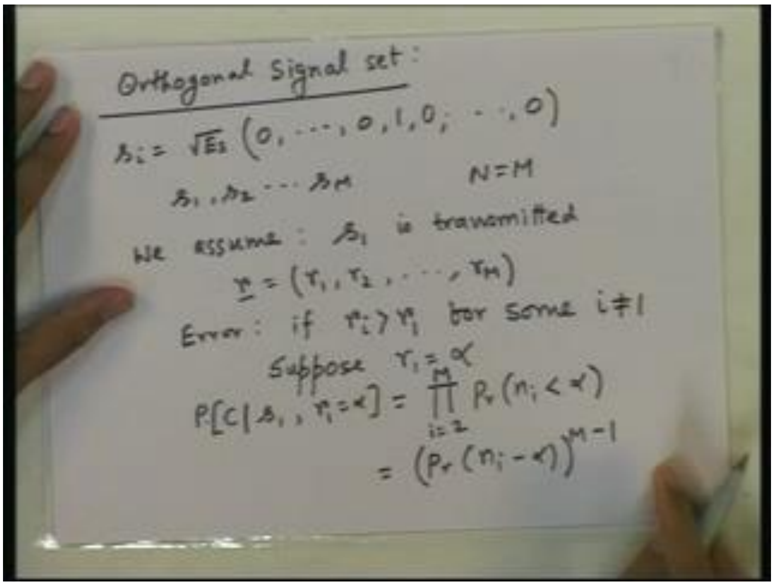

So, what is the set of transmitted points? It is root over Es times the ith transmitted point is all zeros except for the ith component 1 and then that is the multiplied by root over Es. If you multiply this the energy of this becomes Es. So, this is the ith transmitted pointed and there are M such points s1 to sM there are M components here. The dimension is same is as M. So, N is the dimension is same as M. And suppose we received the every component is added by some noise and we received some vector or r.

So since, the signal set is actually symmetric, the probability of error for all each point is same. So, we need to compute only the probability of error for 1 particular point and that will be the probability of error for all the points and as a result it will also be same as the average probability of error.

So, we assume that s1 is transmitted then, try to find out what is the probability of error. Now, what is the deduction scheme? We know that for orthogonal signal set the optimum detection the ml detection simplifies to the following: that take the components of the received vector r1, r2, rM. Choose the largest; choose the largest component and the index of that is the message transmitted that is, index that was transmitted. So, if ri is the maximum out of this then, i si was transmitted.

So, if we now assume that we transmitted s1 and try to find out what is the probability of error; the error will happen if there is some other component here, which is greater than r1. So, there will be error if, ri is greater than r1 for some i not equal to 1. So, let us assume that in this component we have received; suppose, r1 is equal to alpha; some value alpha, suppose we have received r1 equal to alpha then, what is the probability of error? The probability of error that is, the probability of; we will first compute the probability of correct detection and then compute probability of error.

So, probability of correct deduction given that: we transmitted s1 and we received r1 equal to alpha. Then what is this probability? That correct decision will be made if, each ri is less than equal to r1 less than r1 for every i not equal to 1. Now there are, m minus 1 such value of i. For each of them ri must be less than r1. So, what is the probability that ri is less than r1? Remember that, at the ith component we transmitted 0 because, we transmitted the vector s1. So, the ith component we transmitted 0. So, ri is basically the noise component ni.

So, the probability that ri is less than alpha, is same as the probability that ni is less than alpha because, ri is equal to ni for i not equal to 1. And this is for i equal to 2 to n. So, ni is are independent of each other. So, we can take the product. So, this is and this will be same for because, this is a all ni(s) have the same distribution. So, we have probability ni minus alpha power M minus 1.

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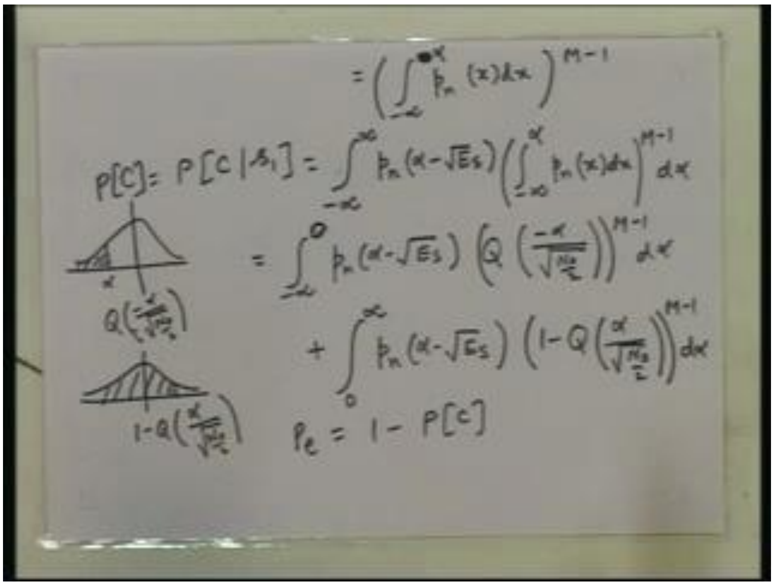

And this can be written in terms of the density function of the noise; all the ni have the same density function and we denote it by this have, density function Pn. Then, this can be written as minus infinity to infinity Pn x dx whole power M minus 1. So, this is less than alpha, minus infinity to alpha. So, ni; value of ni is taken from minus infinity to alpha. So, what is the probability that ni is in this range? So, that is the obtained by integrating the density function in this range.

Now, this alpha itself may be of different value. So, to take the average probability of error given that; s1 is transmitted that is, to compute probability of correct detection, given s1 is transmitted. We have to now integrate average over all values of alpha also that is; we have to take value of alpha from minus infinity to infinity then, take the density of n1 being such that r1 is equal to alpha. Now, when will r1 be equal to alpha? At the first component we transmitted root over es, we transmitted in the first component toward es because, s1 is transmitted, in the first component of the s1 is root over es and we receive alpha as the first component. Then, n1 must be alpha minus root over es.

So, n1 is alpha minus root over es and the density at that point is alpha Pn of alpha minus root over es. So, this is the value of the density at alpha minus root over es. And this time now we have to take this quantity. So, if alpha is received that the first component what, this is the probability that there will be error. So, we have to multiply by this minus infinity to alpha and Pn x dx or M minus 1 d alpha. And this can be now written in to, this can be written in to 2 parts 2 cases; for alpha minus infinity is 0 and for alpha 0 to infinity because if, alpha is positive this value has a is defined to be the Q function 1 minus Q function.

So, let us just see; if alpha is negative, what is this integral? If alpha is negative then, this integral is nothing, but take this density function and then minus infinity to alpha. So, this is nothing, but Q of minus alpha; minus alpha is positive Q of minus alpha by, you have to divide by the standard duration of the noise that is, root over N naught by 2. This is; this integral when alpha is negative. When alpha is positive, what is this area? This is the 1 minus Q of this. So, this is 1 minus Q alpha by root over N naught by 2.

So, here we have for alpha negative; minus infinity to infinity a minus infinity to 0, we have Pn alpha minus root over Es times this quantity is now, Q of minus alpha by root over N naught by 2 this power m minus 1 d alpha plus for alpha positive, we have Pn alpha minus root over Es times 1 minus Q alpha by root over N naught by to whole power M minus 1 d alpha.

So, we have this is actually good enough, but we have written it in terms of our known Q function. But, this does not actually allow us to compute this integration still because, this Q function does not have a closed form expression. We can compute it numerically and there are also tables for the values. So, only way to compute this will be, by numerical integration. And once we get this we know that, probability of error is 1 minus probability of correct decision. And probability of correct decision is same as this for s1 because, it is same all symbols. So, this is 1 minus this. So, this itself is P C and then we get the probability of error this way.

So, this needs to be computed in numerical fashion because, this expression is not really very nice; is quite complicated. By it can be computed quite efficiently using mat lab, by numerical integration. So, in this class we have seen how to compute the probability of error for some rectangular decision region shapes and then used that kind of computation technique to find the probability of error of 16 QAM constellation. And it can be used the same technique to compute any QAM constellation, the probability of error for any QAM constellation. And then, we have also seen how to compute probability of error for orthogonal signal set like, fsk ppm.

And in the next class, we will continue and see some more techniques and also will see when the exact probability of error is difficult to compute; whether we can get a bound which is much easier to compute or expressed in a much simpler way.

Thank you. See you in the next class.