Digital Communication
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Lecture - 11
Bandpass Signal Representation (Part - 2)

In the last class, we started bandpass signal representation. And, in this class, we will also continue the same topic. And, we will start bandpass signal and more or less bandpass system representation in this class. So, in the last class, what we have seen is that a bandpass signal can be represented as a lowpass signal with some… after doing some operation.

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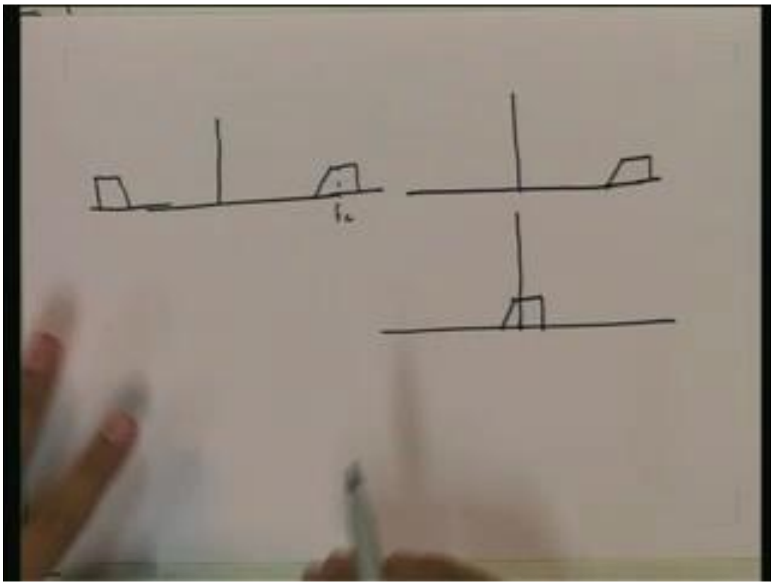

So, what we have seen is that, if we have a bandpass signal with this kind of spectrum – any spectrum; this is f c; then, we can first suppress the negative side by multiplying by two times u of f, that is, the unit step function. And then, we will get again a bandpass one-sided spectrum. The resulting signal will have this spectrum; negative side will have nothing. Then, we can do… We can shift this spectrum here; then, we will get a signal with this kind of spectrum. So, this will be a lowpass signal and this is called the lowpass equivalent signal of this bandpass signal. So, we have seen the relationship between this signal and this signal; and, in particular, we have seen how to construct the bandpass signal from the lowpass equivalent signal and how to construct the lowpass equivalent signal form the bandpass signal.

The bandpass signal is a real signal; but, the lowpass equivalent signal will be complex signal, because there may not be any equivalence; it may not be any symmetry in one side of the bandpass signal. Whatever is there in the positive side, will be also there in the negative side with some phase will be negative, but magnitude will be same. But, in one side itself, there may not be any symmetry. So, when we take only that part and bring it to 0 centre, then we may not have any symmetry. So, the resulting lowpass equivalent signal may be complex. In general, it is complex. So, we have also seen how to construct this signal, what is the real part and imaginary part in terms of this signal, how to construct this signal.

And we have also seen that, for a communication system, though we will be transmitting the bandpass signal through the channel ultimately, the whole system in the transmitter as well as the receiver may not be designed for that particular band, which is lying in very high frequency. So, what we can do is that, we can design the signal as a lowpass equivalent signal. We can the design the lowpass equivalent signal first, because that is the base band signal. That is the lowpass signal with some band width. So, designing that signal is easier because processing required to design that signal, to generate that signal is easier with… It requires low cost electronics. And then, we can do up conversion to get the bandpass signal for that and then transmit that. The receiver on the other hand, we can first take the – receive the bandpass signal, then do down conversion to take the lowpass equivalent signal, and then do required processing to detect the message. So, the main significant processing at the transmitter as well as the receiver may be done in lowpass – on the lowpass signal in the low frequency.

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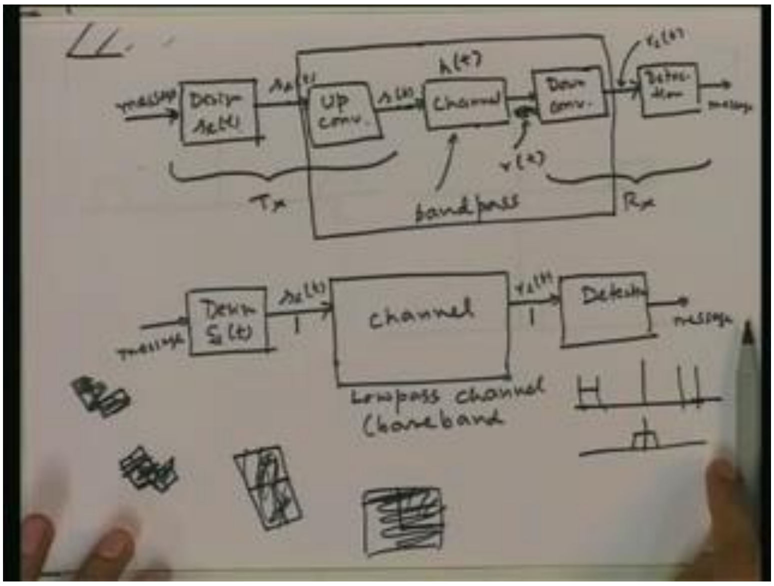

So, with that in mind, we saw that, we can represent a communication system this way. This is the diagram – message in the transmitter; first, this lowpass equivalent signal of the transmitted signal is designed. So, this is design of some lowpass signal. So, this is quite easy. And then, we do up conversion to get the bandpass signal from the lowpass equivalent signal and then transmit it through the channel. The receiver first does down conversion on the received signal to get the lowpass equivalent signal and then does all the processing on this lowpass equivalent signal to detect the message. So, if we represent this whole block – channel including up conversion and down conversion in the transmitter and receiver respectively, we can consider this as a single channel; and then, this equivalent channel takes a lowpass signal as input and gives a lowpass signal as output. So, this channel is now a lowpass channel and it is…

We can consider from now on that, the channel is always lowpass; but, but the channel is now complex, because the lowpass signal is complex; lowpass signal, that is, input transmitted is complex and the output is also complex. So, the channel is complex baseband channel. So, we said that, from now on, we will considered such a channel and consider the task of detection of message from the received lowpass equivalent signal and how to design the lowpass equivalent also. But, there is one thing we have not yet done, that is, we have said that, this is the equivalent lowpass channel. Now, if this channel is given; say this channel has impulse response h t, which is bandpass; then, how to get the equivalent lowpass channel. We are given the bandpass spectrum – frequency response of the channel, which is bandpass. Now, from there we are going to actually consider the lowpass equivalent channel. What is the impulse response of that lowpass equivalent channel then? So, we are going to investigate about that in this class.

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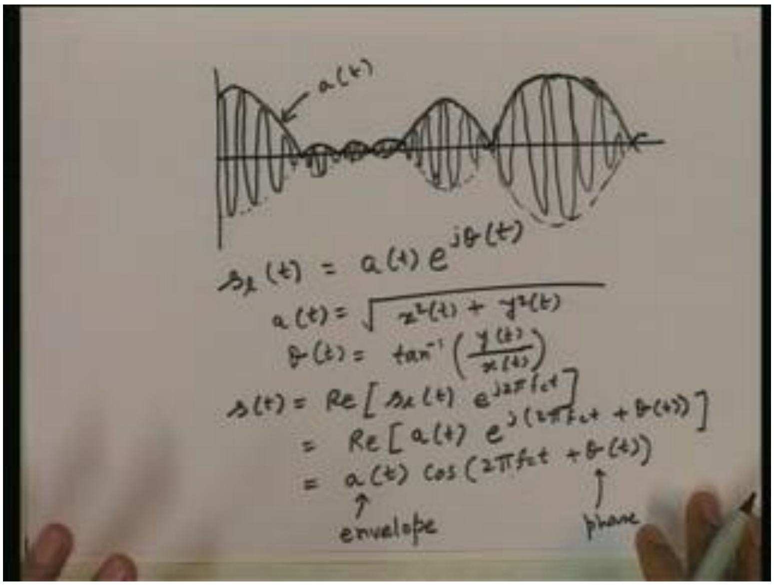

So, before doing that, we will consider also another topic and another small question about the bandpass signals itself; and, that is, how does a bandpass signal really look? If you plot a bandpass signal, how does it look? We have seen that, you can represent it as a lowpass equivalent signal; but, in practices, how does the bandpass signal itself look? So, we are going to first see that. So, usually the bandpass signal will look like something like this. It will have some envelope say something like this. And, of course, the negative side also will be there. Then… So, this is how usually a bandpass signal will look. So, this sinusoid, which is inside this envelope, has about – the frequency is about f c. And, if we… Now, this envelope – this wave form – this is called the envelope of the signal.

Now, suppose we have s l t, we know that, the bandpass signal s t can be represented in terms of a lowpass equivalent signal, that is, s l t. Then, we can write s l t is a complex signal. So, we can write it as something – some magnitude times e to the power j theta t; theta t is a phase and a t is the magnitude. So, now, if we write that way, then we have this representation of the lowpass equivalent signal. Now, what is a t? Here a t is the magnitude – x square t plus y square t. If you remember from the last class, x t is the real part of s l t and y t is the complex imaginary part of s l t. So, the magnitude of s l t is root over x square t plus y square t; and, theta t is the phase of s l t. So, it is tan inverse y t by x t.

Now, once we have that, what is s t? s t is the real part of s l t e to the power j 2 pi f c t. This we have seen in the last class that, the bandpass signal can be expressed in terms of the lowpass equivalent signal this way. And, that is real part of… Take this representation of s l t and put it here. So, we will have a t times e to power j common – then, 2 pi f c t plus theta of t. Now, what is the real part of this? This is a real quantity, a real number, a real function. So, what is the real part of this? Cos of this. So, the real part of the whole thing is a t cos 2pi f c t plus theta t. Now, here is the envelope; a t is the envelope. And, this is the phase.

Now, you can see that, this signal if you plot; this will really have this kind of shape. Cos omega c t; 2pi f c is the angular frequency – omega c. So, omega c t plus theta t; cos omega c t is the kind of the carrier signal, that is, this sinusoidal; and then, it will have theta t phase shift; it will have some phase shift, which will change with time. And then, a t is a lowpass signal; as you can see, it is a real lowpass signal, because it is the real part; it is the magnitude of a lowpass signal. So, it will be positive real signal. So, it will be this kind of signal. So, this will be really the envelope of this whole signal, because this is a very high frequency signal, which is varying very fast. And then, you have the a t times that – will actually change the magnitude slowly; a t is a lowpass signal. So, it will have some slow variation compared to the high frequency carrier signal. So, it will vary the magnitude of the sinusoidal signal. So, it will act as an envelope for the carrier signal. So, here this is the envelope a t. This is the envelope a t. And, phase theta t is actually would… It will change here. So, we have seen what is envelope and phase of a bandpass signal. And, envelope and phase… Envelope specially has a very pictorial pictorial representation. Envelope is basically actually the envelope of this signal as you can see. This signal is the envelope.

Now, we have… So far, we have seen two things: one is… We have seen two representations of bandpass signal. One is as a lowpass equivalent signal. So, it will be complex lowpass signal, which will represent the bandpass signal in a way. So, we can get the bandpass signal from the lowpass equivalent signal; and, lowpass equivalent signal from the bandpass signal. And, we have also seen that, another representation, where s t is represented in terms of an envelope and a phase with some carrier frequency.

So, here you get an idea of how the signal will look like. So, it will basically have that a t envelope. But, we need to also ask – when we do this conversions from the complex – from the bandpass signal to the lowpass equivalent complex signal, how does the energy change? What is the relationship between the energy of the baseband lowpass equivalent signal and the real bandpass signal? When we do this conversion, what happens to the energy? So, that we will now see.

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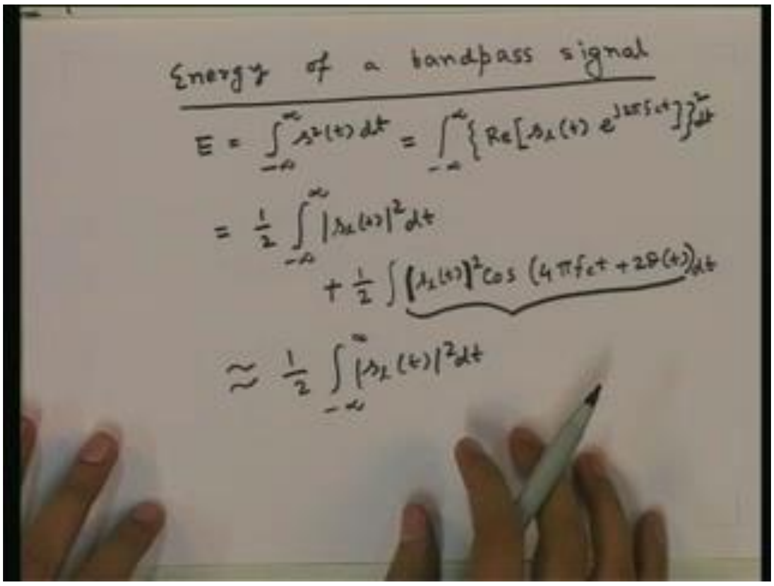

So, let us… So, this is we are going to find out the energy of a bandpass signal in terms of the lowpass equivalent signal. So, the energy is defined this way. This is… Now, we take… We know that, the bandpass equivalent signal is this in terms of the lowpass equivalent signal. If the low pass equivalent signal is s l t, then the bandpass signal is real part of s l t e to power j 2pi f c t. Now, how have we got this? This is basically here; you can see that, from the previous page, we have this. So, real part of s l t e to the power j 2pi f c t; that is real part. So, that is the s t. So, we have here real part of this; basically, this part – this thing. And, this is actually expressed in terms of envelope and phase here. Now, if we take the square of that, which is required; so, here this will be squared. If we take the square of this, what we have is cos square this. Now, what is cos square theta? What is…

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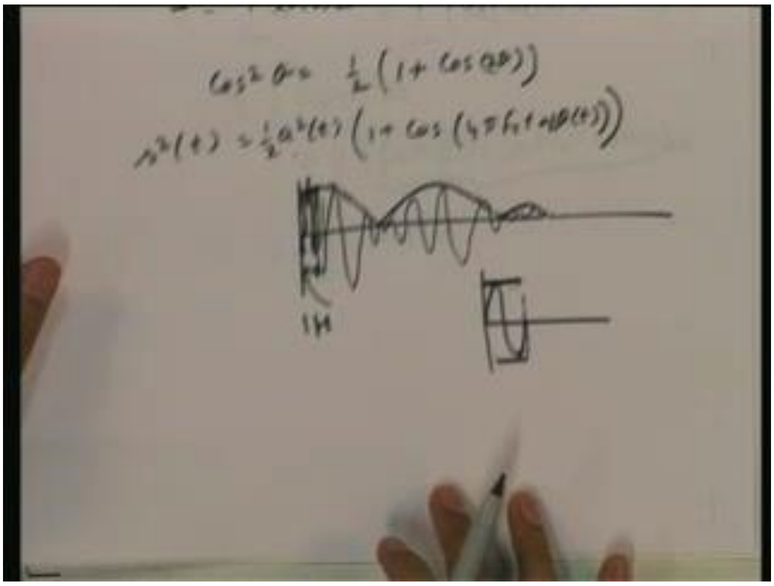

We can express cos square theta as half 1 plus cos 2 theta. So, using that, now, we have this term. Square of this is a square t. So, square – s square t is a square t half of 1 plus cos two times this, that is, 4pi f c t plus 2 theta t. So, now, if we take – integrate separately these two parts as we need here; so, this inside quantity is basically this. This quantity is this. Now, we need to integrate these two parts – half a square t and half a square t times cos of this. So, once we do that, we actually have this. Integrate half a square t; a square t is nothing but s l t square, because a t is the magnitude of s l t. So, we have this part and then half inside s l t square. This is magnitude square times cos of 4 pi f c t, that is, 2 times this angle; this plus 2 times theta t dt.

So, once we have, this can be written approximately as half… – only the first part. Why? Because what is this signal? What are we integrating in the second integration? What we are integrating there is that, if you see the… try to draw the wave form of this inside function, it will also be a bandpass signal, because this a carrier signal with some 2 times theta t; that is, the frequency is 2 f c and the phase is 2 theta t; and, the envelope is s l t mod square – mod s l t square. So, it will look like s l t… mod s l t square is the envelope. And then, this is the signal. So, this is a very high frequency – 2 times f c is usually a very high frequency. And, this is a baseband real signal. So, this is the envelope. And, we are integrating this signal.

Now, once… When we are integrating this signal, we can see that, since 2 times f c is very high, there are many… This oscillation is very high frequency. So, if you now take one cycle at a time, what is the integration? One cycle at a time? If we take only one cycle, because this oscillation is very high frequency, what you can expect is that, in the one period of this oscillation, this envelope is not changing much, because this is a low frequency baseband signal; mod s l t square is a baseband signal; it is lowpass signal. So, if you magnify and draw only this – one period here, you will find it is a signal of this type; that the magnitude is not… This is same as this. So, here magnitude is not varying much. The envelope is almost fixed here, because this is a very small part of this envelope and envelope is a lowpass signal. So, this integration of this signal in this interval will be 0, because it is a sinusoid of period 1 – one period integration of a sinusoid.

So, now, if you continue doing that, in this small interval it is 0 – integration is 0. Similarly, in this integral – small interval, it is 0. This way we keep doing it. The whole signal will give you almost 0. It is almost because the envelope is not really constant here; it is changing its radius – changing very slightly. So, this integration will not be exactly 0, but almost 0. The error will be negligible. So, we can say that, this integration is only this part, because this part will give you almost 0. So, this is almost half times energy of s l t. So, what we have seen is that, the energy of the bandpass signal is almost same as the energy of the lowpass equivalent signal – half of that. So, this is something interesting, because now, we know that, we can… If we treat the lowpass equivalent signal; if we know its energy, then we know that, the bandpass signal’s energy is proportional to that. So, we can keep that in mind. And, this will simplify lot of our analysis. So, we can treat now the lowpass equivalent signal and assume the bandpass equivalent signal to have the energy half times the energy of the lowpass equivalent signal.

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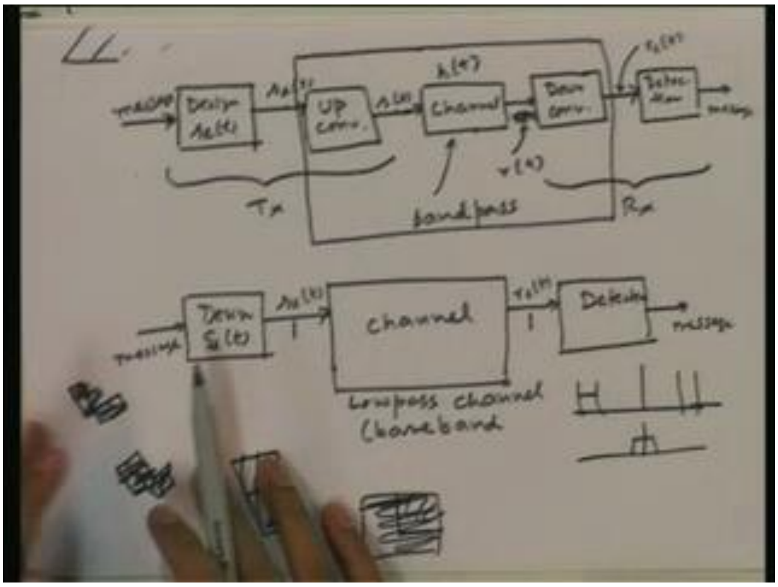

Now, we will go into the representation of the linear bandpass system. So, in the last class we saw that, this is an equivalent diagram of a communication system, where this whole thing is treated as a channel and this is now a lowpass complex channel. But, if we are given the impulse response of this channel, then what is the impulse response of this channel? Now, we will see that relation now. So, to do that development, we need to analyze, represent a bandpass system also just like we represented bandpass signal in terms of lowpass system.

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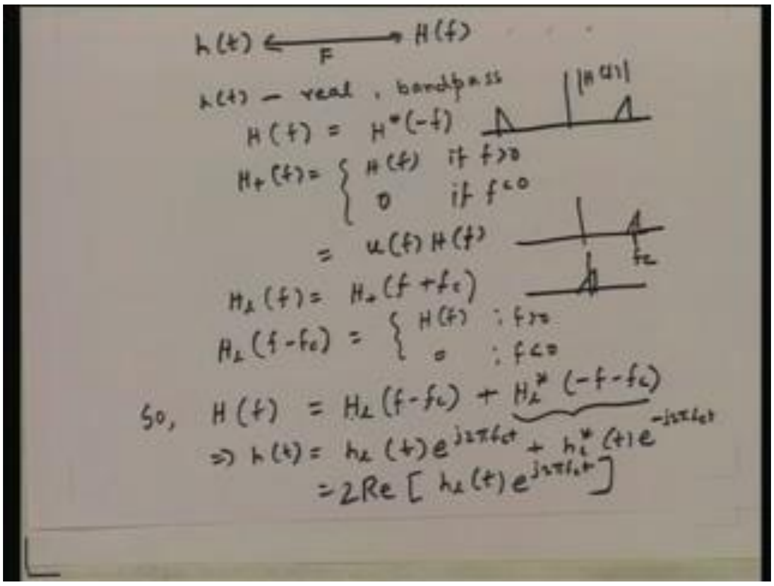

So, suppose the impulse response of the bandpass system is h t. And, it has frequency response H f. Then, we know several things. One is that, h t is real and h t is bandpass.

Now, h t is real; that means that, H of f is H star of minus f. This is a conjugate symmetry. The magnitude is same at f and minus f; and, phase at f is negative of phase of phase at minus f. So, this is the conjugate symmetry. That is the value at f is the conjugate of the value at minus f. Now, just like we did for the bandpass signals, let us say this is the… Suppose this is the H f. We are drawing only the magnitude. Then, just like we did for the bandpass signals, we say that, H plus f is… Take only the positive side. So, basically, H f if f is greater than 0; and, 0 if f less than 0. So, this you can also write as u f times H f.

Now, here for bandpass signals, we actually had two times this here. But, here we are not doing that for some reason; we will see later. Now, again similarly, this is missing as scaling; so, which we will neglect for time being and later we will explain why we have done this. And again, just like bandpass signals, we will shift this. So, this is basically only one side H plus f. Now, H l of f is H plus f plus f c. So, you basically shift it here. So, this spectrum is now. This f c is brought to 0. Now, in terms of H l f; now, we can write H l of… So, what is H l of f minus f c; it is basically H plus of f. So, H plus of f can be… So, we can write this – H of f for f greater than 0 and 0 for f less than 0. So, we can write that, H of f… So, H of f is H l f minus f c. So, H l f minus f c is basically H plus f; that is this. So, if you add this with the negative part of it, which we have removed. So, that part we can get by taking H l star minus f minus f c. So, what did you have here? H f

We want to get this signal from this. This is H l of f minus f c. So, how to get this signal? First keep this signal and then add this part. How do we get this part from this signal? It is basically first invert it. How do you do that? Take this positive side into negative side and negative side into positive side. We simply take the minus of that function. So, H l; replace f by minus f. So, that is H l of minus f minus f c. But, this is not really the negative flipped version of this signal. This has the phase negative of this. So, we have take to conjugate of this. The magnitude we want same; but, the phase we want negative. So, we need to have H l star of that function – flipped function. So, flipped function is H l minus f minus f c. Then, we need to take the conjugate of that. So, that is H l star of minus f minus f c. So, once we add these two, we get this function. This part is basically the positive part; this part is the negative part. So, from here we can also write that, h t is h l t e to the power j 2 pi f c t plus h l star t e to power minus j 2 pi f c t. This is basically the inverse Fourier transforms of these two. So, h l t is the inverse Fourier of H f t. H f –

H – capital H l of f. So, then we can use the well known properties of Fourier transforms to see that; really the Fourier transform of this; inverse Fourier transform of this is this. Similarly, here.

Now, this can also be written as real – 2 times real part of h l t e to power j 2 pi f c t. Now, this is because, this part is the conjugate of this part. So, we can take. Only the real parts will remain; imaginary parts will actually cancel. So, this is what we have. The actual impulse response is expressed in terms of the lowpass equivalent of the impulse response. This is what we also did in the case of representation of bandpass signals. And, we have done the same thing again for bandpass systems. Now, we will see the relationship. Why have we done these two things? We have represented bandpass signals in terms of lowpass equivalent signals; lowpass equivalent signal of the same signal. Then, we have represented bandpass system in terms of lowpass equivalent of that system. Now… So, we knew that, we have a bandpass signal; we can give that as input to the bandpass system. Then, we will get some output. That is also a bandpass signal. Then, we have a lowpass equivalent system of that; lowpass equivalent signal of this signal. Then, if we give this as the input to this signal, what do we get as the output? Would we really get the output as a lowpass equivalent signal of this output? That is what we what to see.

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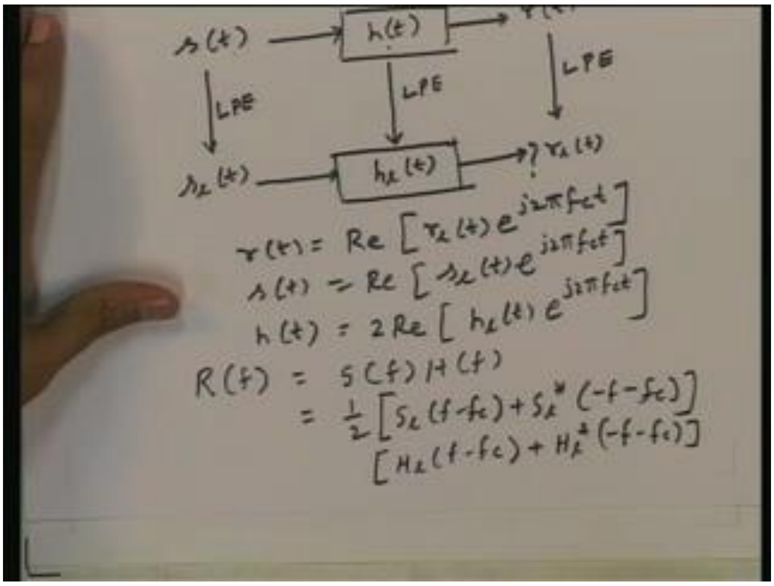

And, this is a very interesting idea. We will use that also later. So, what we are saying is that, we have a bandpass signal s t. A bandpass system h t. We give this as input to this.

And, we get some output say r t. We receive this. From here we have lowpass equivalent; lowpass equivalent of this signal is s l t. And, we have also seen how to get lowpass equivalent system of this system, that is, h l t. It is very similar to this. We have seen just now. Now, we give this input; what would we get as this output? That is the question. Shortly, we will see that, we will actually get the output, which is the lowpass equivalent of this output. So, that is something very interesting. So, this is also lowpass equivalent – lowpass equivalent.

Now, let us see that. So, this we are seeing response of a bandpass system to bandpass signal; response of this bandpass system to bandpass signal. We want to express this whole operation in terms of lowpass signals and lowpass systems; that is what we want to see. So, r t is the output for s t input. So, r t. Now, we express in terms of the lowpass equivalent signal. Now, we do not… We have not yet seen that, the output of this system is actually the lowpass equivalent signal. We do not yet know this. We are trying to see that this is true. So, we know that, r t is a real part of r l t e to the power j 2 pi f c t. Similarly, s t is real part of s l t e to the power j 2 pi f c t. And, again h t is 2 times real part of… Remember there was a difference of scalar in the representation of bandpass systems. So, now, we can also write these things in terms of in the frequency domain. What do we have? In the frequency domain, R of f, that is, the spectrum of this received signal is S of f times H of f. That is the spectrum of the… The frequency response of this system times the spectrum of this input signal.

Now, this can be written as half. What is the expression of s f in terms of the lowpass equivalent signal of s f. So, s f in terms of s l of f. s f in terms of s l f. We have seen just now for systems; it will same thing for signals also. We have seen in the last class; but, it will have only a difference of scalar. Just now, what did you see for systems. We saw that, H f in terms of H l f is this. Similarly, for signals also, we will have the same thing, but there will be half here. So, we have half S l f minus f c plus S l star minus f minus f c times H f. H f just now we saw that, H f is H l f minus f c plus H l star minus f minus f c. So, this is we have just replaced S f in terms of S l f and H f in terms of H l f.

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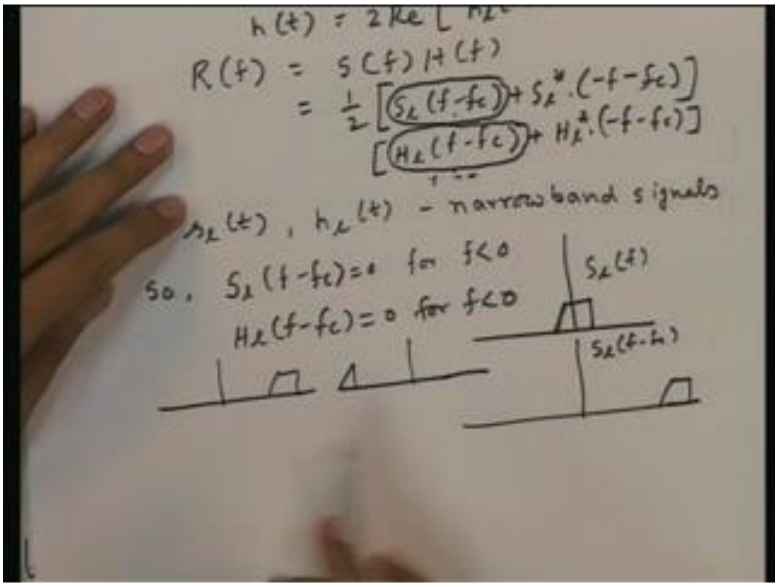

Now, remember that, s l of t and h l of t – the inverse Fourier transform of the S l f and H l f are narrow band signals. So, we can say that S l of f minus f c is 0 for f less than 0. Why? Because S l f is narrow band. It is lowpass narrow band signal. So, it is frequency response of this Fourier transform is like this. It lies here. Something like say this S l f. So, what is S l f minus f c? It is the shift here. So, S l f minus f c. This is S l f. And, this is S l f minus f c. This will be something like this. So, this is 0 on the negative side. So, this is what it is. Similarly, H l f minus f c is 0 for f less than 0.

Now, let us multiply these terms. This is 0 and this is 0 for the negative side. So, this is 0 on the negative side. And, what about S l star of minus f minus f c. It is the flipped version of this and then conjugate. So, flipped version of this will have positive site 0. So, if you take this product; this is at less than 0. This is 0. This is 0 at f greater than 0. So, the product will have 0 everywhere. We have one function, which is 0 at negative side; it has something like this spectrum. And, this part has positive side 0. So, let us say it is like this. Then, the product of these two will be 0, because this is 0 at negative side; this is 0 at positive side. Everywhere the product will be 0. So, only product… So, similarly, this time, this also will be 0. So, only product, which will be nonzero is this times this.

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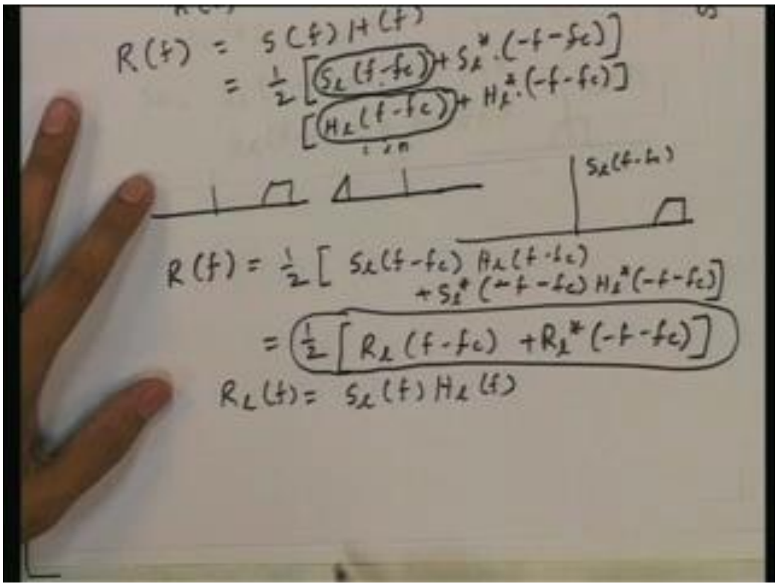

So, we can say that R f. So, R f is half and these two also. Here this both of them have some positive side components. So, these two times these two. These cross terms will be zeros… plus S l star minus f minus f c times H l star minus f minus f c. Now, this is R f. Now, what is this R l f? You can see that, this is R l of f. We have replaced this as R l of f is S l of f times H l of f; that means it is the product of these two frequency responses; that means R l of f is the spectrum of the output signal here. So, that is… So, R l f is the output spectrum of this; and then, R f is the output spectrum of this. This input and this system. So… And, we have got a relationship between them. The output spectrum here and output spectrum here. And, that relation is that, R f is this in terms of R l f. So, this spectrum in terms of this spectrum is expressed here. Output spectrum here is expressed in terms of the output spectrum here.

So… But, what is this really – this thing, which is the output of the bandpass system R f? This is nothing but the up converted version of the R l signal. So, here we can see that, this R l f is really the lowpass equivalent of R f, because it is expressed this way. So, R l is really the lowpass equivalent signal spectrum of the R f. So, we have showed that, this output will really be R l – the lowpass equivalent signal. So, how did you show it? Let us summarize. We have seen now this. We have seen that, this output is r t expressed in terms of the lowpass equivalent. We do not know what is the output of this is really r l t; we are going to verify that. So, we have taken represented r t in terms of r l t; h t in terms of h l t; s t in terms of s l t; then, computed the spectrum of this as product of these two spectrums. Then, taken their lowpass… Represented those two spectrums in terms of their lowpass equivalent spectrums this way and taken that, these two products. And, these two products – the product of the lowpass – these two things, we have got as this.

And then, we have assumed that, this lowpass equivalent spectrum of this and lowpass equivalent the spectrum of this s l t and spectrum of h l t – the product of them we have assumed to be R l of t – R l of h – S l f and H l f, the product is R l f. Then, we have seen here that, this R f, which is the output of the bandpass system is really expressed in terms of the product of S l f and H l f this way. And, from this representation, we see that, R l f, which is the product of these two, is really the lowpass equivalent of R f. So, R l f, which is the product of these two means which is the spectrum here – output spectrum is really the lowpass equivalent of this spectrum; that means the output of this is really the lowpass equivalent of this output.

So, we have verified that, if we have this bandpass system and if we represent the signals and the systems in terms of their lowpass equivalent signals and systems, then we really have this output of the lowpass equivalent system. When it is excited by the lowpass equivalent signal of this input, and the output will also be the lowpass equivalent of this output. This is what we have verified. So, now, we can treat a bandpass system without any loss. We can treat a bandpass system in terms of lowpass system, because whenever there is a bandpass system, it is difficult to suppose implement such a system. We want to implement a bandpass system with some transfer functions. Then, we can take the corresponding lowpass equivalent system and design that. Then, suppose we want to give a bandpass signal as input to bandpass system; we have a bandpass system; we want to implement that. And then, we want to give a bandpass signal as input and then we want to find out the bandpass output.

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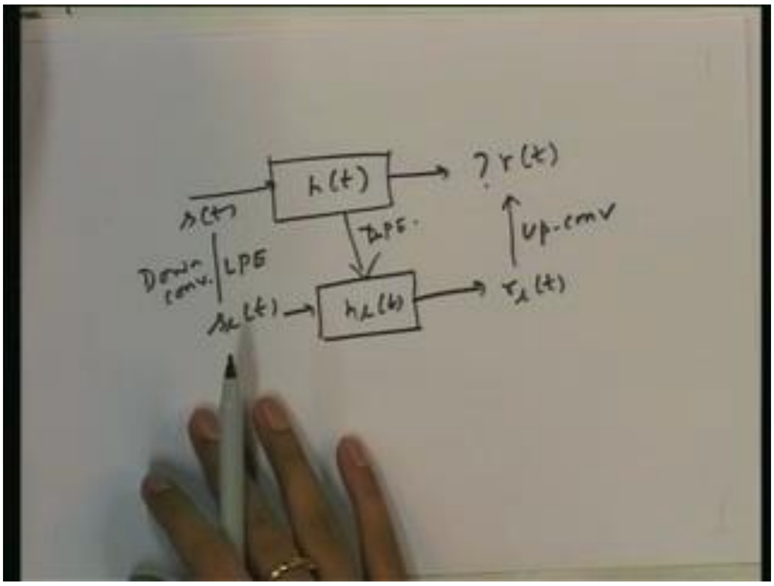

Then, we can do that whole thing in terms of a lowpass system and lowpass signal. This is exactly like we had drawn here. So, we want to implement h t and we want to find out this output. So, what we can do is take the lowpass equivalent signal. If it is difficult to implement this system, then it is not possible to do this way. Then, we can take this. So, do to this down conversion, do this down conversion also and implement its lowpass equivalent system and find out the output. This… Then, do up conversion. So, we will get r l t here and then we can find out r t by doing up conversion. So, this is basically down conversion; this is the lowpass equivalent system. So, instead of implementing this system, we can do this way, because this will be cheaper, because this is a lowpass system. So, we have seen in this class that… What we had seen in the last class; we have actually refined that.

Now, we had seen in the last class that, we can represent this up conversion channel and down conversion as a single channel as lowpass channel. Now, we know now that, there is a relationship between this channel and this channel. If this channel has impulse response h t, this channel has the impulse response h l of t, which is the lowpass equivalent of this impulse response. So, it is something interesting. So, this is what we have seen.

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Now, we will… This… All these things you can study from the book by Proakis. I am also teaching from this book – Digital Communication by Proakis. And, at this point, after these two classes, you should be able to also solve this exercise – exercise 1 from Proakis – fourth chapter; fourth chapter of this book – exercise number 1; that is, the exercise 4.1 from this book Proakis. It is basically an exercise on Hilbert transform. It has these components. Please try this exercise at home. And, I will solve some particular parts of this exercise in the next class. So, it also asks… It actually asks to prove some relationship on Hilbert transform.

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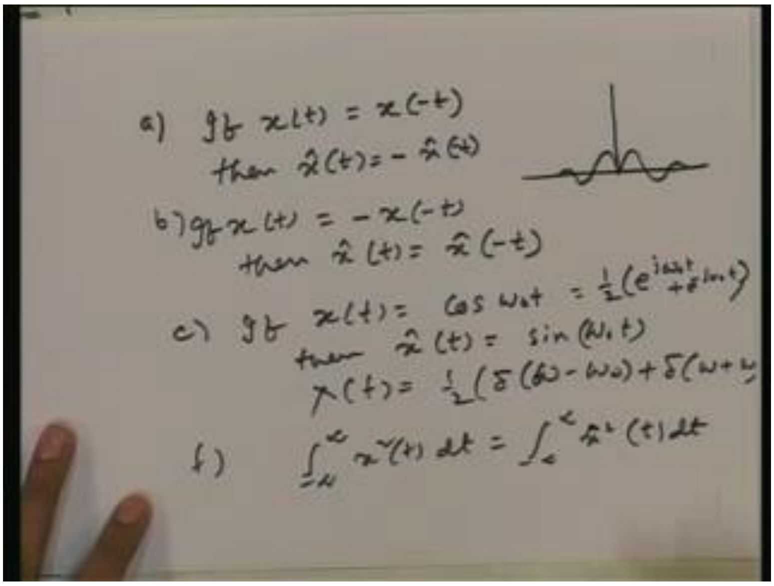

So, for example, part a is if x t is same as x of minus t; that means it is symmetric signal – event; also called event; that means the positive side is exactly the mirror image of the negative side. So, for example, if it is like this, then it is like this; then, the Hilbert transform x hat t is minus x hat minus t. B is similar if x t is minus x of minus t, then x hat t is x hat minus t. So, this can be just use the… For proving these two, just use the definition of Hilbert transform; the convolution of x t with 1 by pi t. That is the Hilbert transform of x t. Use the definition and simple substitution of variable will lead you to the solution of these problems.

And then, there are… c for example, is if x t is cos omega naught t, then x hat t is sin omega naught t. So, for solving this problem, you can use this. And then, solve this problem in frequency domain. You know the spectrum of Hilbert transformer – H f; multiply with this. So, this has X f as half delta omega. So, if omega minus omega naught plus delta omega plus omega naught. Then, multiply by H f and take the inverse Fourier transform; you will easily get this. So, this is how you can solve this exercise. And, there are some more parts not similar to this. But, you can show that, the energy for example, f is actually to show that, the energy of x t and its Hilbert transform are same. I will not give a hint to this. This is actually easier. So, you should be able to do this yourself without any hint.

So, we will see you again in the next class.