Digital Communication
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Lecture - 14
Digital Modulation Techniques (Part – 3)

We have started for in the last 3 classes we have been doing this module and digital modulation as part of this digital communication course. So, first in this module we have done baseband signal representation. Baseband signal and baseband system representation in terms of they are low pass equivalent signals and systems respectively. And in that, context we have seen that a base this is not baseband signal representation, but bandpass signal representation in terms of baseband signals.

So, we have seen that a band pass signal can be converted to a baseband signal by a process we call it down conversion. And similarly, it can be converted back to again the same bandpass signal and that process we call it up conversion. And if, we assume the low pass equivalent signals and systems of the practically a practical bandpass systems and signals. We can we have to take the signals as complex signals. Because, the low bass equivalent signal of a bandpass signal is complex and similarly, same is true for systems also.

And in the last 2 classes, we have done pulse amplitude modulation while we have seen that pulse amplitude modulation can be done in baseband as well as in pass band. So, the only difference we observe is that the pulse that, we take has to be baseband pulse that is low pass pulse for baseband modulation and a bandpass pulse for pass band modulation. So, we took examples like, the baseband modulation we can in fact, get a pass band pulse. Simply, from a baseband pulse by modulating it by a carrier frequency sinusoid say cos omega ct.

So then, we have seen the importance of selecting pulse shape; pulse shape actually dictates the spectrum that we are going to use. The shape of the spectrum that we are going to use. So, in the in 1 class we have seen how to do demodulation for a particular example of rectangular pulse? We have from there, we have seen that demodulation using the modulation using that pulse can be modulated signal can be demodulated using a filter and the sampler. The filter impulse response can be taken, as same rectangular pulse. So, being motivated from that example we have in fact, considered a class of receivers higher the higher the impulse response of the receiver filter is taken to be the matched filter of the pulse. That is that is coming to the receiver filter.

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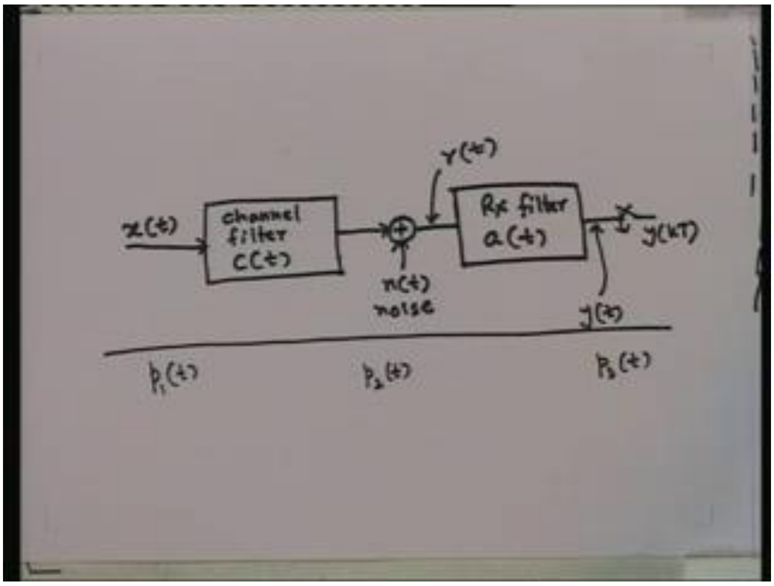

So, we have seen that if these are the pulses. So, if this is the diagram where xt is the transmitted signal and this is channel, this is a noise and here the signal is rt, that is it at the receiver. This is receiver filter at then, this output is called yt and this is sampled at every t T seconds. So, this here xt is summation of some scaled versions of scaled and shifted versions of P1t the pulse and the output here was called for P1t was called P2t. Which is a convolution of p1t and ct.

So, here the corresponding pulse and it goes here it is called p3t which is p2t convolution at. So we have seen that, if you take this at to be same as p2T minus t I mean such a filter is called said to be matched to this pulse. So, if you take this particular filter that particular filter we have taken we have seen that, we can demodulate the signal from this this sampled values.

So, at yk plus k plus 1 times T we will have something we have seen that, we will have ak times some constant plus noise time. So, from there we can estimate ak. The this ak which is which actually tells us what was transmitted at the transmitter.

So, if we if we are able to detect this ak able to estimate this ak. Then, we are we will be able to estimate the beats that were transmitted in that symbol also. So then, we have seen if we take such a filter if, we take such a matched filter. We should actually, sample at every t seconds because that will give us the maximum SNR.

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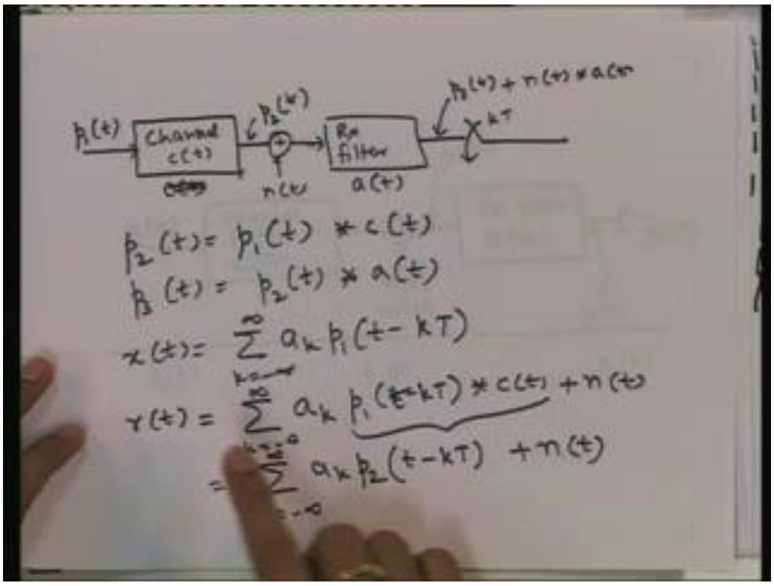

There the p3t has the maximum amplitude like, we saw for 2 different examples for rectangular. We saw that the P3t will have maximum amplitude here. So, this will give the maximum scaling of the symbol. So, the SNR the noise is same, the signal is scaled maximum. So, it will give us maximum SNR.

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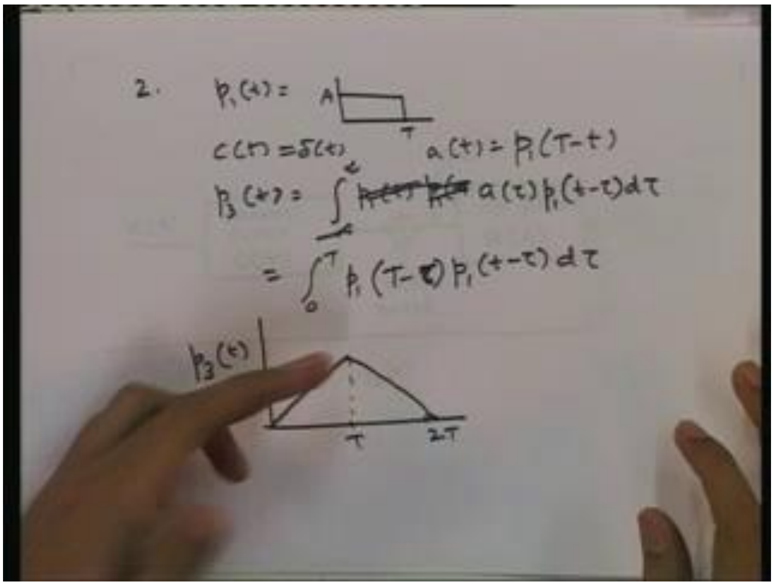

So, similarly for another example for this pulse for this, pulse triangular pulse we have seen that this is P3t again it is maximum at T. So, we have seen that sampling at every T seconds is the most beneficial in terms of SNR. Then, we started proving that; in fact, out of all the possible filters match filter is the best.

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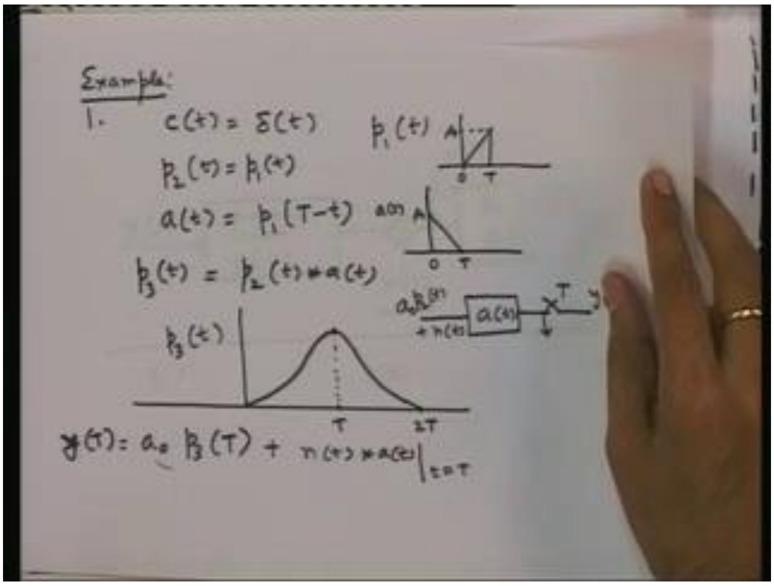

So, in that direction we have first characterized the SNR. We are seeing that here this component y (T) after sampling this has 2 parts: 1 is contributed by the signal. So, a naught t this part depends on a naught T. So, this part has an information about a naught t whereas, this part does not have any information about a naught.

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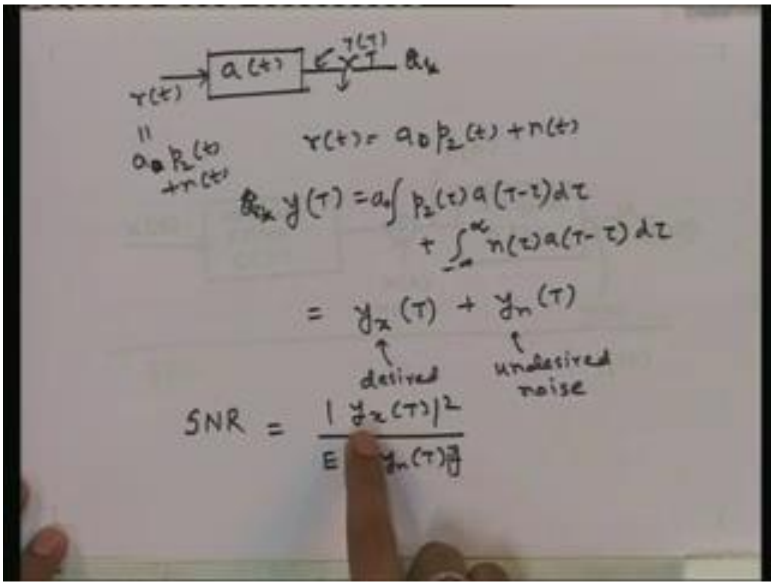

So, not a naught T, but a naught. So, this is simply noise. So, this is not desired this is the desired part. So, we are interested in the this energy of this part versus the energy of this part; expected energy of this part. Because, this is noise this is random. So, SNR is the energy of this by expected energy of this, that's what is written here. So, that is how we have defined SNR? It is a natural definition and then, we wanted to maximize this quantity.

So, we need to first calculate this. So, if we assume that noise is wide Gaussian noise with power spectral density 2 sided power spectral density N naught by 2. We have seen that, the this radians is given by this. Now, once we have that we can now see what filter? What kind of filter at will give us the maximum SNR?

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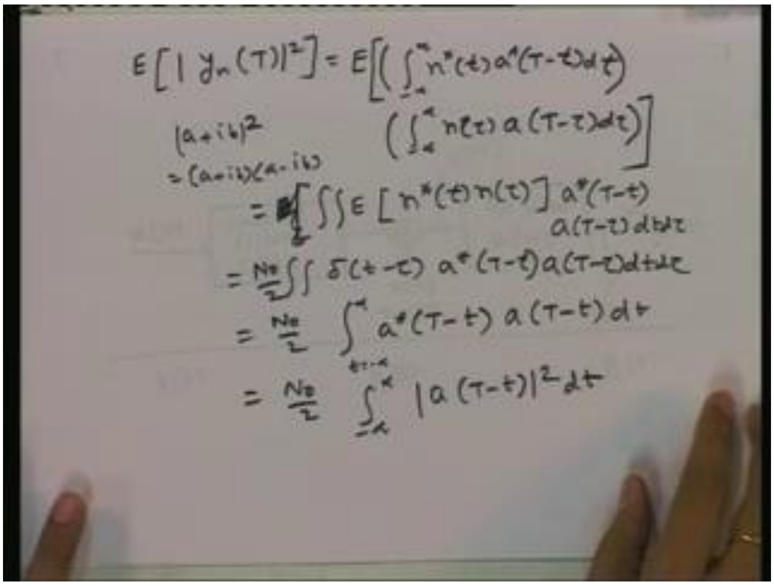

So, once we have this expression; what is the SNR? Or SNR is the signal energy by the noise energy. So, SNR is now signal energy by noise energy. Signal energy simply, Mod Square of this part. So, where have we got this from this is basically, this yt we have seen that the signal part.

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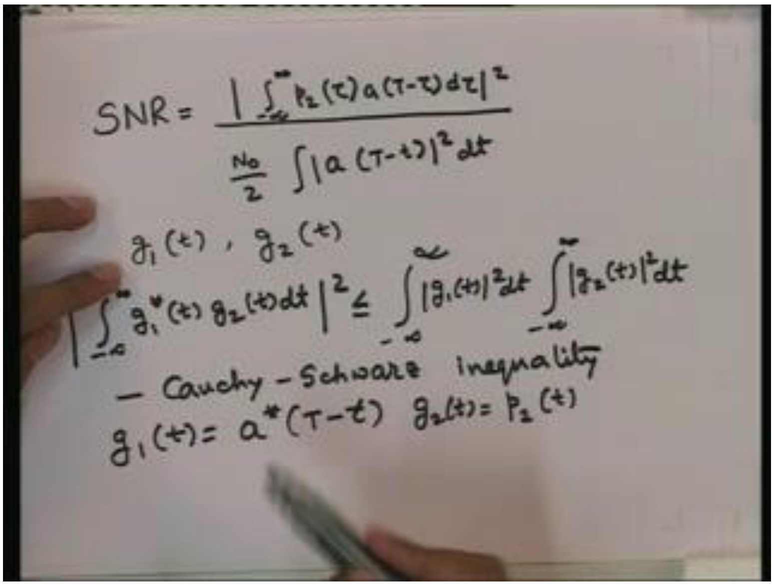

The signal part is this. So, we can neglect this N naught is actually it does not depend on the filter N naught is something that which are transmitted. So, we can as well maximize this part; the energy of this part by energy of this part. So, energy of this part by energy of this part we can maximize that. So, if neglecting this part we have seen already this. So, energy of this is nothing, but this so, this is the energy of this part.

Then, expected of energy of this we have just now we have developed in the last class we have derived in the last class to be this. So, you have put in the denominator. And now, we have a known inequality by it is called Cauchy-Schwarz inequality. The inequality says that, if we have 2 signals g1 t and g2 t both complex; may be dl also. Then, this is this holds minus infinity to infinity 1 star t g2 t dt square is less than equal to. So, this is the product take the integration and then take the mod square.

Instead of doing this if you do take energy of each. Then, that will be bigger than this quantity. Energy of g1 is g1 t mod square dt times; energy of g2 t is g2 t dt. So, this is the this is the Cauchy-Schwarz inequality. So, using this inequality we can do something here. Take g1 to be take g1 to be p2 tau and g2 to be a T minus tau. Then, what can I say? We can say that, we can using; this we can say that this square this is just this part.

So. In fact of we have previously yes, a this is p2 yes, so we take g1 t to be take a star of T minus tau. And g2 t to be p2 t then, check; what is this? g 1 star t g1 star t is a T minus tau. So, g 1 star is this star goes a star conjugate of a conjugate is, the complex number itself. So, a T minus tau this and then g2 is just p2 t. So, p2 times is a T minus t, this is t.

So, we have basically this; the integration variable tau. What t does not matter? Because, that is the integration variable dummy variable.

So, this side with this this g1 and this g2 becomes this numerator. And then, denominator this part; this part is in fact, g this part g1 mod g1 t square integration. Because, Mod a star or Mod a are same things. So, we have here using this we have Cauchy-Schwarz inequality tells us, that p2 tau a T minus tau d tau whole square. This will be less than equal to minus infinity to infinity p2 tau d tau minus infinity to infinity a T minus tau d tau.

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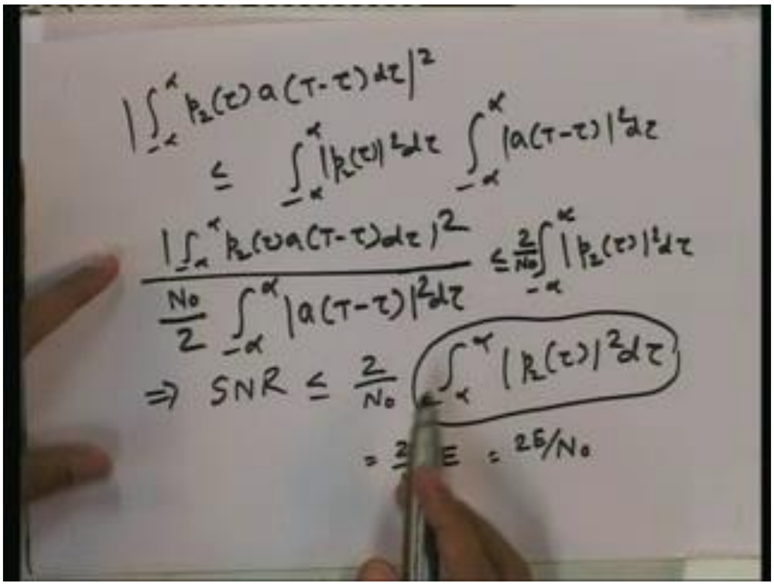

So we put a star or not it does not matter because, when we take Mod square it is the same thing. So, now if you take this part on this side we get this quantity by this quantity that is this by minus infinity to infinity Mod a T minus tau d tau this is less than equal to this part. So, now, see the SNR formula; this is what we have got from Cauchy-Schwarz inequality using g1 t equal to a star T minus t and g2 equal to p2 t.

So, difference between this part and this part is just this factor. Otherwise, everything is same. So, now if you put now N naught by 2 here, we can just divide this by N naught by 2. So, 2 by N naught we can write. This part is less than equal to this and Cauchy-Schwarz inequality also says that, this inequality will always hold and this equality this will be equality only if g1 t is the constant multiple of g2 t. So, g1 t will be when g 1 t is just a constant multiple of g2 t then, this equality hold and that is the only case. When this will be equality otherwise, always it will be strictly inequality; strict inequality.

So, this equality will come only when g1 t and g2 t are essentially same except for a constant factor. So, we have here from this using Cauchy Schwarz inequality that SNR. This is the SNR is less than equal to 2 by N naught t N naught and minus infinity to infinity p2 tau square d tau. Now, what is this? This is the energy of p2. So, this is 2 by N naught the energy of p two. So, that is let us call it E then it is 2 E by N naught.

So, SNR is less than equal to 2 E by N naught that is true given. That results from using Cauchy-Schwarz inequality. Now, when will this be equality? If, this is equality that is the maximum SNR. If, it is possible to get it in equality then it that is the maximum value we can have. So, when will this be equality? That is given by again Cauchy-Schwarz inequality as we said.

So, what is the condition g1 t and g2 t should be just constant multiple of each other. So that means, our g1 t was if you remember g1 t is g1 t is a star T minus t and g2 t is p2 t. So, if you say now g1 t is some constant C times g2 t then. So, without loss of equality we can take just equality also.

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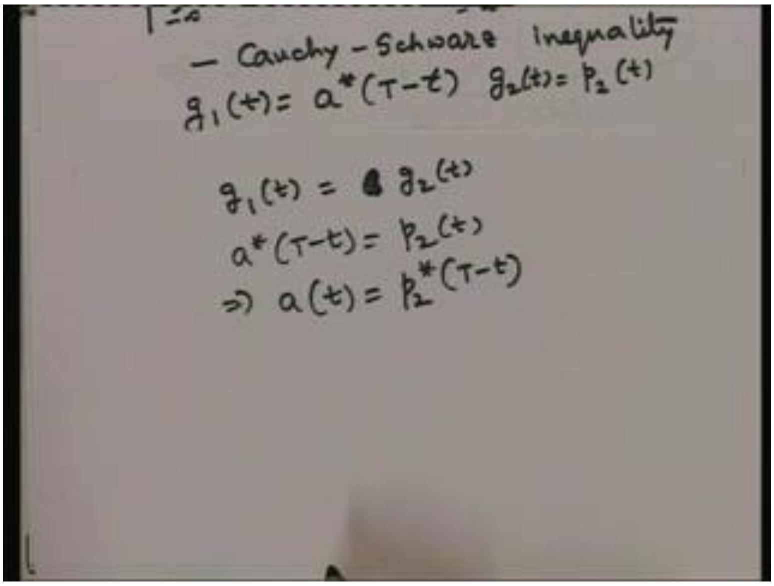

Then, also they will hold the equality will hold. So, what will that mean a star T minus t is p2 t. That will mean, that a t is p2 star T minus t. And that’s what is the that is the matched filter. So, here we have seen that we have first characterized the SNR; we have seen that SNR is always less than equal to a maximum value 2 E by N naught and equality we assume the equality only when g1 t is a constant multiple of g2 t.

So, instead of taking constant multiple you can always take the constant as 1 and then, we have taken 1 particular case g1 t equal to g2 t equal to g2 t. When, we will have the maximum SNR. And that means, we have seen that g1 t equal to g2 t mean that at that is the receiver filter is matched filter match to p2. So, we have seen here that matched filter maximizes SNR; that means, matched filter is optimum. So, we have seen that sampling if we use a matched filter previously we have seen that using a matched filter it is best to sample at multiples of T seconds.

And now, we have seen that match filter itself is a optimum filter. Because, that will give us the maximum SNR. So, now in all this treatment we have always taken only 1 pulse as input and analyzed, what is happening? And using that we have seen, what is the best filter to choose? And what is the what are the time instances when you should filter? All this things we have analyzed by taking only 1 pulse as input, we have not taken many pulses.

Now, this may not give a generalized situation. This may actually, create problem in practice because, in practice there are going to be more pulses same to 1 another. So, what may happen is? That the effect of a previously, transmitted pulse may affect the reception of the next pulse. Because, the channel is has some channel is not ideal filter. Channel has some response and it may spill the pulse may spill over to the next symbol.

Then, they may overlap when they grew to the channel and then, there will be some interference from the next symbol. Such a situation, such a phenomenon is called intersymbol interference. And presently, we will see exactly how intersymbol interference really come into picture? When you do this matched filter receiver? Or using any filter? So, the question we choose ultimately is how to choose this p1 t? How to choose P1 t? How to choose our receiver filter at? Such that, there is no intersymbol interference also.

So, if we choose p1 t we will know p2 t because channel is fixed. We cannot change the channel, but when we receive it we can also design at and we have seen that at should be actually p2 of T minus t. So, the only freedom we have actually is to design p1 t. So, to design the p1 t now we have to see, that we should not have intersymbol interference. 1 symbol should not distort or should not disturb the detection of the next symbol.

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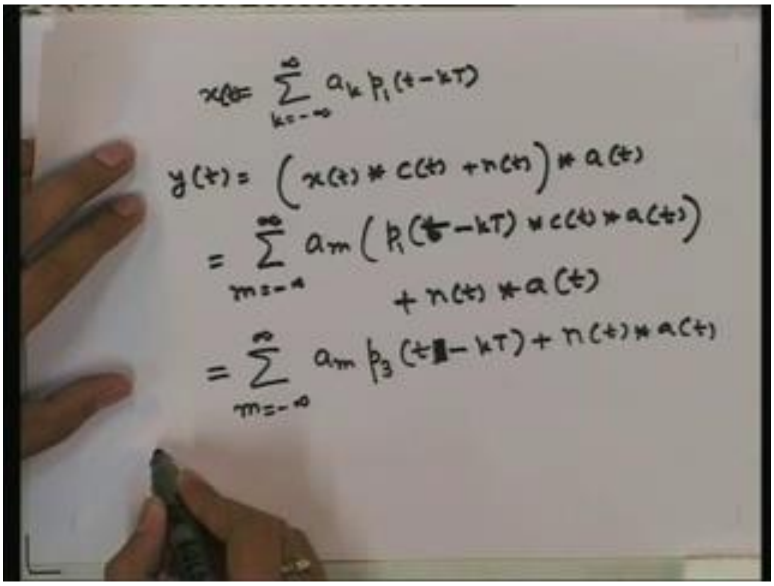

So, let us see P2 t we have seen is p1 t convolution ct and now, we consider this transmitted signal which is the transmitted signal for pm. So, previously we were taking only 1 term and what analyzing the affect. What is happing at the receiver? Now, we will treat this whole input. We will take this whole input and see what is happing? So, yt for this is xt then, yt is again remember this remember. This diagram this is not the so, we have this channel, then the noise, then the receiver filter.

So, yt is actually the convolution of xt with channel impulse response and then plus noise. And then, that convoluted is the receiver filter. So, xt convolution ct plus nt this is the received signal, this is convoluted with receiver filter impulse response at; xt is this. So, if you put now xt equal to this what we get? The summation can be taken outside instead of k we put m as running variable we will use k later. So, we put m here. So, we have am then p 1 k of t minus kT convolution ct convolution at plus nt convolution at.

So, this is equal to am p3 t at t minus kT. So, remember p1 t convolution ct is p2 t and that convolution at is p3 t. So, when we have shifted here that resulting output will also will be shifted. So, this is p3 t minus k capital T. And then the noise term remains nt convolution at. Now, we will be sampling this output at every capital T seconds. So, let us see what the k’th sample is. So, k'th sample is ykT. So, the k'th sample is y of we evaluate it at

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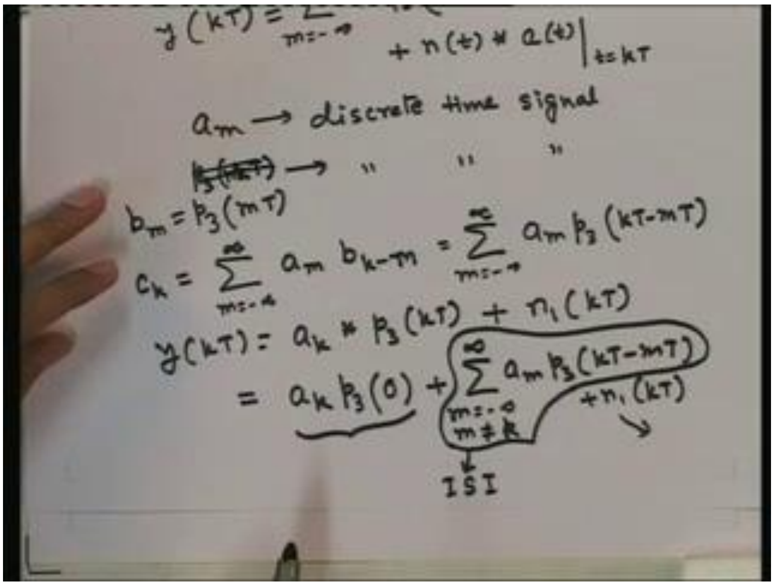

y kT; ykT is so, we have to take m here. This is we have replaced k by m. So, here it is m here also this is m. Then, now we are introducing k variable here. So, kT this is sampled at kT time. So, this will now replace in place of t we place kT. This convolution also evaluated at t equal to kT. So, this is the sampled value at time kT. Now, let us see what is happening in this term? What is there in this term? This am itself is a sequence it is a discrete time sequence. There is N naught a1, a2, a3.

So, if it has if it is it started be much before it will have a minus 1 a minus 2 and so on. So, there are these samples these are also some discrete time signal. And this, also as k values it is a this p3. So, am is a discrete time signal and similarly, p3 kT T is constant. You are varying only k or m. Let us, say so m or k whatever. So, this is also a discrete time sequence, discrete time signal.

Now, let us see what is the convolution of these 2 sequences? This is the running the variable here is k or you can have same variable also m or k whatever. Let us, use the same variable in both the places p3 mT. Now, if you convert these 2 and take the value at k. what we would we get? This is suppose, this sequence is we call this bm. So, we are taking the convolution of these 2.

What is the convolution discrete time convolution of 2 discrete time sequences? So, the convolution at k suppose the convolution is ck; ck is m equal to minus infinity to infinity am bk minus m. This is the discrete time convolution. Now, in place of b bm we put this.

So, bk minus m is am then p3 in place of am we put k minus m. So, kT minus mT. So, this is the convolution of these 2 sequences evaluated at the k'th sample.

So, this term whole term is a convolution. So, this we can write as this is ak convolution p3 kT. So, this sequence and this sequence convolution and take the k'th sample of the convolution plus call this signal n1 t. Then, n1 at kT this is the sampled value at kT. Now, this is equal to take different values of k and we will write individually. So, for k equal to 0. What we get? For So, kT so p3 of kT. We get for so, this term is basically this term.

So, this term will write for individual m. For m equal to k, what we get? For m equal to k we get ak then, p3 0. Then, plus other terms remain m naught equal to we take m naught equal to 0 m naught equal to k. Then am p3 kT minus mT plus n1 kT as it is. So, we have just taken this output at k times T this is a sampled value at k'th sample. And this interval this convolution is broken into 2 parts: 1 is m equal to 0 Then, the other is that the rest of the values. Then, we have ak times p3 t p3 0 plus this term.

Now, this term is the gives us the k'th symbol. So, when we do this term actually gives us information out ak. But all these terms are effect of other symbols on this symbol. So, we are going to actually detect ak from this sample, but what has happened is? All the other terms also, all the other symbols are also, interfering with this. So, this term this is noise we have analyzed this and in the context of noise filtering and we have seen that, this we can do we can minimize the energy of this.

We cannot really do, we cannot eliminate this, but this intersymbol interference can you eliminate this. So, this is also undesired thing. So, this is also undesired; this is actually intersymbol interference: ISI. So, we would like to avoid this term also. So, from now on p1 t p2 t all these terms will not come in this analysis. So, will have only p3 t 1 pulse and we will denote that by g t.

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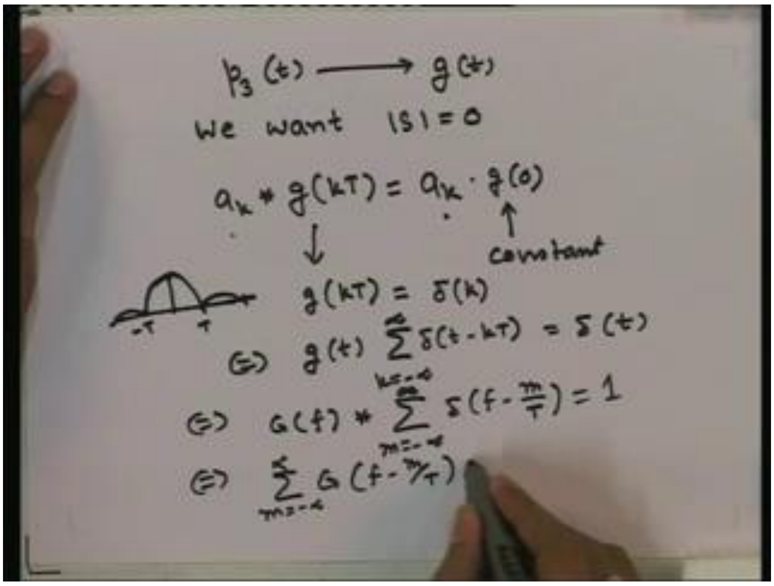

This will denote that by gt then, so we want the ISI term to be 0. We want so, we are going to see what to do? To ensure that, the ISI is 0. So, we have separated the ISI part from the signal part. The main symbol part from which, contains the information about the symbol. And we are going to see, how to remove the ISI? So to say that, all these terms are 0s it means that, this convolution has only 1 sample non 0 all the other samples are 0s.

So, that means that so this is this convolution this is not 0, but at the k'th symbol at the k'th sample we will have ak times p3 0 as the value. So that means, we should have ak convolution g kT equal to ak times g 0. So, remember this is the sequence, this is the sequence, this is also a sequence, but this is a constant. So, this is really a sequence; a constant times the sequence ak.

So, when do convolution of ak and g kT these 2 sequences we get a scalar multiple of ak. So, that is what? We want. So, this will be true if and only if, this sequence is the delta sequence that is it has only 1 nonzero value that is the g kT. So, it is basically at k this value will be k equal at k equal to 0. This value will be 1 and for all other case it will be 0. So, it will look like, the g should be such that it is like this.

So, at 0 it is it has 1 value and then, it should come back to 0 at T minus T. And then, it may have some other value, but at 2t again it has to be 0. So, that is the restriction we are seeing that gt should be such a signal, such a pulse. So, now this is the condition we need to ensure that ISI is 0. So, this again means that gt. What is this sequence? This sequence is basically, the sampled version of gt at multiples of T. So that means, g this is gt times this is equal to delta t.

This is basically, sampling at that is the output signal is still represented in in terms of continuous time signal. So, this delta t minus kT when this is multiplied to this it will pick up only the value of gt at kT time. So, this is this is means this term the this expression. So, that if you take now Fourier transform; what is the Fourier transform of these? It is let us, say G of f and this is product. So, product will be convolution when you take Fourier transform and what is the Fourier transform of this?

Fourier transform of this is known to be again delta f minus m by T the fundamental frequency is this is a.so, there are deltas spaced 1 by t apart: 1 by t is a frequency, t is a time. So, take at a multiples of 1 by t frequency there is a delta function that is the Fourier transform of this. And what is the Fourier transform of delta t? It is 1. So, now if you take the convolution; what is the convolution of Gf? And delta f minus m by T it is Gf minus m by T. These are all standard results from Fourier transform this is 1.

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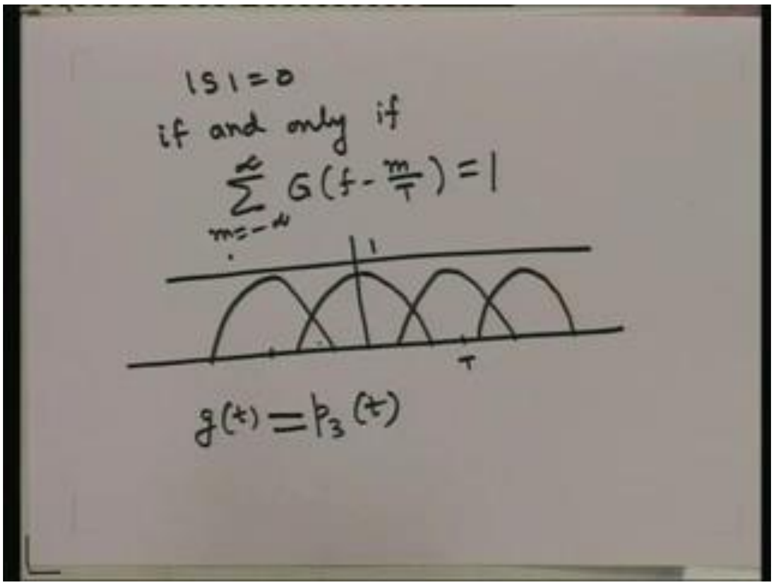

So, what does this mean? We have got that ISI is 0. If and only if and only if summation m equal to minus infinity to infinity G of f minus m by T is one. So, this means let us see let us plot each of them. And see, what it means? Suppose, we will plant each of these functions and then, add all those functions for different m then, that should be 1 that is what it means.

Now, what is this function for m equal to 0? It is just Gf suppose, Gf is you know something like this. And here is t then, Gf minus 1 this is Gf minus 1 by T physically shifted version of this at 1 by T. So, it will be like this again at minus T it will be like this. So, we will have lot of these and then when we add it should be 1. That is what it means, so we add these curves it should give us this 1 value. So, we have seen now what is the condition these pulse p3 t should satisfy? Remember, gt is actually we have used this as a different notation for this p3 t itself. So, gt should satisfy this condition. That means, this its spectrum we should take we should shift its spectrum at multiples of 1 by t frequency and then, we add those frequencies it should give us a constant value 1 that is the condition for having 0 ISI.

Now, we told that we should actually see how to design p1 t. Because, once we design p1 t p2 t is nothing, but p1 t convolution channel impulse response. Which we do not have control on and then, p3 t is nothing, but convolution of p2 t and at. And what is at? We saw that, at the best at is to take the filter match to p2 t. So, at also we have already decided that whatever, this p2 t we are going to choose a filter matched to that. So, only thing we have to decide is p1 t.

So, we should now take p1 t in such a way that the overall this output pulse should satisfy this condition. It should satisfy the condition that, it's you take the spectrum of p3 t you shift it by 1 by t 2 by t and so on. And minus 1 by t minus 2 by t and so on. And then, you add all those different curves, different functions add them you get you should get a constant value 1 everywhere. So, that is the condition for 0 ISI

Now, let us see different cases when how is it possible to achieve the 0 ISI? How to actually take p3 t? How to construct p3 t? What kind of spectrum Gf should be? So, that its shifted version should add up to 1 so, that we will see. So, first suppose that Gf has Gf is band limited to b. It is band limited to say suppose, band limited to frequency from minus p2 plus p that is Gf is 0 for f greater than B and f less than B less than minus B.

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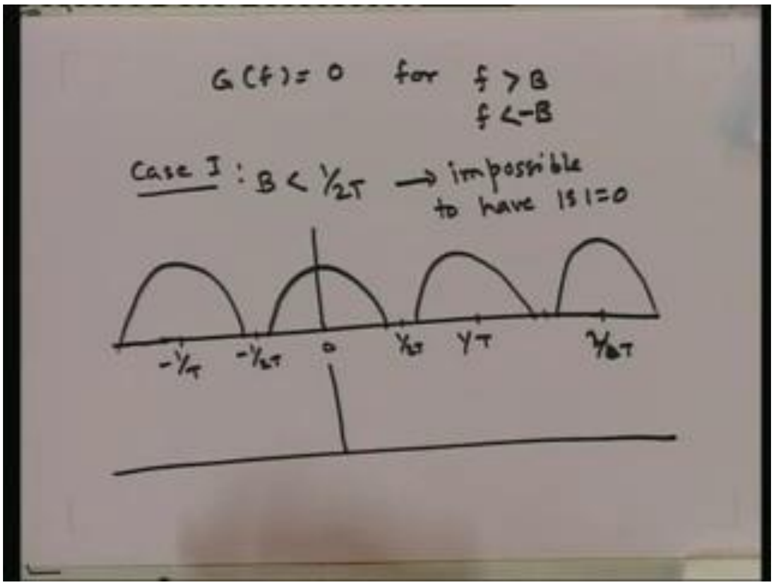

So, there is sum before which this will satisfy, but what may happen is? B is infinity that may be also happen. So, there is sum B now, case 1 is B less than 1 by 2 t. That means what? That means, f 0 T this is 1 by T, this is this is frequency axis. So, 1 by t,1 by 2t, 1 by 2 by t, this is 1 by 2T, this is minus 1 by 2T, this is minus 1 by T and so on. So, this case says the Gf is its bandwidth is below 1 by 2T. So, it is something like this. So, when we shift it to 1 by 2. What happens? What we get? we get this similarly, we get into this and this side.

So, when you add this are you going to get? Amplitude 1 everywhere, we are not never going to get. Because, here there is no nothing so it is the value here will be 0. This is the addition is basically this so, it is not constant 1. So, this case it is impossible to have 0 ISI. This is impossible to have 0 ISI for this case. Now, let us see all the other cases B equal to 1 by 2 T this is case 2

So, p equal to 1 by 2t the here also, what will happen is? In this case, it is strictly less than 1 by 2T, but for equal to 1 by 2T. What will happen is? This n's will touch, but there will be no suddenly. So, the addition will be just in this section the addition will be just itself, in this section the addition will be just itself So, for the addition to be 1 everywhere this pulse itself should be 1 everywhere in this range.

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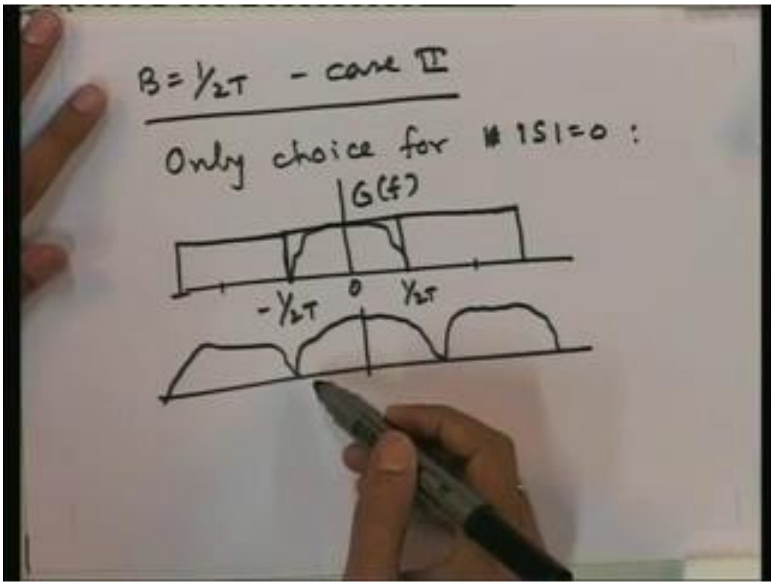

So, the only possibility is only possibility for 0ISI is to have this Gf this is 1 by 2T this is minus 1 by 2T. Then, what will happen? When we shift it to 1 by T it will be like this It will be like this when we shift it to minus 1 by T it will be like this. So, it will give us constant 1 when you add all these, but if it is was not like this if it is like this then, we will get something like this like this.

So, it will be not 1 and also, if it is more than so if it is band limited to this if the shape is otherwise, these shapes also will be the same shapes. So, it will not be 1 everywhere. Now, this is the second case now we have another case that is P is greater than 1 by 2T. So, case 3 B is greater than 1 by 2T.

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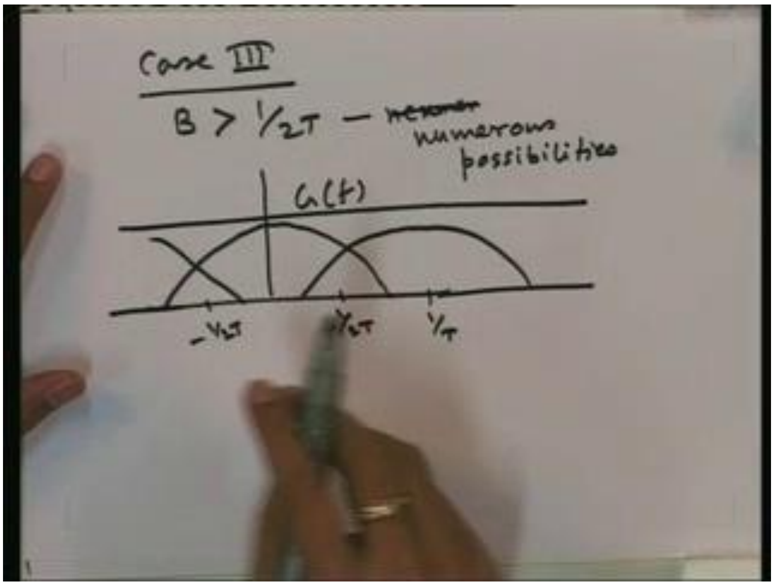

In this case what may happen is? The GF is this is 1 by 2t, this is 1 by T. So, it may be something like this, but when we shift it to this it will overlap here. Tt will overlap here, but when we add this this can be still 1 this can still be 1 even If, the overlap this this add these things this is decreeing this is increasing. So, they may add up to 1. So, there are in fact numerous possibilities for this GF for GF in this case.

So, numerous possibilities and in the next class we will see that see a very standard pulse shape. In fact, we can we will in the next class we will see that a very standard class of pulse shapes which fall in this category. So, in this class what have we done? We have we have completed the proof that match filter is the optimum filter that will give us the maximum SNR for detecting the so, that will give us the maximum accuracy a minimum probability of error for detecting the pulse amplitude modulated signal.

And then, we have started seeing what is the pulse shape we should use? So, here the ideas is that for match filter derivations we have used only 1 pulse transmission. And checked what will be the SNR? What is the maximum SNR? We can get. So, there we have we have basically considered only noise, but when we transmit 1 after another symbols 1 symbol when it goes to channel? It’s in its duration will elongate. So, it will overlap with the next symbol and as a result we this next symbol is getting distorted.

So, we will not be able to detect demodulate the symbol accurately. So, this phenomenon is basically ISI and we have seen how to design these pluses. So, that even if the overlap will have no ISI no intersymbol interference So, we have derived this condition and we have seen that, the resulting pulse that we should design p1 t such that p3 T should have a spectrum satisfying some condition.

And that condition is that if you take the Fourier transform a of p3T and then, shift its shift those spectrum by 1 by T and multiples of 1 by T and add all those spectrums then we should get a constant amplitude of 1 everywhere. So, its value should be 1 not only the amplitude its value should be 1 that is phase is also 0. So, that is the only way to get 0 ISI. But for this Gf itself this conditions to satisfy this condition, there are many possible pluses. There are numerous choice for p3 T. We have seen 3 cases: 1 is the bandwidth of the p3 T pulse is less than 1 by 2T in which case it is not possible to satisfy that conditions.

It is less than it is equal to 1 by 2T in which case there is only 1 choice that is the rectangular pulse from minus 1 by 2T 1 by 2T And then, we have seen that for B greater than 1 by 2T there are numerous choices. We have not really found examples of such things we will see in the next class, but we have seen we have discussed why intuitively it should be. So, why it is possible in such a case to have a different p3 Ts? Which will satisfy this that condition? So, in the next class we will start with this pulse which I said falls in the case 3 and this is a very that pulse shape is very commonly used pulse shape. So, it is a very important thing. And then we will start other modulation techniques, other generalized techniques then pm. See you again in the next class.