Advanced Optical Communications

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Lecture No. # 33
Non – Linear Schrodinger Equation

We are discussing Non-Linear Fiber Optics. In the last lecture, we saw that when the light intensity increases, the higher order susceptibility terms in the induced polarization of dielectric material have to be taken into consideration. We also saw that for a material like glass, the second order susceptibility contribution is negligible. So, the non-linearity primarily is because of the third order susceptibility, which leads to what is called the Kerr non-linearity.

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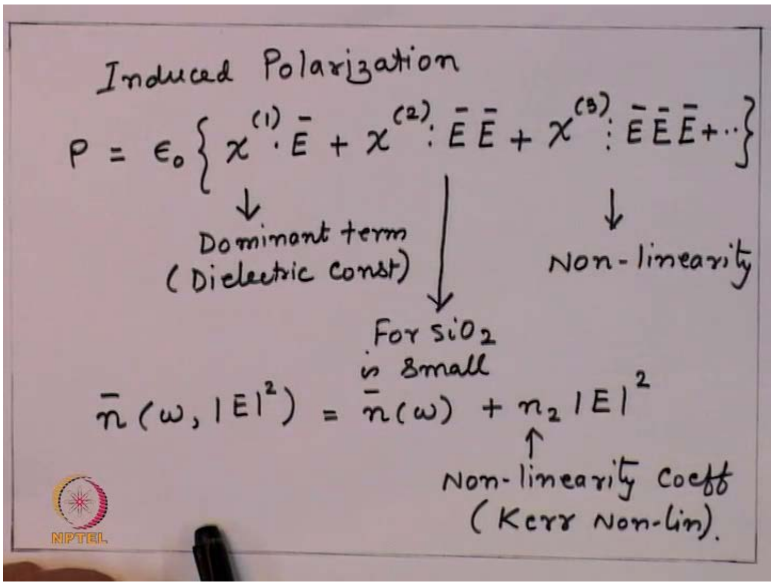

So, we showed that the polarization has these terms here; this is the dominant term which contributes to the dielectric constant. And that is the term which contributes to the non-linearity; so for the Kerr non-linearity, the refractive index of the material has a term, which is proportional to the square of the electric field or the power density in the material. So, essentially we are investigating the pulse propagation inside an optical fiber in the presence of this non-linearity.

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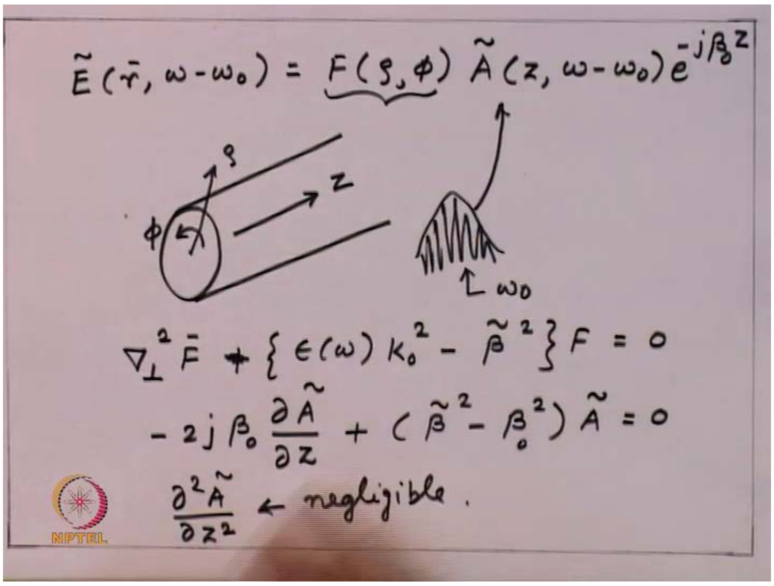

Then starting with the simple Maxwell's equations and defining a function, which has a cross sectional field distribution and a pulse, which is evolving in the direction of propagation z. We wrote the wave equation in the structural domain around frequency omega 0 and then we got these two separate equations. One which governs the transverse distribution of the field, which is this function F and other one is the evolution of the pulse envelope which is given by this, in the direction of the optical fiber.

We also saw that, since the evolution of pulse is slow, the second order derivative of A as a function of z can be neglected. So, you have a simple first ordered differential equation for the evolution of the pulse, whereas, for the transverse field distribution you have this equation. Now, note here that this equation is exactly identical to what we have already solved, when we investigated the field propagation or the modal distribution inside the optical fiber.

So, when the non-linear effects were not present, essentially this envelope function we had not taken, we simply had seen that the electric field is given in terms of this transverse field distribution and the phase function; and where beta 0 is the phase constant at frequency omega 0. So, we are assuming here that even in the presence of non-linearity, the modal field distribution is practically unchanged.

So, that means if I consider let us say l p 0 1 mode or h e 1 1 mode, that distribution is more or less same, whether we include non-linearity or we do not include non-linearity.

Only thing is this field distribution will evolve as this field distribution would start propagating along the optical fiber, which is given by the second equation. So, our focus is only on this equation the second equation, because this equation, we already solved and we know the solution to this equation comes in the form of Bessel functions, modified Hankel functions, you apply the boundary conditions and so on and so on.

So, let us focus now, primarily on this equation and see what more we can we can do how can we simplify this and get some more simplified version of this equation; and see physically what different terms mean in this equation. So, first of all since, this quantity beta tilde, which you see here this quantity, since this is the very, very close to quantity beta naught, because we are talking about a narrow band frequencies, one can make a approximation to this.

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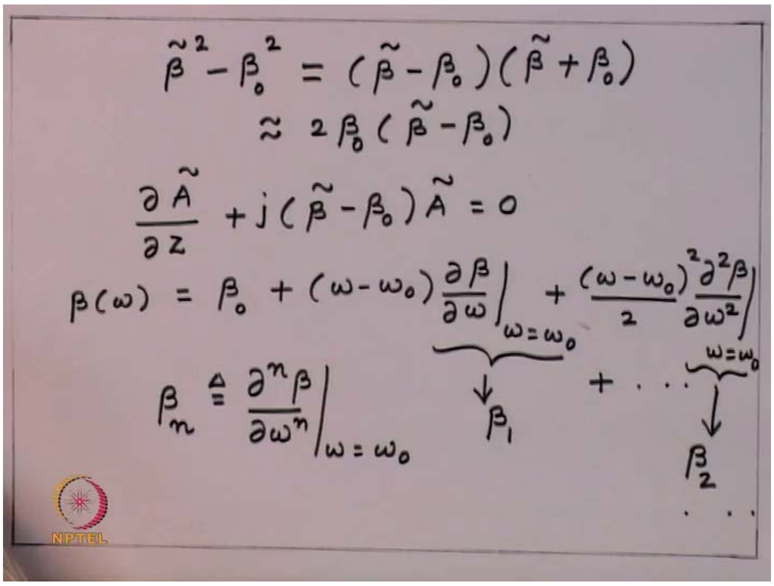

So, one may say first of all that this square minus beta naught square, this approximately let us say first do not make approximation this is beta tilde minus beta naught beta tilde plus beta naught. And since this beta tilde is very close to beta naught, we can say that this quantity is almost same as beta naught. So, we can say that, this is approximately equal to 2 times beta naught, which is this term times beta tilde minus beta naught.

So, I can take this thing now and substitute in to this equation (Refer Slide Time: 05:46), so this quantity here, we can say 2 times beta naught into beta tilde minus beta naught; so 2 beta naught will cancel. And then this equation will be modified to by d z plus j beta tilde minus beta naught A this is equal to 0. Now since we are having a band of frequencies, the beta is not same as beta naught everywhere but there is a small change with respect to this beta naught, because we are having a band of frequencies.

So, what one can do is one can sort of have a Taylor series expansion to get a value of beta around beta naught, in terms of the derivatives of beta naught as a function of omega. So, we can say that this quantity beta as a function of omega, that is equal to beta naught, which is beta at omega equal to omega 0 plus omega minus omega 0 d beta by d omega at omega equal to omega 0 plus omega minus omega 0 whole square upon 2 d 2 beta upon d omega square again that omega equal to omega 0 plus and so on.

For the brevity reason, if we say that this beta n is defined as d n beta divided by d omega to the power n beta divided by d omega to the power n at omega equal to omega 0. Then this quantity can be denoted as beta 1 this quantity here is denoted as beta 2 and so on. Now the dielectric constant of a medium as we already said is having a linear and a non-linear component.

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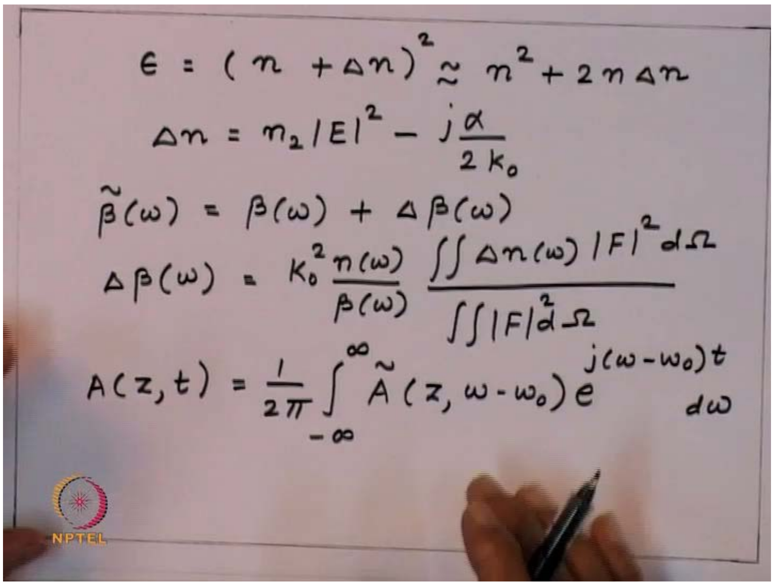

So, the epsilon for the medium is given as n is the function of omega plus the delta n whole square, if you know from our very basic relation, that the the epsilon is now equal to the refractive index square. So, this one we can approximately say is equal to n square plus 2 n to delta n, you are again we are assuming that this delta in term is negligibly small.

So, the second ordered terms can be neglected we retain only the first order term of delta n and this delta n is as we have seen earlier, this is because of two factors. Now one is the because of non-linearity which is given as this and the lost term, because we have seen that the first order susceptibility is complex. So, you have a imaginary term which is minus j alpha divided by 2 times k naught.

So, you have the refractive index, which is for a dielectric constant considering only first order susceptibility, without the loss term. And then you are having a small contribution which is coming from non-linearity and the loss in the material. So, this quantity in this will now (Refer Slide Time: 10:18) this quantity beta tilde as a function of omega, we can write as beta omega plus a small change in beta omega, because of this terms so delta beta omega.

And this delta beta omega can be obtained essentially from this, which is given as delta beta omega that is equal to k naught square n omega divided by beta omega integral of delta n is a function of omega. The field distribution square delta omega, where this is the the cross sectional area divided by integral F d omega, F square; if we now define the inverse Fourier transform for the envelope.

So, I say this is a now z of t, which is equal to 1 upon 2 pi minus infinity to infinity A tilde, which is the envelope function as a function of z and frequency e to the power j omega minus omega naught t delta omega. And substitute now (Refer Slide Time: 12:38), this into this term this equation here of this equation here should we have go for all various terms after expansion and so on.

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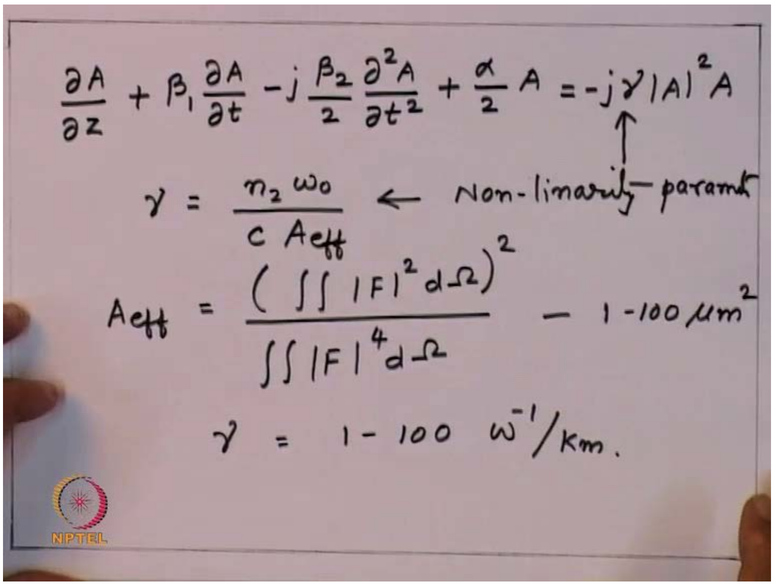

We get a equation in terms of this A which is in time domain as dA by d Z plus beta 1 dA by dt dt minus j beta 2 upon 2 d 2 A by dt dt square plus alpha upon 2 A equal to minus j gamma mod A square into A. Where this quantity gamma is the non-linearity parameter and that is related to the the change in refractive index (Refer Slide Time: 14:02), because of non-linearity this quantity n 2 non-linearity coefficient and also the confinement of the of the light.

So, where this quantity here gamma, which you get is n 2 omega 0 divided by c A effective that is the effective area over which the the light is confined inside the optical fiber. And when you do this algebra here this quantity A effective, which you get that is given as integral of F square d omega whole square divided by F to the power 4 into omega. So, if I know that transverse field distribution of the mode propagating inside the optical fiber, we know this quantity F.

So, essentially by substituting this expression we can get the effective area over which the light is confined inside the optical fiber. And once we get that parameter then, one can substitute in to this; I know the non-linearity coefficient for glass, so we can calculate this parameter gamma, which is a non-linearity parameter parameter. Now, the typical value for the effective area inside the optical fiber may range from about 1 to 100 micrometer square, depending upon you know what the core size and and what field distribution you are taking and so on and so on.

And the corresponding value of this parameter gamma, may be in the range of 1 to 100 watts minus 1 per kilometer. So, this is the equation finally, which has evolved now, which tells how the, which is the envelope function is going to evolve as the pulse travels on the optical fiber. Now, let us look at the various terms here and let us try to get a physical meaning, what they are we actually representing this terms here, so let us take the first simplest possible term.

Let us say there are beta 2 is negligibly small, let us say this quantity is not there let us say there is no loss, so alpha is also equal to 0, and let us say there is non-linearity also, even this quantity is equal to 0. So, if I if I neglect this term (Refer Slide Time: 17:18) if I neglect this term, if I neglect this term, then I have got this quantity which is dA by d z which is equal to minus beta 1 dA by d d t. So, from there essentially we get d z by d t, which is equal to 1 upon beta 1 and that is what we know, that what does this quantity d omega by d beta represent (Refer Slide Time: 17:47).

We know that d omega by d beta represent the group velocity of the signal that means, it is the effective velocity of the pulse on the optical fiber; so, essentially d beta by d omega, which is beta 1 that is nothing but 1 upon the group velocity; so this quantity as we have seen here this is 1 upon the group velocity. So, what this thing is simply saying that, if the loss was neglected, if this quantity is neglected, if non-linearity is neglected then, you have got a simple equation, which is this. Which is simple saying that the velocity of the pulse is equal to group velocity, that is what the simple thing, we can get.

Let us say, we consider now a situation, where we are moving with the pulse; that means if I just take a situation, where the pulse is not stationary but pulse is moving and that is moving with with velocity which is the group velocity; so we move with the group velocity. So, if I talk about now a a new time reference, which is in the moving time frame, the frame which is moving with group velocity along with the pulse, that is obvious that this term essentially has to go to 0. Because then there is no question of velocity, because we are seeing now, we are actually riding the pulse, we are measuring all the times with respect to that moving frame, which is respect to that pulse. So, this term essentially goes to 0, so first thing, what we can note is that if I define a new time frame.

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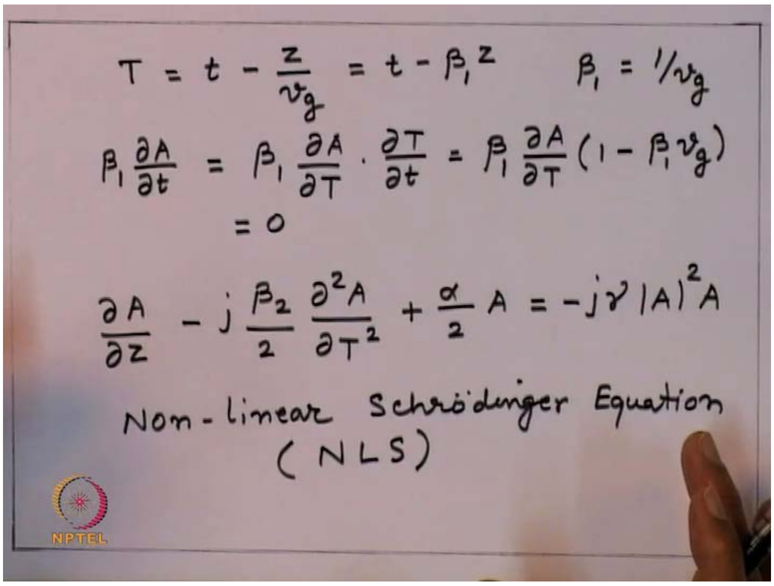

Let us say I define a new time T, which is the old time t minus z divided by v g, which is same as t minus beta 1 times z and what is beta 1 as we have seen beta 1 is equal to 1 upon group velocity. So, this term here beta 1 times dA by d t, that you can write as beta 1 dA by dt dt times dt dt by dt dt and from here you can write as this is beta 1 dA by dt dt into 1 minus d z by dt dt which is nothing but group velocity. So, you got here beta 1 times v g but beta 1 times v g is equal to 1, so this quantity 1 minus 1 is 0 that means this quantity, if I take this equation in the new time frame, which is this capital T this quantity will be equal to 0. So, what that means is now, that we can substitute now this in this equation, the new time (Refer Slide Time: 21:42) and in that time this term will not be there, we have all other terms but this terms will will vanish.

So, we will get the new equation now, which is dA by d z minus j beta 2 by 2 d 2 A by d capital T square which is the time measure in the moving frame, plus alpha by 2 A is equal to minus j gamma mod A square into A. This equation resembles the Schrodinger equation therefore, this one in the non-linear fiber optic technology, this equation is referred to as the non-linear Schrodinger equation (No audio from 22:58 to 23:14) or in short it is written as NLS.

So, the pulse evolution on the optical fiber essentially is governed by this equation, which is at approximate form of the wave equation for the evolution of pulse. So, two thing to note here, what we have done is we started with the wave equation; in the wave equation we made certain approximations. We assumed that the transverse electric fields or the magnetic fields transverse field distribution is practically same as what was there in the absence of non-linearity only when this field distribution travels along the optical fiber the envelope undergoes an evolution. And this non-linear Schrodinger equation gives the behavior of the envelope as a function of distance on optical fiber.

So, essentially now if you want to study the evolution of the pulse in a comprehensive fashion, we have to look for the solution of this non-linear Schrodinger equation. However before we get in to that, let us first try to understand what is different terms in non-linear Schrodinger equations are representing. So, first of all let us let us take the simplest term which is this term here, which is alpha by 2 into A, so let us assume that the beta 2 is negligibly small, non-linearity is not present and I have a equation which is simply dA by d z plus alpha by 2 A that is the equation, which I am having.

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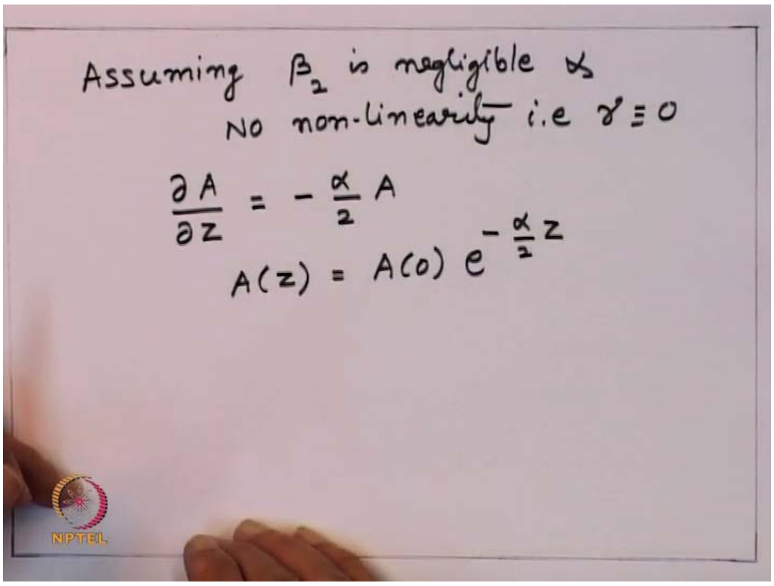

So, in the simplest form let us say assume beta 2 is negligible and no non-linearity, that is gamma is equal to 0, in this situation then one can get an equation which is simply dA by d z, which is equal to minus alpha by 2 into A. So, solution to this equation is very simple that the A as a function of z is at some initial location A 0 e to the power minus alpha by 2 z. And as we know the alpha as we have taken is the attenuation constant on the optical fiber.

It simply says that this pulse whatever we are launching, is going to exponentially die down at with this attenuation constant alpha; is like a typical medium; which has a loss alpha the signal when it propagates on the structure it exponentially dies down with the attenuation constant that is what a simple solution which you get. So, this term here this quantity here (Refer Slide Time: 26:44) is representing the loss in to the medium and if other effects are absent, then you have a exponential decay of the signal on the optical fiber; so this term corresponds to simple loss term.

This term as we already said depends upon the non-linear coefficient, so this is coming because of non-linearity. What is this term representing now, this term as we can see is proportional to this quantity which is beta 2 and what is beta 2? Beta 2 as we already seen is d 2 beta upon d omega square; that means that is the derivative d by d omega of d beta by d omega. And d beta by d omega is nothing but the group velocity; that means this term is telling you the change in group velocity as a function of frequency.

That means different frequencies are going to travel with different velocity different group velocity and that is what actually is captured by this quantity here. And what is that phenomena, that different velocity is travelled with different group velocities, that phenomenon are nothing but dispersion. The dispersion is defined as the different wavelengths are travelling with different velocities, we have seen already different types of dispersion we have seen dispersion which could be just intrinsically because of material properties.

That means the refractive index changes as a function of frequency, so different frequencies travel with different different velocity or it could be because of different modal propagation. But essentially when different modes are not there, if the fiber is simply single mode, then the different frequencies are going to travel with different velocities, which we call as the monochromatic dispersion. That is what actually is captured by this term here (Refer Slide Time: 29:03), so this term in this equation essentially represents the dispersion.

So, now what we are saying is that, this non-linear Schrodinger equation which we have got, it takes into consideration the effect of dispersion on the pulse evolution. It takes in to consideration the loss on the optical fiber and it also takes in to consideration the non-linear effects on the evolution of the pulse. So, normally if you really look at this equation to solve this equation complete is a quite tedious task, it require numerical methods to really solve this equations but in principle essentially what we can do is, one can define this two operations.

You see, this is the which is going to take place right into say second derivative of time and this can be easily done if I take the Fourier transform this equation. Forget let us say delete these two quantities, then you will see that solving this equation the Fourier domain is much easier; because the second derivative and time is reduces to simply multiplication of the frequency square. So, if I look at the phenomena of dispersion that can be easily investigated in the frequency domain, because different frequencies are travelling with different velocities.

If I look at this phenomena, which is the non-linearity phenomena then, that has to be considered only in time domain, because this quantity depends upon this mod A square and A is an envelope function in time. So, if you want to solve this equation this equation has to be simultaneously solved in frequency and time; however since both these effects are rather weak effects. One can solve, while solving this one can assume that, when the dispersion is playing a role the non-linearity is negligible, when non-linearity is playing a role the dispersion is negligible. And considering the effect of one at a time one can essentially understand the evolution of the pulse.

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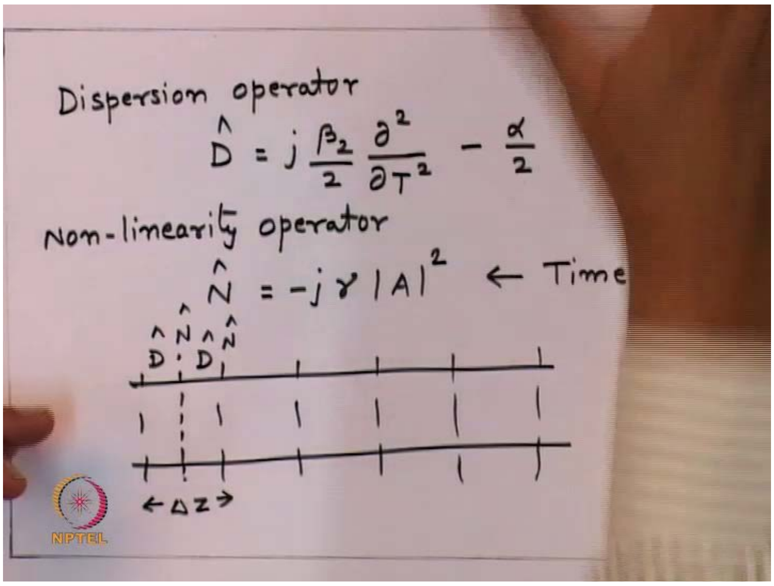

So, what one can do is now for numerically solving this problem, one can define the two parameters or operators, let us say define a dispersion operator. Let us say define the let us some D cap that is equal to j beta 2 by 2 d 2 by dt dt square minus alpha upon 2. And you define a non-linearity operator, let us say N cap that is defined as minus j gamma mod A square. And as we have already said applying this operator in frequency domain is easier and applying this operator in time domain is easier time domain.

So, now the evolution of the pulse now can be studied by using what is called a split Fourier step method. So, here what we do is we have a medium on which we want to study the pulse evolution let us divide this into the small small sections, so let us say we divide this in to this sections here. So, let us say this size is some some delta z over which we assume that the effect of non-linearity and dispersion both are small and it may be adequate to consider one effect at a time.

So, what one can do is, one can solve this this equation by applying this D operator assuming that the non-linearity is not present, so let us say I take half of the step, from to here I apply the operator which is D cap. So, the spectrum is modified, because I am doing in frequency domain, take a Fourier transform come in time domain and apply this operator at this location, that the pulse evolve again for this distance, which is D cap then apply again the N operator.

So, in this region you are solving the wave equation in the Fourier domain in the spectral domain say you get a spectrum, when the spectrum is there you come at this location you take this Fourier transform, you got a time function. Once you get a time function you are applying time domain this operator, because now time function is known; so this quantity is known, so I know this operator now so I can find out what thing have to be applied.

So, I applied this quantity here whatever new, function I got I take this Fourier transform back go to spectral domain again apply this operator and you go on doing it repeatedly over this, so that you get evolution pulse on on the optical fiber. So, that is the way this will be numerically solved, though for certain specific kind of pulse shapes, you may get analytical solutions. If you consider the things one at a time but if you consider a general complex pulse function, then you essentially a (( )) by numerical techniques and by applying the operator sequentially, I do all the small small steps and then finding out how the pulse evolution is going to take place.

So, that is what is the basic formulation for getting (Refer Slide Time: 36:06) the solution of the wave equation in the presence of non-linearity on an optical fiber. What one can do is one can now; define certain parameters, which characterize these three quantities. What one can do is one can define some two parameters, which sort of characterizes the lengths or optical fiber, related to dispersion and the non-linearity, so let us consider now a situation for a Gaussian pulse propagation.

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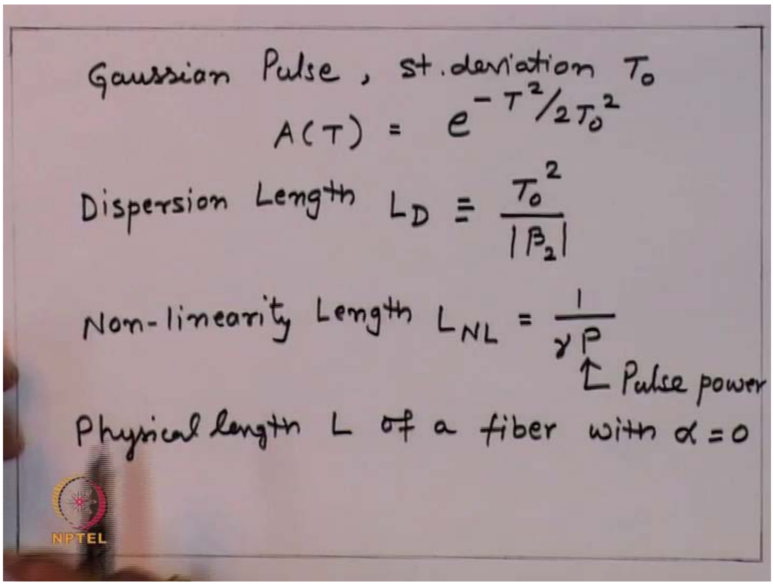

So, let us say I take a pulse which is Gaussian and this pulse is having a the sigma which given by T naught, so you have a standard deviation for this Gaussian, which is given as T naught. So, the pulse Gaussian pulse let us say A as a function of T is given as e to the power minus T square upon 2 T naught square. Then first this pulse, we can define the characteristic lengths on optical fiber, so one can define the two characteristic lengths one is what is called the dispersion length. And that is defined L D which is equal to T naught square divided upon mod of beta 2.

One can define the characteristic length which is called the non-linearity length, which is L N L and that is defined as 1 upon gamma into P, where P is the pulse power. And gamma is the non-linearity parameter as we have defined in the non-linear Schrodinger equation. So, once we define this two characteristic lengths (Refer Slide Time: 39:06) then the solution of this non-linear Schrodinger equation, can be approximated depending upon the situation of the pulse propagation on the optical fiber.

So, there will be various possibilities that if I consider a physical length of optical fiber over which the propagation is taking place. This physical length could be much smaller compared to this quantity (Refer Slide Time: 39:34) could be much smaller compared to this quantity, could be much smaller compared to this. But much larger compared to this or much larger compared to this, much smaller compared to this and vice versa.

So, essentially you have four combinations, that if I consider the physical length L physical length L of a fiber and for simplicity, let us assume that the loss is negligible loss is not playing any role.

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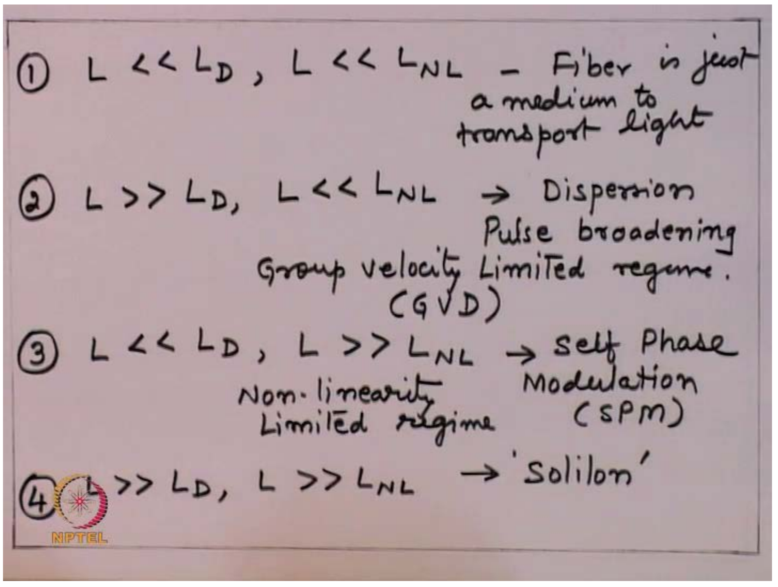

So, for physical length L on a fiber with let us say alpha is approximately 0, you may have three main possibilities or let us say four main possibilities; one is this length L, so situation 1 is L is much, much less than L D dispersion length L is much, much less than L N L. So, firstly what is this two two length they are telling us (Refer Slide Time: 40:43) this length is telling us that over this distance the dispersion effect becomes significant the L N L is the length over which the non-linearity effects are significantly observed.

So, if I can consider a situation that the physical length or the propagation length is much less than L D; that means the length is not enough even to observe the dispersion effect L is much, much smaller than L N L also that means the length is not enough to even see the non-linear effects. So, in this case the optical fiber becomes a simple bear medium of transportation of pulse form input to output, so there is no modification of pulse either in the spectral domain or in time domain; the same pulse whatever you are launching essentially appears from the other side.

So, in this case the fiber is a bear medium, just a medium to transport light the signal does not undergo any modification at all. The second possibility as we said is that L is much, much greater than L D but L is still much, much less than L N L when this this situation occur. And if you consider a very narrow pulse that time this L D will be small so the pulse will be travelling over a distance, which is much larger than L D that means dispersion effect will start seen on the optical fiber. But the power in the pulse is still not very large, so non-linear effects are still not seen.

So, in this case what we will see, you will see what we have been discussing in the optical communication so far, that you will see the pulse broadening phenomena or the dispersion phenomena. So, in this case you will see dispersion of the pulse or pulse broadening and this regime then, we call is the group velocity, limited regime or GVD limited regime. So, normal optical communication, when we talk about normally most of the time this is the situation, that the length is much larger than L D but non-linear effects are not significant and you see pulse broadening of the on the optical fiber.

Third possibility is that L is much, much smaller than L D but L is much, much larger than L N L. When will that happen, that will happen if I consider a reasonably broad pulse, so that the L D is large, because L D is proportional to t naught square. But the power in the optical pulse is large enough, so that the non-linear effects are really observed. When I do this that time, we see a situation what is called the self phase modulation and we will discuss these phenomena in in detail little later.

But what it simply saying is, now the pulse broadening is not taking place because of dispersion but I am having a pulse and this pulse is going to see the different refractive index at different location. So, within the pulse the frequencies are going to travel with different velocities and which creates some kind of the phase function. So, the pulse has a tendency to modify itself in phase function, because the refractive index is a function of the pulse shape; so this phenomena is what is called the self phase modulation.

Because this self phase the modulation of the phase is taking place, because of the pulse itself, because the pulse intensity is going to modify the refractive index. This refractive index is going to change the velocity of light and different frequencies will travel with different velocities and as a result you will see there is a some kind of a variation of phase you know within the pulse itself; so, that phenomena is what is called the self phase modulation. So, in this regime what we call as a non-linearity limited regime, we see the phenomena of what is called self phase modulation or in short it is called SPM.

The last case if that the pulse is narrow enough so L is much, much greater than L D, so dispersion is showing effect L is also much, much greater than L N L the power is large enough, so that non-linear effects are also visible. Now, you will see the interplay of the dispersion and non-linearity both. And that in the situation we get a very special kind of propagation what is called solitonic propagation, so this will give solitons; so what we will do when we meet the (( )) essentially we will take one by one.

This case is of course, very, very simple this we already investigated in great detail, when we talked about just the simple optical communication. So, we do not have to investigate this case, we will investigate this case which is the dispersion limited regime; so we will say the non-linearity is not present and then see, what way the pulse evolves what happens to the internal structure of the pulse at so on and so on.

Then we take the second case, when we say that the dispersion is negligibly small only self phase modulation is there and then, we will see how the what happens to the pulse as in envelope, as in internal structure, as a frequency spectrum; and then we will see where both are present what will happen to this pulse.

So, essentially taking one by one then we will we will have a complete comprehensive understanding of the evolution of the pulse of the optical fiber. And that will eventually lead to what is called the solitonic propagation on the optical fiber. So, let us summarize what we did, we started with the wave equation, we defined the fields which are having transverse distribution, which is same as what the fiber has in the absence of non-linearity. But now the field distribution has an envelope function which evolves as this function travels around the optical fiber; and then we got was proper making certain approximations.

Then we got an equation, what is called a non-linear Schrodinger equation; the non-linear Schrodinger equation is a comprehensive equation, because it takes into account the loss in the optical fiber, the dispersion into the optical fiber, the non-linearity into the optical fiber and also the group velocity of the pulse on the optical fiber. Then by using simple coordinate transformation define the time in the moving frame, we got rate of the term, which corresponded to the group velocity term. And then we got the non-linear Schrodinger equation with the terms with non-linearity dispersion and the loss.

And then we also saw that we can define certain characteristic parameter corresponding to the non-linearity as well as dispersion, and then depending upon the physical length compared to these characteristic lengths. You may see different phenomena on optical fibers, you can either see dispersion which is pulse broadening; you can see self phase modulation, which is because of non-linearity and also a phenomena, which is interplay of this two, which will lead to what is called the solitonic propagation.