Digital Communication
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Lecture - 02
Sampling

Hello, everyone. In this class we will discuss sampling an essential part in a digital communication system; where the source generates an analog signal not a digital signal. So, when we have a source which generates analog signal but we want to transmit using digital communication techniques; first we have to make the signal digital. And, for doing that we need to convert the signal in some form where per second there will be a finite number of bits that will represent the signal. So, since in time an analog signal has infinite number of values, for every time incidence there is a value; it is not possible to digitize a signal without first discretizing the time. And, discretizing the time is called sampling; later on we also have to discretize the amplitude. Because any sample value continuous sample value has infinite amount of information.

So, we need to discretize the amplitude also. So that we can represent every sample by finite number of bits; that will be called quantization and we will discuss it later. So, let us start sampling. So, we have a continuous time signal.

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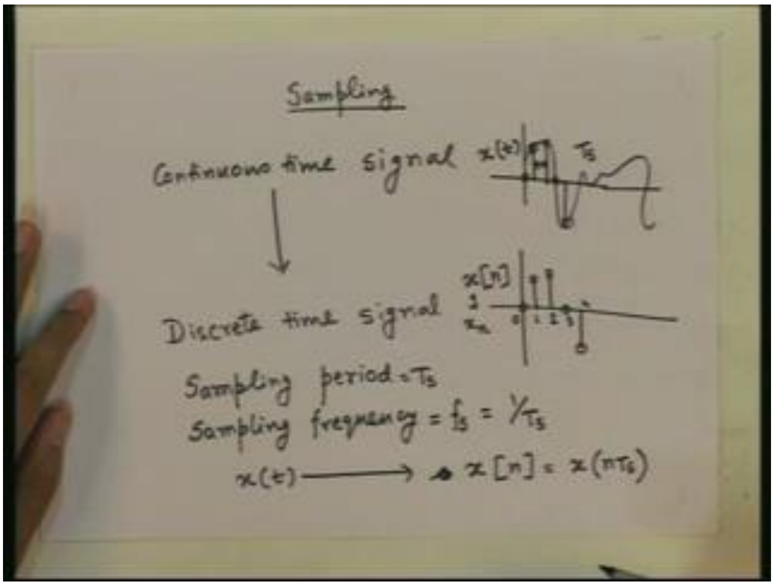

So, we are going to discuss sampling; we have a continuous time signal which may look like this and this is converted to a discrete time signal. So, we take a regular interval; we take values like this. And, we have a sequence of values; in between these sample values there is no data in the signal. So, the time is discretized in this signal. So, if this signal is represented by x (t) then this will be usually represented by x [n] this is 0, 1, 2, 3, 4 and so on or sometimes x subscript n. So, this is a discrete time signal. Now, while you discretize the signal first we have to fix a gap between the samples. So, that is the sampling period; this is T. So, the sampling period we will denote it by T s, it is denoted by T s. And, then 1 by T s is called the sampling frequency which will be denoted by f s.

So, in other words we have if we have a signal x (t) after sampling we have a discrete time signal denoted by x [n]; which is nothing but x this signal at n T s. Now, am important question here is whether we can recover this signal from the discretized signal; because we have removed all the values in between 2 consecutive samples. So, can you recover all the intermediate values? So, that we can get back the original analog signal from the discrete time signal that is an important question. And, we will try to answer that question for some cases. In other words when we discretize a signal that is when we remove all the intermediate values between 2 samples is there any loss of information? If there is no loss of information in principle it should be possible to recover those values of the of the intermediate values from the samples we have. On the other hand if there is some loss of information then it is not possible to recover all the values from the samples.

So, the question is it possible to recover the analog signal from the discrete time signal? We will answer that question for a special class of signals called band limited signal.

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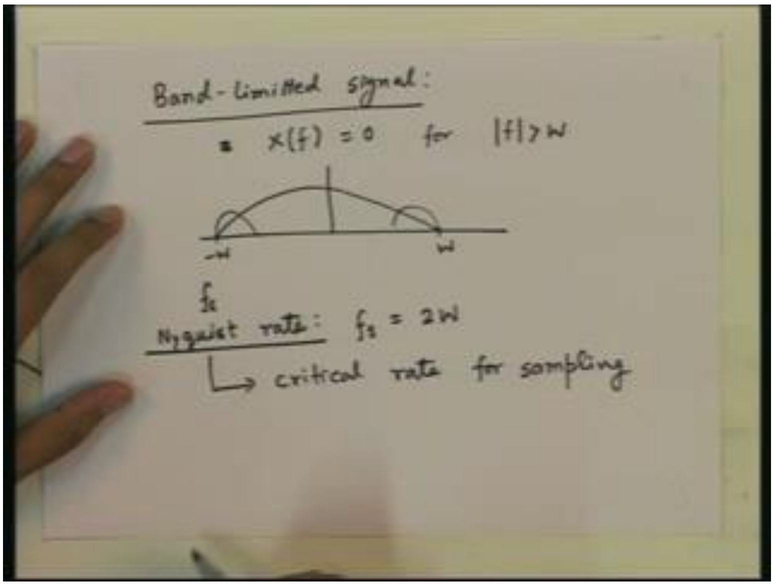

So, what is the band limited signal; it is what we know basically as low pass or a band pass signal. A band limited signal is so that the Fourier transform of the signal which is denoted by x f is 0 for all f which is greater than w or less than minus W that is for all f whose absolute value is greater than W. So, what it means is in the frequency domain if you take minus W to plus W range; the Fourier transform of the signal is 0 outside this range it is nonzero only in this range. So, it may be either a low pass signal or it may be a band pass signal like this, like this or a signal like this. So, this is band limited signal. Similarly, a stochastic process is also called band limited if the power density spectrum satisfies the same condition. So, a stochastic process is band limited if the power density spectrum is band limited.

So, suppose we know that a signal is band limited in minus W to W range; then what is the rate at which we can sample so that we can get back the original signal from the sample signal; that is can you say that given that the signal is band limited in minus W to W, what is the minimum sampling frequency? So, that even after sampling there is no loss of information. So, let us try to answer that question; before we prove or we get an insight into why what rate is sufficient let us first define what is called as nyquist rate?

The nyquist rate for a band limited signal band limited within minus W to W is the sampling frequency f s equal to 2 W. So, 2 W is actually the bandwidth of the signal if it is low pass. So, then the sampling frequency is 2 times the maximum frequency. So, that is the nyquist rate 2 times the maximum frequency; and this nyquist rate is the critical rate for sampling. In other words if we sample a band limited signal above nyquist rate we can recover the original signal from the sample signal. And, if it is below the nyquist rate we cannot recover the original signal from the sample signal. And, we will try to first get an insight into why it should be true and then we will also see how to actually recover the original signal if the sampling frequency is above the nyquist rate.

So, let us try to first get a feel of why nyquist rate should be sufficient for recovery of the original signal in the frequency domain? Let us first see it in frequency domain.

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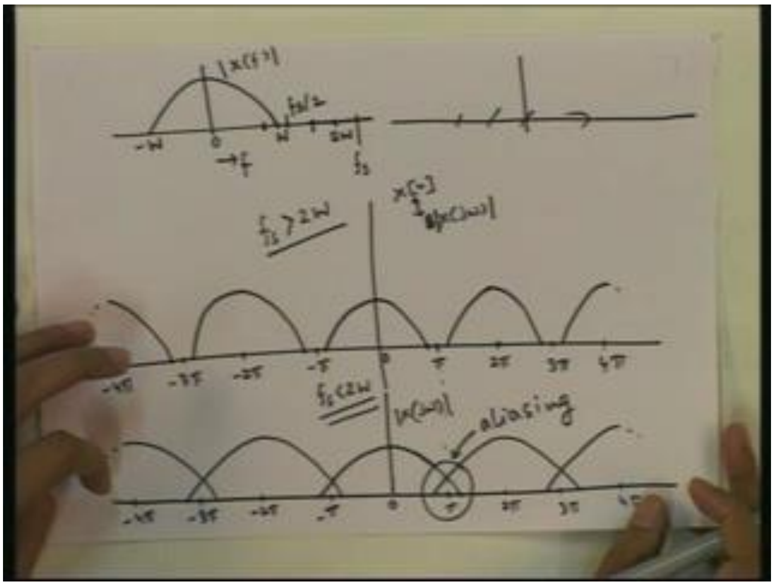

Suppose, we have the signal x (t) whose spectrum is whose magnitude spectrum magnitude is. So, this is frequency, this is 0 frequency, this is W, this is minus W. So, this is the this is the band limited spectrum of band limited signal and we want to sample it at some frequency. Now, the nyquist rate is 2 times this that is somewhere here; this is f s this 2 times W but we can sample it either at a lower rate than 2 W or a higher rate than 2 W. So, let us first see what happens if we sample the signal at a higher rate than 2 W? So, this is 2 W but we sample it at let us say here then what will be the discrete time Fourier transform of the sample signal; we know that just like we have Fourier transform of continuous time signal, we have discrete time Fourier transform which we abbreviate the abbreviate it as d t f t of a sample signal or a discrete time signal.

So, when you the sample the signal x (t) you get a discrete time signal; then we can take d t f t of that signal and how will that d t f t look like; given that the analog signal x (t) has this spectrum this Fourier transform it will look like this. First of all d t f t will be a periodic will be periodic with period 2 pi. So, let us draw it here. So, this will be pi minus pi, 2 pi, 3 pi, 4 pi and so on. So, the d t f t of the signal that is if the signal is denoted by x [n] and it has d t f t x j omega then it is magnitude will look like this. So, how will we get that this W this f s will be mapped to 2 pi frequency and 0 will be mapped to here; and then the rest of the frequencies will be placed linearly accordingly.

So, 2 pi is f s and so this will come here. So, this since the since f s is greater than 2 W this W will be less than f s by 2 which is somewhere here, f s by 2 is somewhere here. So, and f s by 2 will be mapped to pi. So, it will be like this. So, W frequency will be mapped to somewhere here and then this same thing will repeat after shifting by multiples of 2 pi. So, this will be shifted to this 0 will be shifted to 2 pi. So, we will get a copy of this here and at 4 pi here, then at minus 2 pi like this, then and minus 4 pi like this and so on. So, this the periodic d t f t of the sample signal. Now, as we can see in frequency domain at the d t f t of the signal has this spectrum as part of it. So, we can actually this whole spectrum is still present in this signal separately; there is no mix up of this these different copies. So, we can still recover this signal should be still possible to recover this signal from here. Because we just take this part and some do some processing and take this part and convert it to analog signal. So, we will see how to do it later. But we at least see that there is apparently no loss of information when you go from here to here; we get only multiple copies of this in this joined frequency locations.

On the other hand if we sample the signal, original signal at a rate less than nyquist rate that is if f s is below 2 W what do we get? So, this is case when f s is greater than 2 W; if f s equal to 2 by 2 W this 2 will touch at pi even then it is recoverable; there is no mix up what is call the aliasing, there is no aliasing. Now, let us see when what will happen if f s is less than 2 W that is f s is somewhere here and f s by 2 will be somewhere here less than W in the same frequency scale pi, 2 pi, 3 pi, 4 pi minus pi. So, this f s by 2 now which is less than W will be mapped to pi. So, the copy of this here will look like this, here this f s by 2 is mapped to pi. So, W will be mapped to something greater than pi and then this will be shifted to 2 pi this center 0 will shift to 2 pi this whole thing; and then we will get something like this.

And, then it should also be shifted to 4 pi. So, we get something like this then 2 minus 2 pi something like this and minus 4 pi will get something like this. So, this is this will be x j mega magnitude for f s less than 2 W this is for f s greater than 2 W. And, we see that here there is some mixing between different copies of the original spectrum. So, in this part there is some mix up and as a result it may not be possible to get; and what we get that this is actually the sum of these 2 components here. Here, this will be the spectrum will look like this and then some something here which is the sum of these, here this is just to magnitude. So, when we take sum it we have to take the complex numbers and then add. So, the magnitude of that will be will have some shade and it may not be possible to get these values from that sum value. And, as a result we may not be able to get this original signal back from this signal. Because the spectrum has some mixing of the different copies of the original spectrum; and this is called aliasing.

So, now suppose that the sampling frequency is greater than 2 W; then how to recover the signal that is also another question that we will try to answer later. But let us also try to see here we have seen in frequency domain why f s by f s greater than 2 W is required for recovery of the original signal and why f s less than 2 W is not sufficient; it may not be possible to recover the original signal if f s is less than 2 W. We will also try to get a feel of the same thing in time domain.

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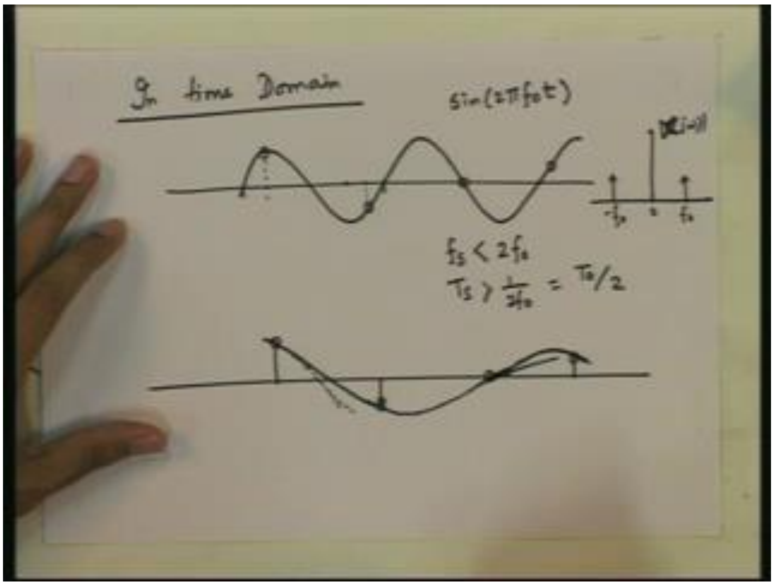

So, in time domain let us just consider a sinusoidal signal for simplicity; we have a sinusoidal signal of frequency say f naught. So, this is sin 2 pi f naught t; and let us choose a frequency f s which is less than 2 times f naught the maximum frequency of this what is the spectrum of the sinusoid signal? The spectrum is just 2 deltas this is the magnitude spectrum 2 deltas. So, this the signal is sometimes denoted by x (t) sometimes s t. So, here with the convention we are following we take the signal to be x (t) then this will be x. And, this will be at f naught and this will be at minus f naught, this is 0; this is the spectrum magnitude spectrum of the sinusoidal signal 2 delta functions. Now, sampling frequency is less than 2 f naught. So, what does it mean in terms of the sampling period T s is greater than 1 by 2 f naught. And, that means T naught by 2 1 by f naught is the period of this from here to here. Now, if we choose a sampling frequency less than 2 f naught what happens in the time domain; let us just sample this at less than this side.

Let us say that one sample falls here the peak this. So, say here and then so we sample; so the T s is greater than T naught by 2 that is the half period it is greater than half period. So, from here the half period is here. So, the next sample will come later than this. So, let us say somewhere here. So, this is the next sample. So, we get here a value this and then another T s later. So, T naught by 2 is here and then later let us say here we get another sample. Then, another T s which is greater than T naught by 2 this is T naught by 2 from here; and then let us say here we get another sample, these are the sample values we have got. Now, if we try to recover the signal from here; the estimate of the intermediate values will be obtained by interpolating these values. So, we can see the shape of the signal that we are likely to get it is a signal like this, this is the signal we will get we try to recover from the sample. And, we can see that this signal is quite different from what we originally sampled. So, this has a lower frequency for example; and what frequency we will get also can be actually inferred from the frequency domain. Let us see this what is happening here in frequency domain?

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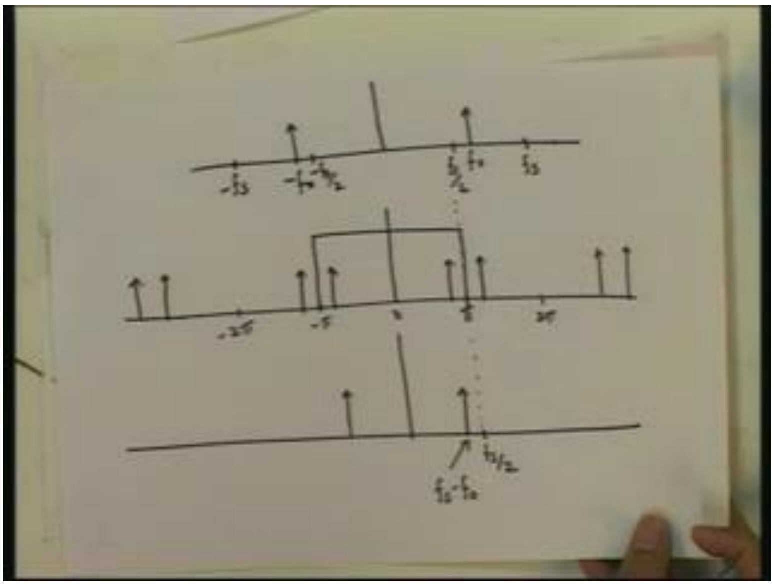

We have the signal spectrum, original signal spectrum like this f naught and minus f naught and f s is less than 2 times f naught, 2 times f naught is here and it is less than this. So, f s is here. So, f s by 2 is somewhere here. So, this f s by 2; similarly here minus f s by 2 and here we have minus f s. Now, when we discretize the signal the d t f t will look like this; we will have this shifted to multiples of f s and then added and of course the frequency will be scaled. So, that f s will be mapped to 2 p i. So, this 2 pi is the angular frequency actually. So, the frequencies f s will be mapped to 1 if we normalize; the frequency in d t f t. So, the spectrum will be so this f s by 2 will be mapped to angular frequency pi f s will be mapped to 2 pi this is 0, this is minus pi, this is minus 2 pi. So, we have one copy here the original spectrum; then this will be shifted to 2 pi.

So, once this is shifted to 2 pi we get another copy of the same thing here; this will this will come here and this will come here. And, similarly when this is shifted to 4 pi will get something here and another component later. And, when this is shifted to 4 pi minus 4 pi will get one component here and another component here. So, this is the d t f t of the sample signal. And, what is the recovered signal; if you recover the signal again this pi will be mapped back to f s by 2 because we know that f s is the sampling frequency. So, we will recover it in such a way that this frequency is mapped to f s by 2. So, this is f s by 2; then what we will get is this part of the spectrum. So, we will get our signal will be like this and what will be the frequency, this frequency will be so that pi is mapped to this then this is mapped to what that is frequency we will get. And, that frequency will be simply f s minus f naught.

So, you see that f s the signal frequency that we have got after a recovery of the signal is different from the original signal frequency. So, we have not recovered the signal correctly. Now, on the other hand if we had sampled at a frequency greater than 2 f naught then we would have got the same frequency back; we leave this as I leave this as an exercise for the audience. Now, let us see how to reconstruct the original signal if the sampling frequency is greater than the nyquist rate.

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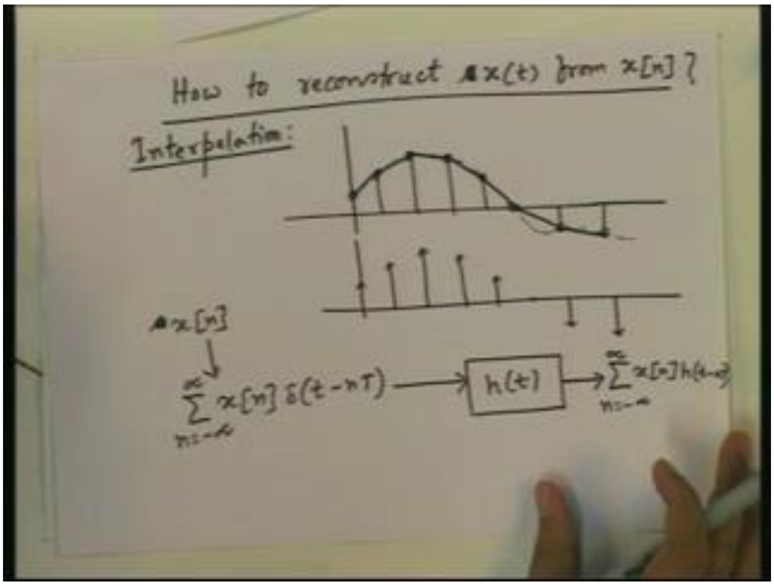

So, the question is how to reconstruct x (t) from x [n]? The answer obvious answer is interpolation; that is we have to estimate the intermediate values in some way or other from the sample values that we have. So, we let us say we have a signal like this and we have the samples at regular intervals. Then, we need to estimate we have only these values, these values we need to estimate the intermediate values because you want to make this again a continuous time signal. So, one obvious way is to just join the values; that is join like this, join by line straight line we will get some estimate of the intermediate values in this is. And, then we can of course to smoothen the signal we can pass it through a low pass filter; and that will be a reasonably good estimate of the original signal. Now, in general we can do interpolation by linear time in variant filters. Now, suppose we have what we do is we have x [n] first so we have this these values; instead of considering them as values we first convert it into a signal with delta functions of proper magnitude placed at these locations.

So, this we make this is a delta function that is we take from x [n] we get summation n equal to minus infinity to infinity x [n] delta t minus n T; this is the direct delta function. So, this is an analog signal now and then pass it through an analog filter L T I filter with impulses ones h t this is the impulses function; h [t] is the impulses function of the filter. Then, what we get here is the convolution of h [t] with this then the convolution can be taken inside the summation and then we have x [n] times this convolution h [t]. And, we know that when h [t] is convoluted with the shifted delta we get a shifted copy of the h

[t] itself. So, here we get x [n] h [t] minus nt. So, let us take some example and see what kind of signal this is; the output signal is.

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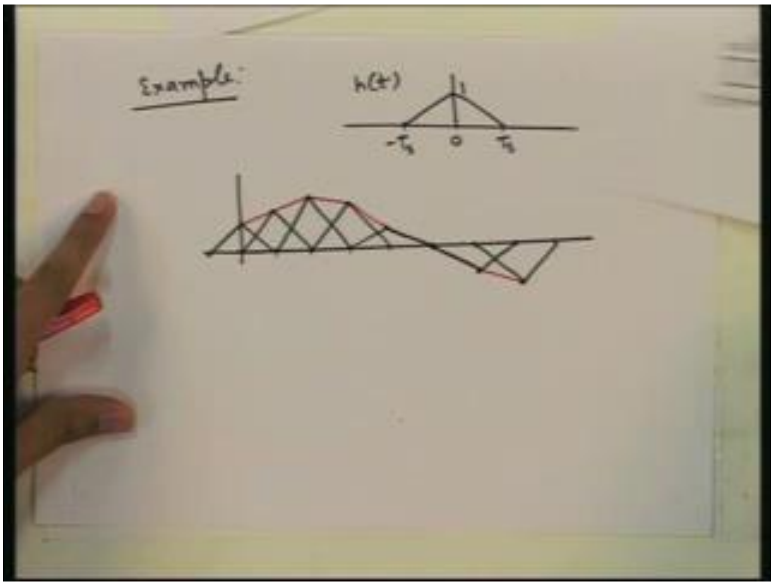

Let us take first the triangular impulses response. So, h (t) is say this is the sampling period T s and this is minus T s and h (t) is this and this amplitude is 1; if this is h (t) what we will get at the output; let us see what we get at the output. So, at the input we have this signal and we pass it through the filter with impulse response this. Now, when you convert this with h (t) what will happen? This h (t) will be placed at every delta because convolution of this with this delta is nothing but this shifted at that position. And, so the output signal will be sum of all such shifted copies; shifted and multiplied by the amplitude of this delta function. So, from this the output will be so this is these are the sampling positions and these are the values. Then, h (t) will be placed here that is the convolution of this delta this delta with h (t).

Then, convolution of this delta with h (t) is this convolution of delta here and h (t) will be this; just delta h (t) shifted to this position after multiplying by the minus 0 of this data. So, all these deltas will be added because there is a summation here. So, after the convolution will be multiplied by this delta; then the magnitude is 0 here. So, there is no copy of h (t) here. And, then now once we add this triangles; what we get here is nothing but these tips added by straight line, when we add these 2 graphs we get this straight line here. Similarly, we add these 2 we get a straight line here; if we add these 2 we get a straight line here. So, this is the signal we get and this is nothing but the simple interpolated we considered first that is joining the sample values by straight lines. So, that interpolation can be done by using a linear time invariant filter; first converting the discrete time signal into a pulse delta train. And, then pulsing it through a suitable L T I filter. But this is not the only choice of the impulse response; we can change this impulse response in different ways and get different ways of interpolating the intermediate values.

And, what h (t) is the best then, what impulse response should be chosen? So, that we get a good recovered signal recovered signal should be very near the original signal.

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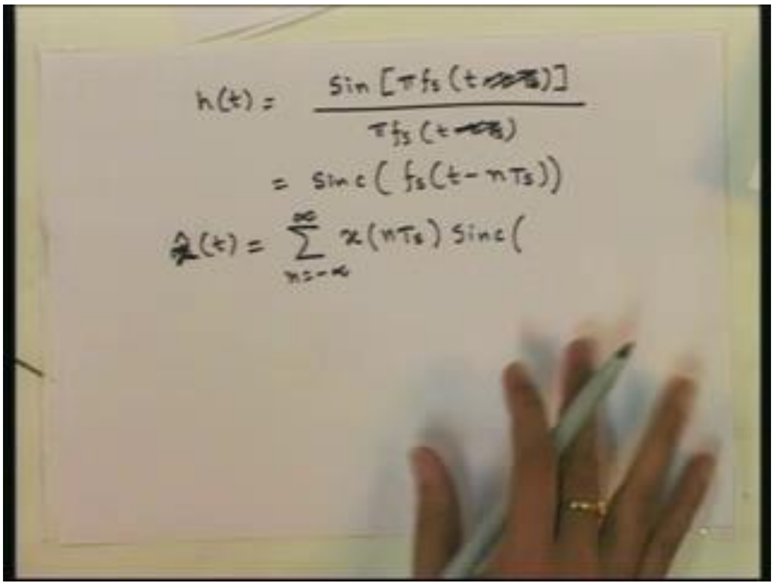

The answer is that we should take the following impulse response which is nothing but sin c f s t minus n T s. Now, if we take this impulse response the recovered signal is let us by our notation this is x; then x n T s sin c; right there is no sorry this is after shifting. So, if we do not shift it we get simply t a slight it replace phase.

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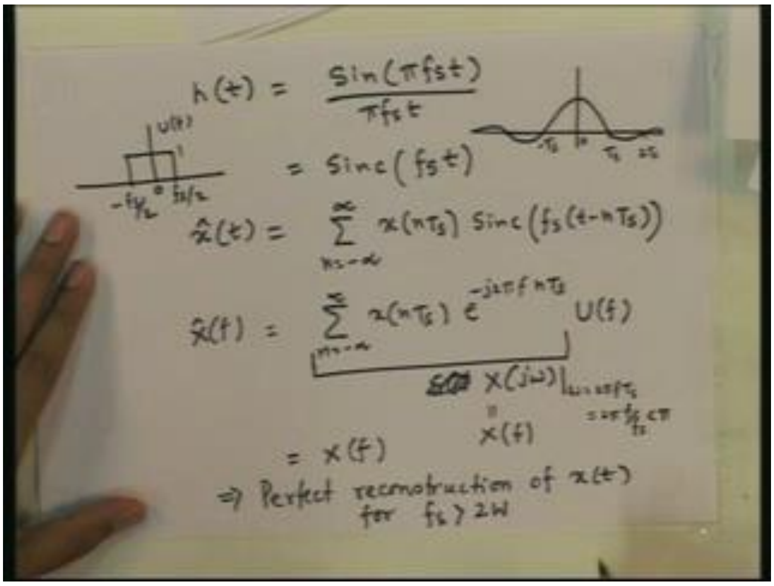

If we take the following h (t) the sin pi f s t by pi f s t then we will get the original signal back; if the original signal was band limited to band limited and the sampling frequency was greater than the nyquist rate we will see that now. So, this is nothing but sinc f s t and then the recovered signal at the output of the filter is x hat t; these are the sample values and then the shifted impulse response. Now, consider the Fourier transform of this. So, here we have taken a particular impulse response whose shape is like this is T s, this is 2 T s, 0, minus Ts, this one.

So, if we take the Fourier transform of this; what is the Fourier transform? X hat f is this is just the constant. So, summation n equal to minus infinity to infinity x n T s; then the Fourier transform of this if the Fourier transform of the sin c multiplied by e to the power because there is a shift it will be multiplied by e to the minus j 2 pi f n T s. And, then Fourier transform of the sin c is the rectangular function which is this is U (f) 1 from f minus f s by 2 to f s by 2 this is U (f). Then, the this Fourier transform of this is this that this is also h (f) the frequency response of the filter and now this is independent of n. So, this comes out and what is this is nothing but our d t f t of the discrete time signal. And, we have already discussed that if the nyquist rate is greater than if the sampling frequency is the greater than the nyquist rate then the d t f t does not have any mix up of shifted components of the original frequency response.

So, in the range of minus pi to pi the frequency response the d t f t is exactly same as the Fourier transform of the original signal. So, this is the d t f t of the x [n] of the sample signal; that is this is S this is X j omega at what frequency; the frequency is 2 pi f T s which is 2 pi f by f s which is less than pi. So, in minus pi to pi range this is this gives the this is the d t f t. And, this d t f t is same as the Fourier transform of the original signal; that is X f if f is less than f s by 2 for f less than f s by 2 we get this. And, when we multiply this by U (f) we get the same thing X (f) because f is less than f s by 2 it is in this range. So, in this range of f this is same as X (f) that is the idea. So, what we have got is the recovered signal has the same spectrum as the original signal.

So, this has of course this d t f t has multiple copies but then when you multiply by this it picks up only the copies entered at 0. So, you get the original signal back. So, in other words we get perfect reconstruction of the analog signal for f s greater than 2 W; if the sampling frequency is greater than nyquist rate then this filter gives us perfect reconstruction. Now, this is also true for band limited stochastic processes; as we said a band limited stochastic process is defined to be a stochastic process for which the power spectrum density power density spectrum is band limited; that is it is 0 outside minus W to W.

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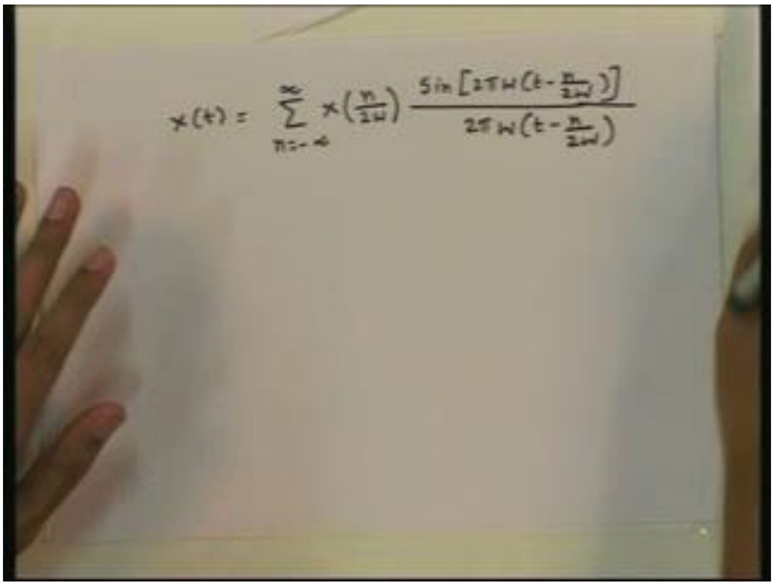

If that is satisfied just like a band limited signal the stochastic process X (t) can be expressed as in terms of the samples at n by 2 W, 2 W is the nyquist rate. So, this is the sample signal. And, then the h (t) minus n T s is nothing but sin 2 pi W t minus n by 2 W previously we wrote 1 by 2 W as T s and 2 pi W t minus n by 2 W. So, where these are independent samples, independent random variables for a stochastic process X (t). So, any band limited stochastic process X (t) can be expressed in this form. So, in other words if we sample a band limited stochastic process we can recover the sample function from the sample values. So, in this class we have discussed sampling we have first. So, what is the purpose of sampling; its purpose is to discretize a signal in time. Because there are infinite number of time instances where the analog signal has values and we cannot store all those values if we want to discretize the signal. And, so we need to pick up samples at regular intervals, regular instances and stored the stored the values of the samples. So, discretize a signal in time at a rate one rate f s which is 1 by T s; where T s is the sampling period, f s is called the sampling frequency.

So, if we have the original signal as X (t) the sample signal X (n) is nothing but X (n) time T s the ns sample is the value of the analog signal at time n T s. Then, what is the band limited signal? We defined a band limited signal to be a signal for which the Fourier transform of the signal is 0 outside minus W to W; for some W if this is true then the signals called band limited. And, what is the total bandwidth of the signal it is 2 W; 2 sided bandwidth. And, that is also the nyquist rate which is the that is the critical rate for sampling; if we sample the signal at greater than 2 W rate we can recover the original signal back from the sample signal. And, if we sample at less than 2 W rate we cannot recover the original signal. Then, we have seen why it should be true in the frequency domain; when you discretize the signal and we take the d t f t of the signal, the d t f t is obtained from the original signal by shifting the spectrum of the original signal at multiples of 2 pi after scaling; after scaling so that f s is mapped to 2 pi.

And, all the other frequencies are mapped to intermediate frequencies linearly if f s is mapped to 2 pi f s by 2 will be mapped to pi and so on. So, we first scale the frequency here and then we repeat it every 2 pi. So, that is the d t f t of the sample signal. Now, when we do this there may be mixing of different components and that is called aliasing. And, that mixing will happen if the sampling frequency is less than the nyquist rate. And, so that is not desired because in that case we may not be able to recover what these values are from the mixer of these 2 copies. So, on the other hand if there is no mixing then we can recover this original spectrum from here. And, we saw how to do it; we also saw how in time domain why nyquist rate is necessary for recovery of the original signal. And, if the original signal is greater than the if the sampling rate is greater than nyquist rate we have also seen how to get the original signal back from the sample signal.

We have seen that we need to take this impulse response and pass the delta train obtain from the original sample signal through the filter with this impulse response. If we do pass the delta train through this impulse response after convolution we get the original signal. Because the in the frequency domain we see that the spectrum of the output signal is same as the spectrum of the original signal, original analog signal So, we get we reconstruct the signal perfectly from the sample signal. And, this is possible because the sampling rate was chosen to be equal to or greater than 2 W.

Thank you.