Advanced Digital Signal Processing – Wavelets and Multirate
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Module No. #01
Lecture No. #09
Iterating the Filter Bank for phi and psi

A very warm welcome to the ninth lecture on the subject of wavelets and multi-rate digital signal processing.

We continue in this lecture to build further on the relationship between the filter bank and the scaling function and wavelet functions. Let me put before you some of the important conclusions that we had drawn towards the end of the previous lecture. We had said that there is a generic dilation equation that relates the filter bank to the scaling function and the filter bank to the wavelet.

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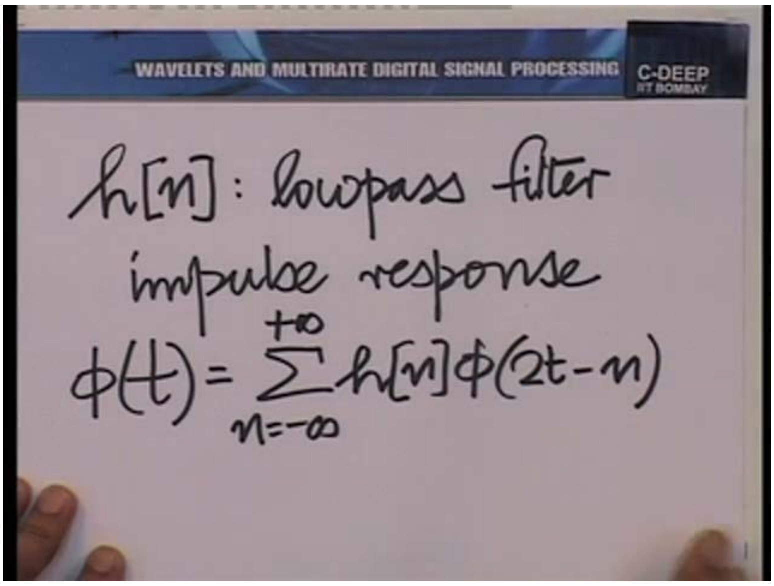

In fact, if hn is the low pass filter impulse response, we had said that phi t obeys a dilation equation like this. As far as the wavelet is concerned, we had said that if we take the high pass filter in the filter bank, say if g of n is the high pass filter impulse response, then psi of t is summation n going from minus to plus infinity, g of n phi 2t minus n again phi note not psi.

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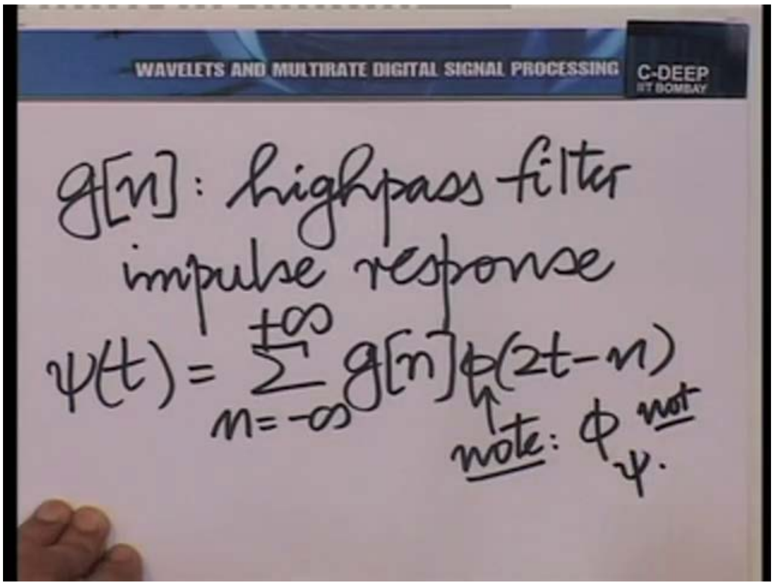

So, this is not surprising. What we said was after all phi t belongs to v1, psi t also belongs to v1. So, therefore, both phi t and psi t should be expressible in the basis of v1 and that is what we are essentially written down. What is noteworthy is that the coefficients of the impulse response of the low pass filter and high pass filter act as the coefficients in the expansion in terms of the basis.

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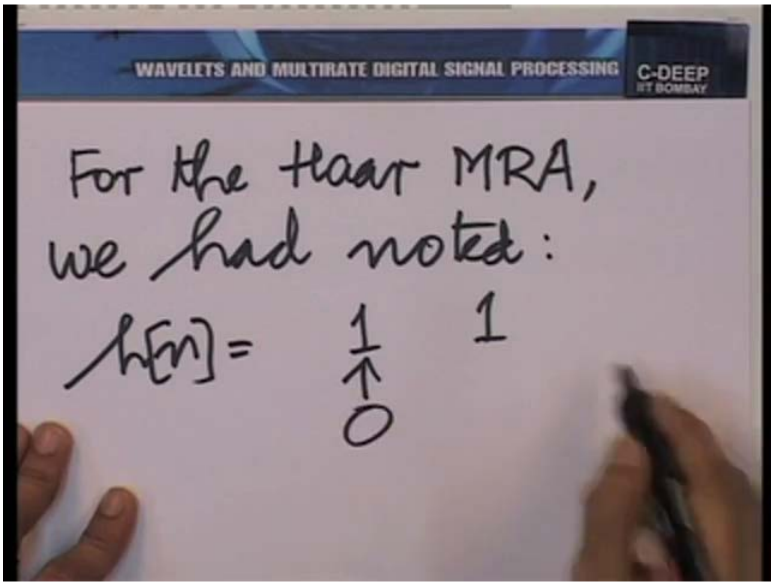

Now, in particular for the Haar MRA, we had noted hn is this sequence. Recall that this is a way of denoting finite length sequences and this means that add n equal to 0. The value of the sequence is 1 and then points after and before take values as shown. So, for example, here if this is n equal to 0, this is going to be n equal to 1 and of course, other points which are not shown are automatically 0.

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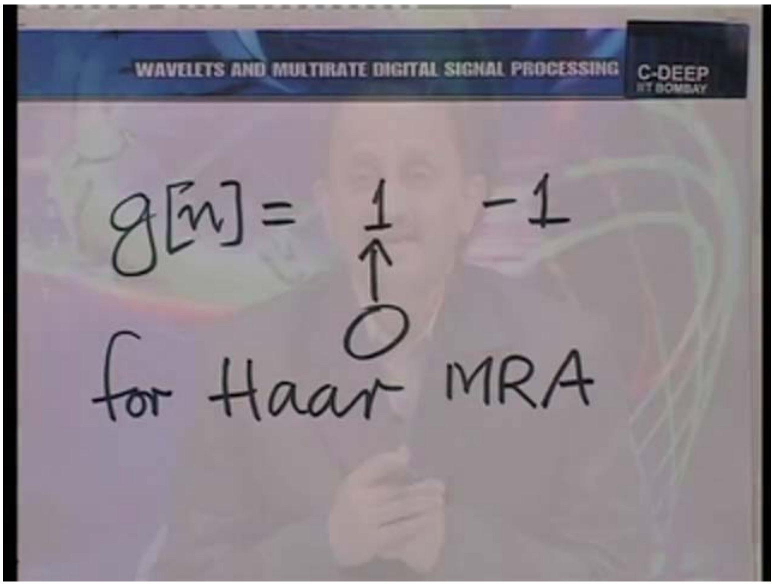

GN is this for the Haar system and in fact, we said that what these equations told us was something much deeper than the containment of phi t and psi t in v 1. In a sense, these equations tell us how to go from the filter bank to the wavelet and from the filter bank to the scaling function. We are just hinted at this in the previous lecture but now, we make that idea very very concrete.

So, let us begin by looking at the Fourier domain. As I said last time, we need to take the Fourier transform because that is where we shall see something very interesting. So, let us take the first of the two dilation equations. You know incidentally just as you have differential equations, you have difference equations, you have dilation equations here.

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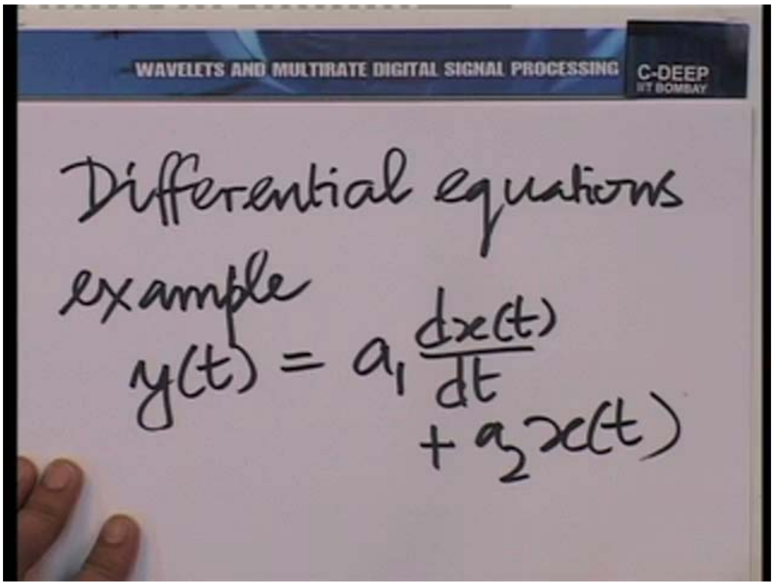

So, you often encounter differential equations. Example could be yt is a1 dxt by dt. Let us say plus a2xt. So, differential equation, we have difference equations.

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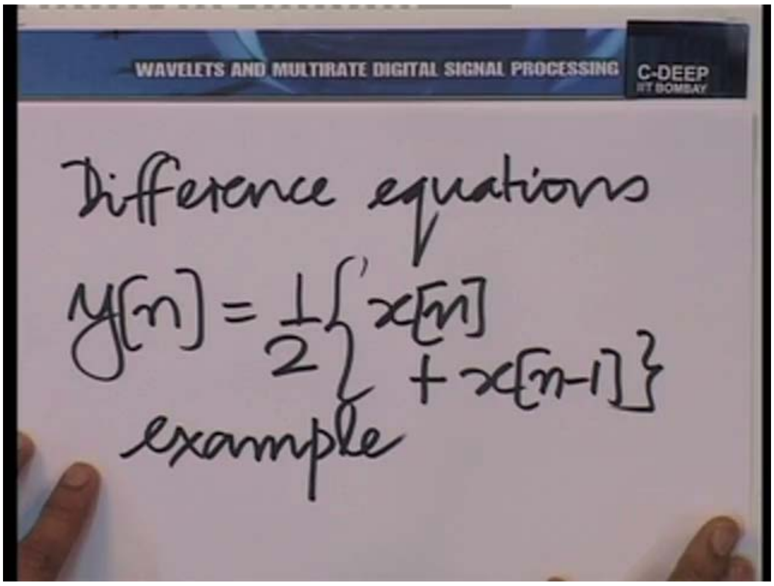

For example, y of n is half x of n plus x of n minus 1 is an example of a difference equation which describes the discrete system and now, we have a dilation equation.

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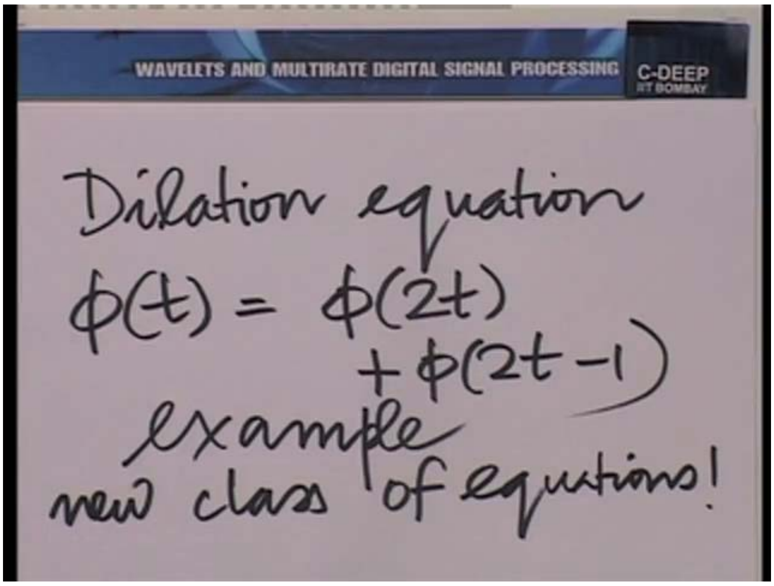

This is a new class of equations. It is a new class of equations and this new class has a reason from our discussion of wavelets. In fact, from the relation between wavelets and multi rate filter banks. Anyway, with this little aside, let us come back to the issue of relating that filter completely in generative terms to the scaling function and the wavelet.

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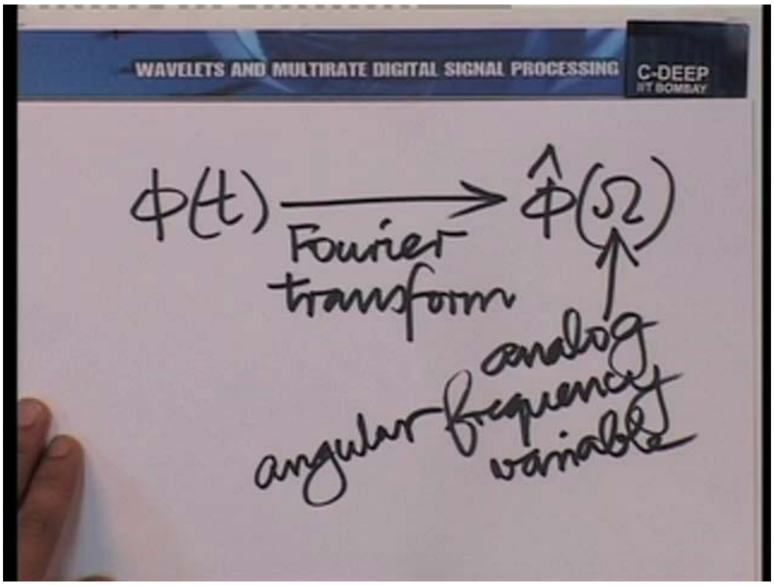

So, let us take this very general dilation equation, phi of t is summation n going from minus to plus infinity h of n phi of 2t minus n and we take its Fourier transform on both sides. Indeed, let us denote the Fourier transform of phi t as phi cap omega.

Now, remember this is the analog frequency variant or the frequency variable corresponding to the continuous time context. So, I should say analog angular frequency variant, to be very precise and we know the relation between phi t and phi cap omega. So, we have phi cap omega is integral from minus to plus infinity phi te raise the power minus j omega tdt and we operate this on both sides.

So, we write down integral from minus to plus infinity summation hn over n times phi 2 t minus n e raise the power minus j omega t dt and integrated all the way from minus to plus infinity. So, we have this integral here. Now, let us you see if converges which is does because it is the Fourier transform of phi t. We could interchange the order of summation integration.

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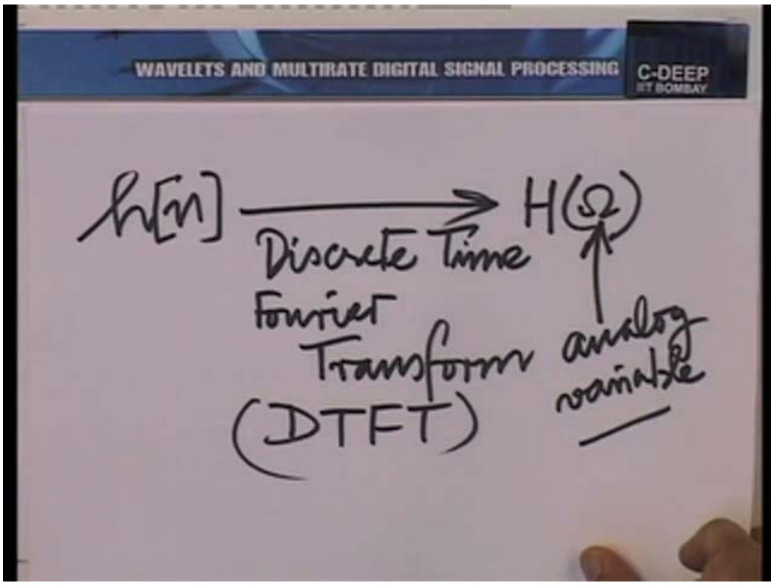

So, we would have this is equal to summation n going from minus to plus infinity h of n integral from minus to plus infinity phi 2 t minus n e raise the power minus j omega tdt. So, we isolated the part that operates with dt here. Let us evaluate that part separately.

Put 2 t minus N equal to lambda where upon we have t is equal to lambda plus n by 2 and of course, one can also write down dtdt is essentially half d lambda and substituting this, we have the integral becomes integral from minus to plus infinity phi of lambda e raise the power minus j j omega lambda plus n by 2 d lambda and a half outside. We can do a little more work on this.

So, we keep the terms dependent on lambda inside and we have j lambda rather j capital omega here, n by 2 emerging outside this is j capital omega n by 2 and this is minus infinity to plus infinity phi lambda e raise the power minus j omega by 2 lamda d lambda and this is familiar. This is essentially phi cap evaluated at omega by 2 as one can say the Fourier transform evaluated at the point capital omega by 2. So, now, we have a very beautiful relationship. You see what we are saying in effect now is that we can express the Fourier transform phi cap omega in terms of itself which is not surprising because you have a recursive dilation equation on phi t.

So, there is a corresponding dilation equation on the Fourier transform. What is that dilation equation? That dilation equation is phi cap omega is summation n going from minus to plus infinity h n times half e raise the power minus j omega by 2n times phi cap omega by 2. Now, you know this part of the summation that involves n is familiar to us again. Indeed, we note that summation n going from minus to plus infinity hn e raise the power minus j omega n would be essentially the DTFT, the discrete time Fourier transform of h evaluated at capital omega.

So, all that we have done in this expression is that we have replaced capital omega by capital omega by 2 from this point and we will give it, we will again use the notation that we have been using. So, we are saying if h of n has the discrete time fourier transform or DTFT given by capital H of omega. Now, note here I am using the continuous or analog variable that is because I want to retain my discussion in the analog domain or in the continuous time domain.

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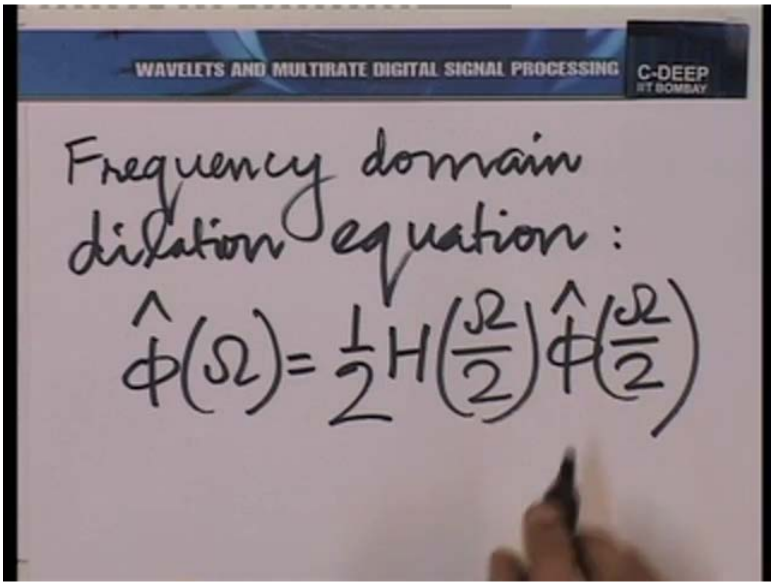

So, I am substituting small omega by capital omega here for the sake of consistency in notation and if hn has the discrete time fourier transform given by capital H of capital omega, then what we have here is the following dilation equation. The frequency domain dilation equation is phi cap capital omega is half capital H evaluated at omega by 2 times phi cap evaluated at capital omega by 2.

You see the beauty is that a dilation equation which involved summation over minute term has now become a dilation equation involving of a simple product. Then how do we interpret this? The Fourier transform of phi t is the same Fourier transform evaluated at omega by 2. So, evaluated omega is equal to evaluated omega by 2 times the DTFT.

Now, the beauty is that what we have done here to go from phi cap omega to phi cap omega by 2 can be done to go one step lower. So, the same equation can be re-written at capital omega replaced by capital omega by 2 and doing that, we would have phi cap evaluated at capital omega by 2 is half H evaluated at omega by 4 times phi cap evaluated at omega by 4.

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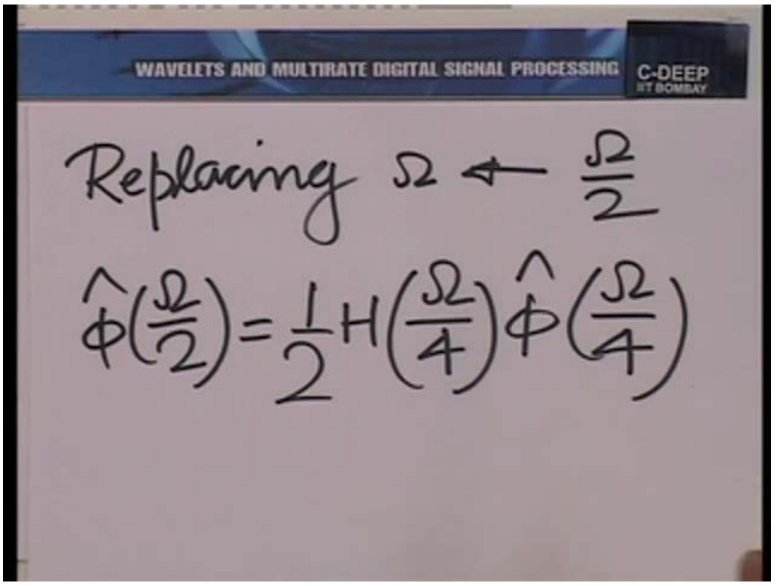

So, now we have a recursive process. Every time, you have phi cap omega by 2, you replace it in terms of a product of phi cap omega by 4 and then a DTFT. So, ultimately we have something like this. We have phi cap omega is like a product, it is a product m going from 1 to n capital N if you like, half H omega by 2 raise the power of m, this product and then multiplied by phi cap omega by 2 raise the power capital N.

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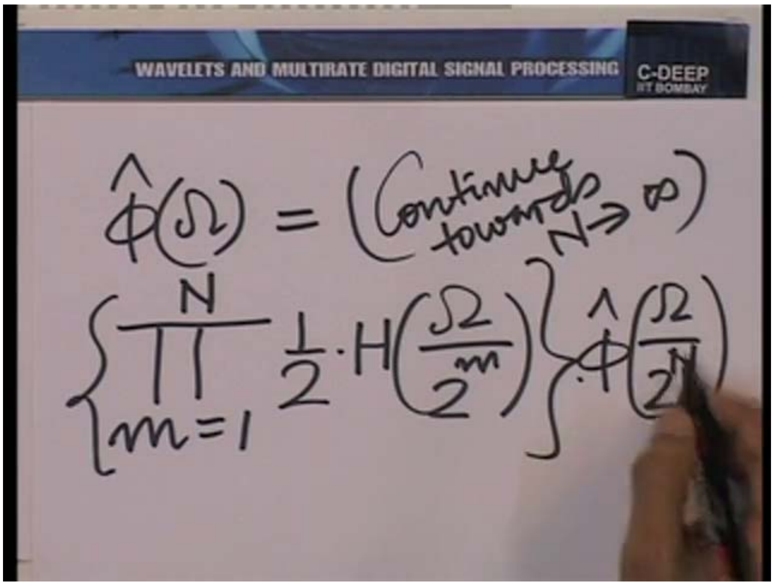

So, we have a product of these discrete, so called Discrete Time Fourier Transforms. Here, the only catch is, now we need to use the analog frequency variable because we are dealing with analog frequencies here and here. Now, we can take the limit or continue towards N going towards infinity.

Now, what is going to happen when you make capital N go towards positive infinity here? Any finite capital omega is going to be taken closer and closer and closer to 0. Again, if you wish to be very finicky, you should use the opponent proponent model where you say no matter how small I ask this argument to be, I can make it small enough and so on but I think we understand well enough that you can make capital N as large as you desire and you get a larger and larger number of terms in this product. You can take this argument to as small a value as your desire whenever capital omega is finite.

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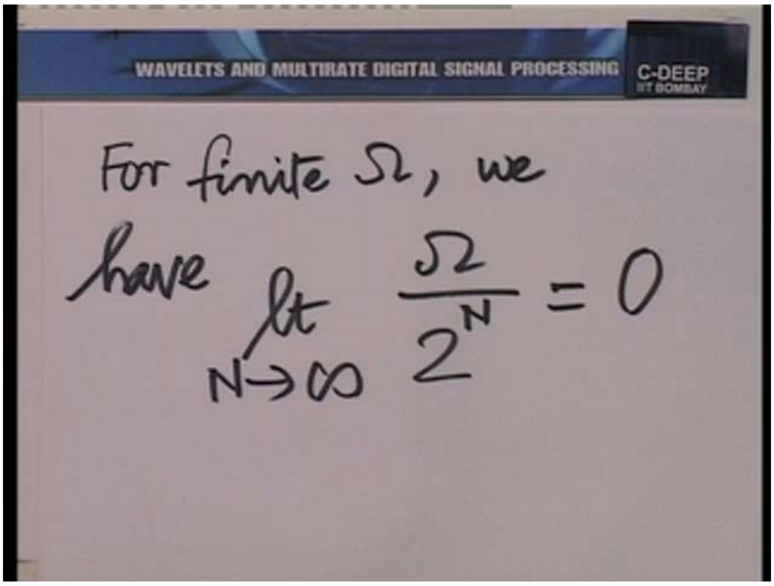

For finite capital omega, we have the limit as capital N tends to positive infinity of capital omega divided by 2 raise the power of N equal to 0. So, therefore at least on the finite frequency axis, the left hand side is equal to the right hand side well and the right hand side has essentially the fourier transform of the left hand side at the point 0.

So, what do we have here? Let me write that down mathematically phi cap omega. Therefore, it is essentially a product m going from 1 to infinity, positive infinity. Remember, the half occurs with each of these terms in the product.

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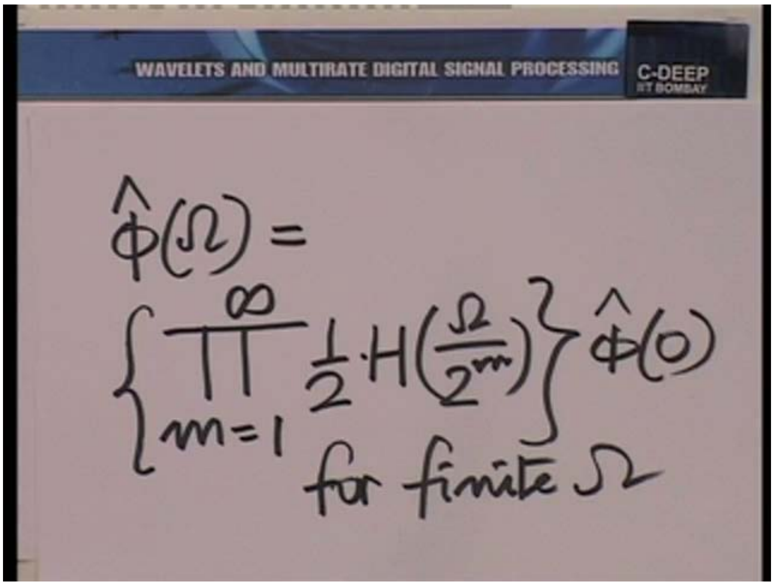

Now, we have to be careful and say, for finite capital omega but that is not a very serious problem. You see if you look at the Fourier transform of the Haar scaling function for example, let us look at it. Let us look at the fourier transform of phi t in the Haar context.

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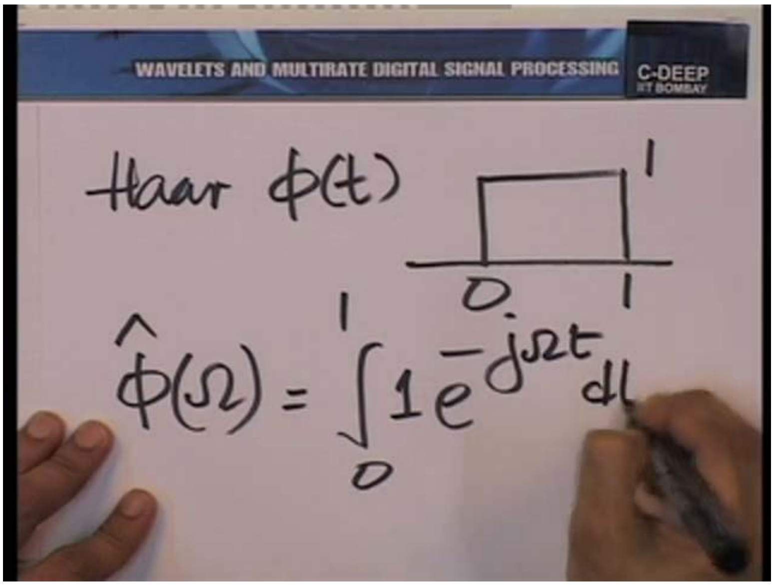

Essentially, it is this and the Fourier transform is easy to calculate. Now, we can simplify this using the standard trick of taking out e raise the power minus j omega by 2 term and then we note that we have a sine hidden there and doing away with the j's we will have ((Audio not available: 24:16-24:42)).

Now, as you can see this Fourier transform goes towards 0 as capital omega goes towards infinity. So, the Fourier transform vanishes as capital omega goes towards infinity and in fact, we can even sketch the magnitude to get a feel of this.

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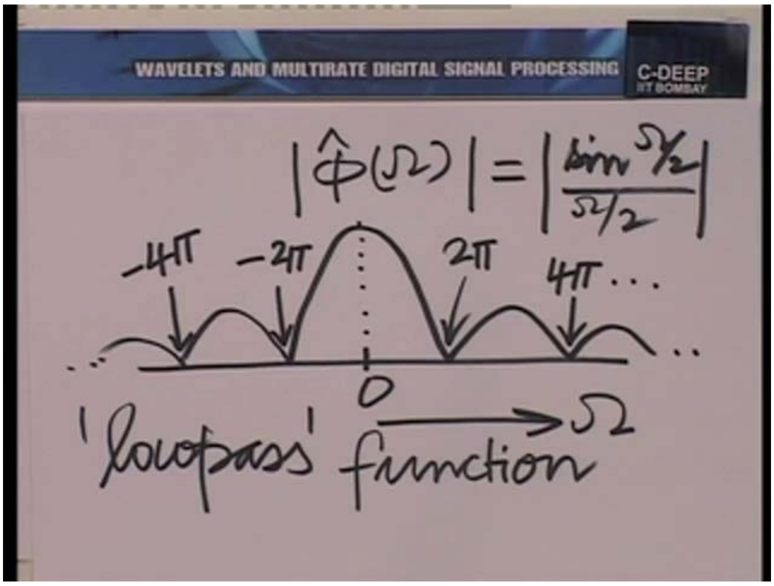

The magnitude of phi cap omega looks like this. You see at omega equal to 0, you will notice that one can use Lapthal's rule and show that this is equal to 1 in magnitude. So, it has a pattern like this. Of course, we know where this comes? This will come at omega by2 equal to phi or at 2 phi. This would come at 4 phi and so on. This is how the magnitude looks.

Now, you know this very clearly shows that phi has a concentration around 0 frequency. So, this is interesting. We begin from the low pass filter, we construct a dilation equation, a recursive dilation equation starting with the low pass filter and we get a low pass function. Phi t is essentially a low pass function.

A low pass function means it is predominant in the frequency domain around capital omega equal to 0. Low pass function of course, is an informal term. You may always argue with that after all, it does have bands at higher frequencies too but the point is its prominent bands are around 0 and the farther you go away from 0, the more the spectral magnitude drops off. In that sense it is low pass.

In fact, just as we try to build the idea of ideality in a filter bank, we also would like to build the idea of ideality in this phi t. The ideal phi t is actually the ideal low pass function. So, low pass function which is like a brick wall.

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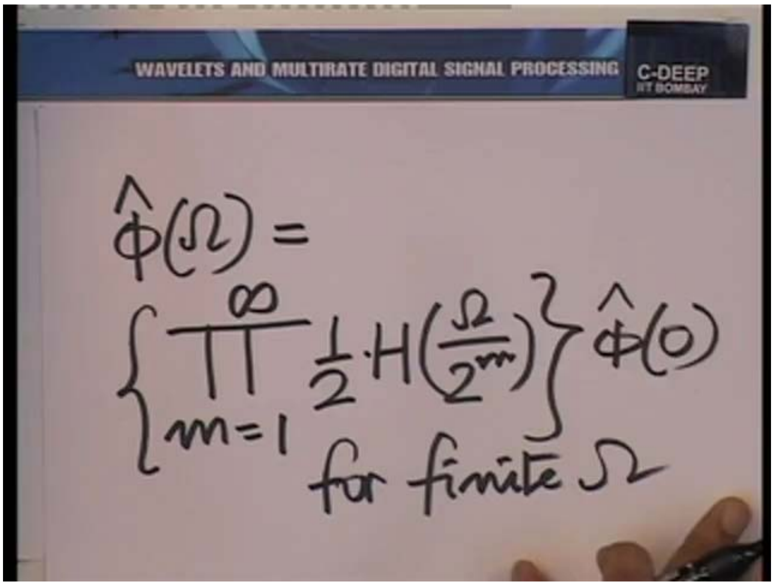

So, in some sense phi t in the spectral domain or phi cap omega for that matter is moving towards the ideal. You know this is where we are moving and where we are is somewhere here, very far away from it of course. Not surprising after all that is what we saw in the Haar multi-resolution analysis too, ideal here and actual there.

Now, once again there is the same conflict that drives the engineer, the scientist or the mathematician. We know that we want to go towards the brick wall ideal but we also know the brick wall ideal is unattainable for various reasons. The reasons are similar to what I talked about last time for the unattainably of the idealism in a filter bank.

So, you would not need to repeat them once again here. However, what we will now do is to see, what is the relationship? Whether, ideal or practical. What is the relationship between phi cap omega and the discrete time Fourier transform of the filter bank? Low pass filter in terms of construction.

So, in another words we have written down a dilation equation in the frequency domain but we need to translate that dilation equation into a constructive step. How do we construct phi t given the low pass filter impulse response? In a way, if we do that we have answered the question. How is the design of the filter bank related to the design of the multi resolution analysis?

So, let us do that. Towards our objective, let me put before you once again that infinite product here. So, you know now, you also understand why phi cap 0 should not be 0, phi cap 0 must be non 0. In fact, phi cap 0 is very often the maximum value of the magnitude of phi cap omega because of that low pass character.

So, we have seen this low pass character in the Haar context. Now, we shall assume it to be true of most multi-resolution analysis and proceed from there. So, this is just a constant, you know a non 0 constant to be vary but what we need to identify is this.

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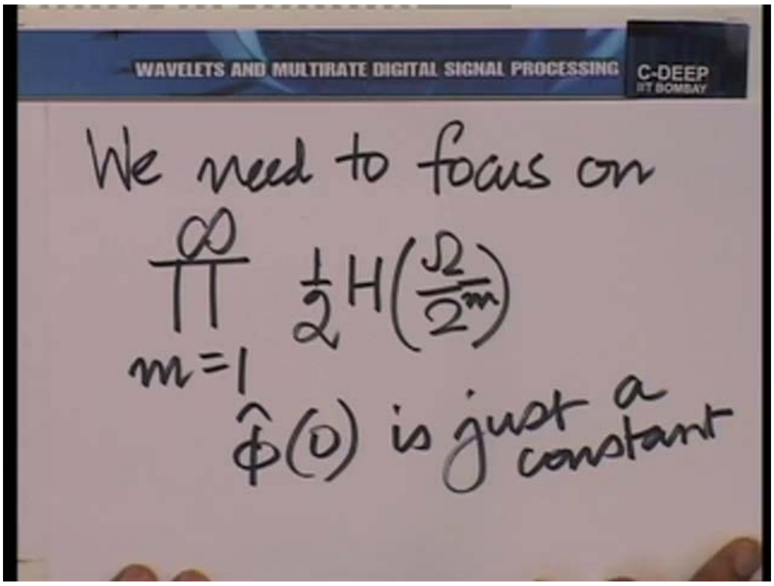

So, we need to focus on this infinite product here, phi cap 0 is just a constant. So, let us take just two terms in this product instead of infinite terms. In fact, you know now we need to interpret this continuous analog variable little more carefully here. When we bring in the idea of a continuous analog frequency variable here, then we need to remember that we are taking a Fourier transform of a continuous function.

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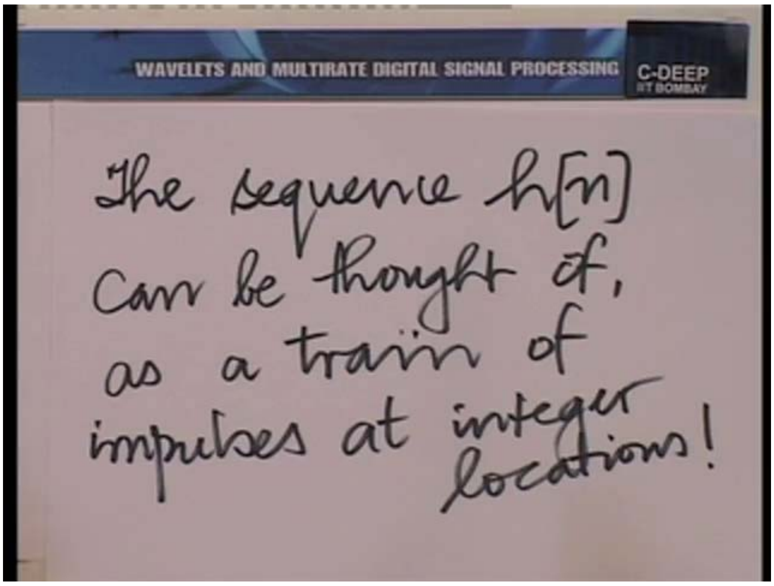

Now, what is the idea of the discrete time Fourier transform which is of course, of a sequence becoming the Fourier transform of a continuous function. Well, that is simple. So, suppose you thought of the sequence that train of impulses located at the integers. So, the sequence h n can be thought of as a train of impulses at the integer locations and the train of impulses, therefore is of course a continuous function.

So, you can take its Fourier transform and use the continuous analog frequency variable. Now, one must interpret capital H of capital omega in that sense.

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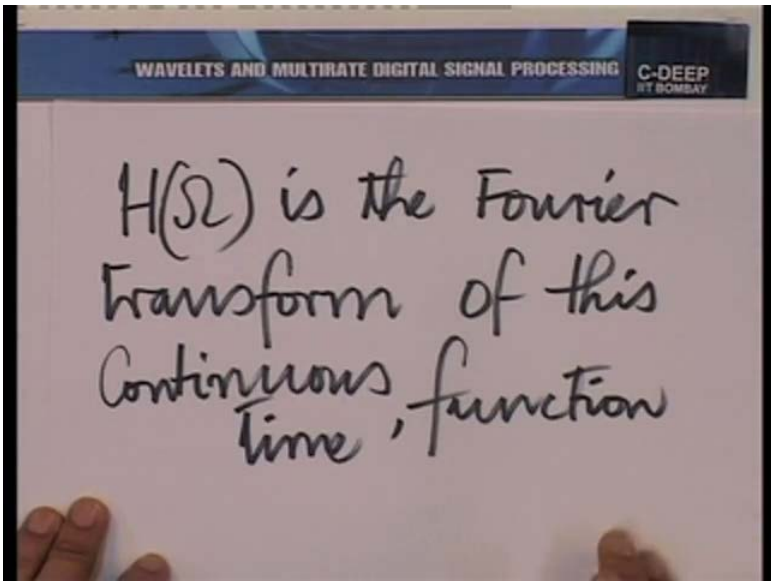

So, capital H of capital omega is the Fourier transform of this analog function or a continuous variable function. Maybe, I should say continuous time function to be precise and now, what is half h omega by 2. Then for that purpose, let us assume that we have a function h of t whose Fourier transform is capital H of omega.

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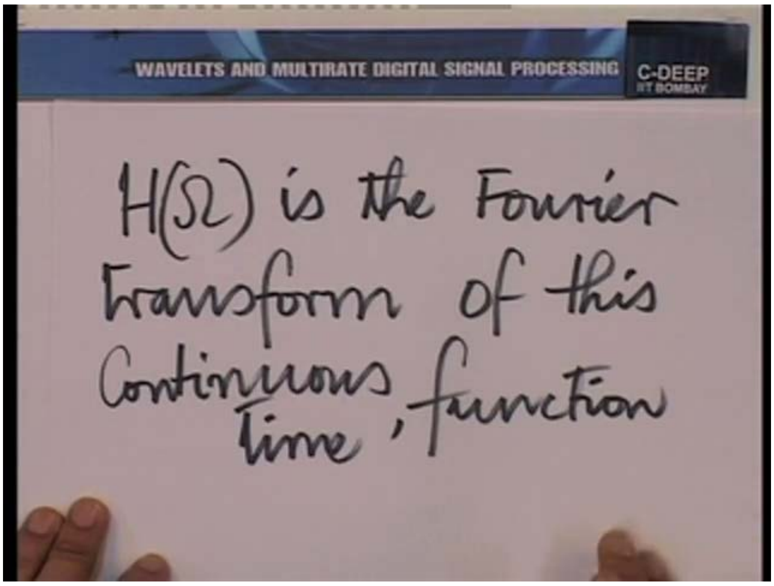

Of course, we know capital H omega is integral from minus to plus infinity h t e raise the power minus j omega tdt and if we happen to consider alpha times h omega by alpha with positive alpha. So, what I am saying is consider alpha times h alpha times omega with alpha positive.

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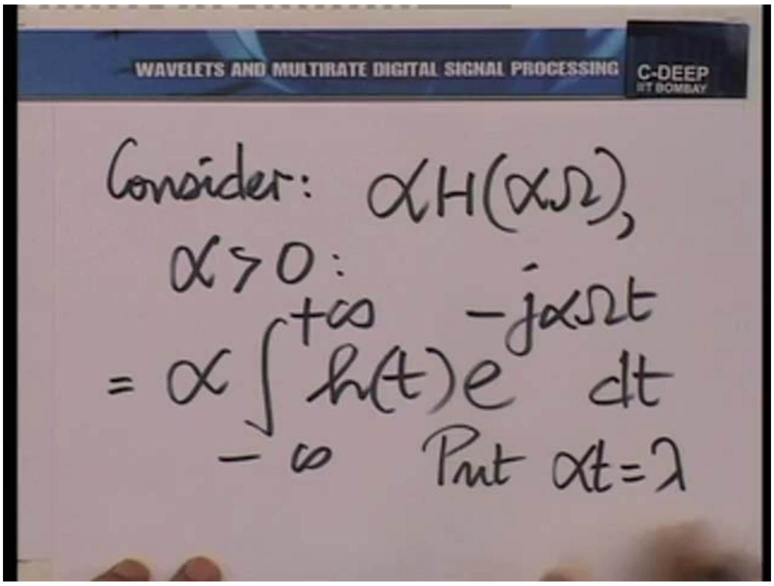

It is equal to alpha times integral from minus to plus infinity h t e raise the power minus j alpha omega t d t. Now, we have a simple step that we can perform. If we simply put alpha t equal to lambda and we would get, you see alpha t equal to lambda, alpha is strictly positive. So, when t runs overall from minus to plus infinity, lambda also runs from minus to plus infinity.

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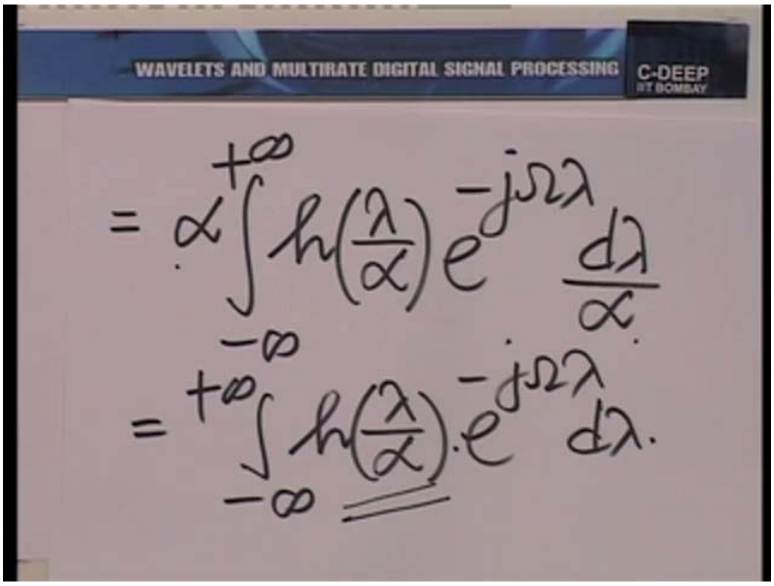

So, therefore we would have this is equal to alpha times integral minus to plus infinity h lambda by alpha making the substitution e raise the power minus j omega lambda.

Now, d t is d lambda by alpha and now, if we just cancel the alpha here and the alpha here, we get integral from minus to plus infinity h lambda by alpha e raise the power minus j omega lambda d lambda which is essentially the fourier transform of h of lambda by alpha as the argument. So, we have divided the argument by the positive number alpha.

So, what we are saying in effect is if h t has the fourier transform h of omega, then ht by alpha has the fourier transform alpha times h of alpha omega where alpha is of course, strictly greater than 0 here. Now, of course one can generalize this for alpha real and negative. All that one needs to do is to take a modulus outside and no modulus inside but I leave that as an exercise for you.

We do not immediately require it. One can easily generalize but coming back to the point, then what h omega essentially means? What h omega by 2 essentially means with a factor of half outside is a dilated version of the train of impulses.

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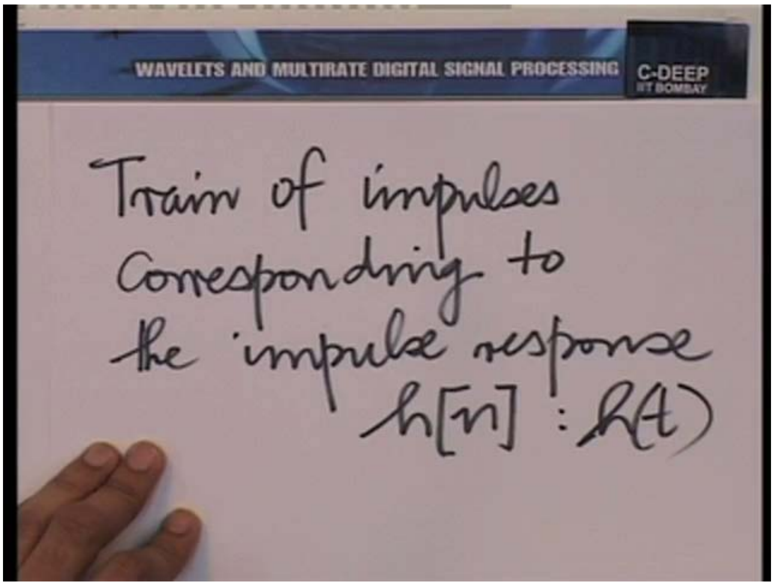

So, we have this train of impulses corresponding to the impulse response h of n which we have called h of the continuous variable t.

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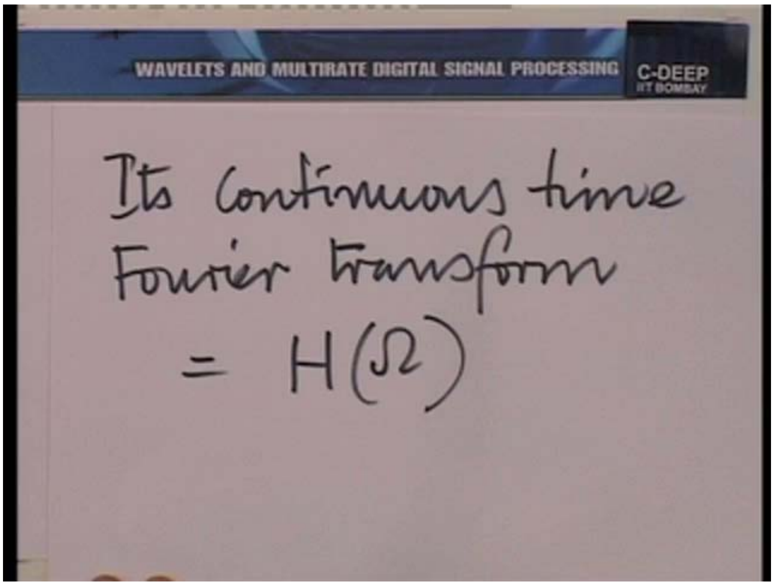

It is a continuous time fourier transform or analog fourier transforms. So, to speak is capital H of capital omega and then half capital H of capital omega by 2 is then the continuous fourier transform of h of 2 t. That is easy to see because you have chosen alpha equal to half there.

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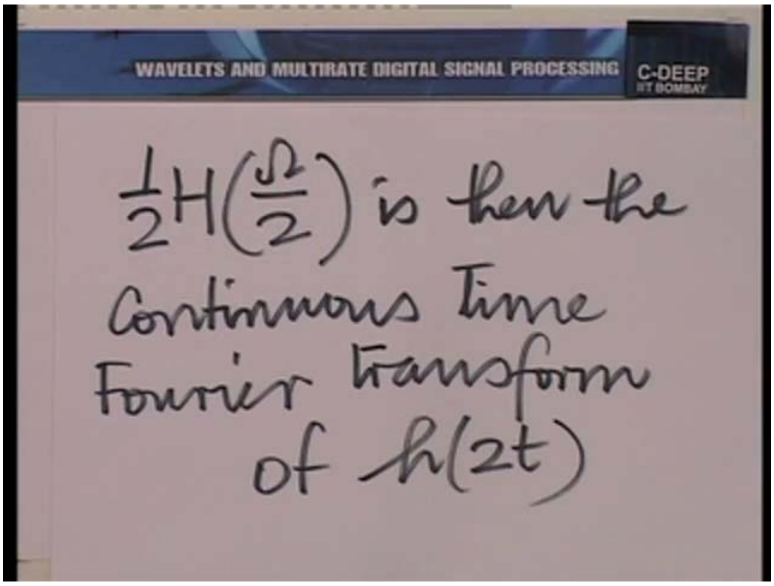

What you mean by h of 2t? h of 2t means you have squeezed ht by a factor of 2 on the time axis on the independent variable. So, you have brought the impulses closer. Now, when you multiply 2 fourier transforms, the corresponding continuous functions are convolved.

So, essentially you may think of h of t here. So, to speak for the Haar case, h of t looks like this. There is an impulse at 0 and a impulse, a continuous impulse at 1. These are impulses as understood continuous time and h of 2t will look like this. You know if I really wish to be finicky, I should be putting down the strengths of the impulses carefully to but let us not get that finicky. This is what h of 2 t will look like.

There are impulses 0 and half H of 4t for example, will look like this. Now, this one squeezed again by a factor of 2 that we an impulse at 0 and an impulse at 1 by 4 and so on and so forth. Of course, the rest of it is 0 just 2 impulses. So, what do we have now? We have a product. Let us just take two terms in that product.

So, if you take just the first two terms half capital H capital omega by 2 times half capital H capital omega by 4, it corresponds to h of 2 t convolved with, this is continuous time convolution here. Convolve with h of 4t.

Now, as I said I am being a little careless about constants but if you really wish to be finicky you can. I am more interested in getting a feel of the shape of the convolution. I am not so concerned about the precise heights and so on right. Anyway, let me convolve them and show you. So, let us put back h of 2 t and h of 4 t as we add them here. So, we had h of 2 t here.

Essentially, 2 impulses located at 0 and half, we have h of 4 t with 2 impulses located at 0 and one fourth. Now, what will happen when you convolve these? You know when you convolve a continuous time function with an impulse; a unit impulse gives you back the same continuous time function.

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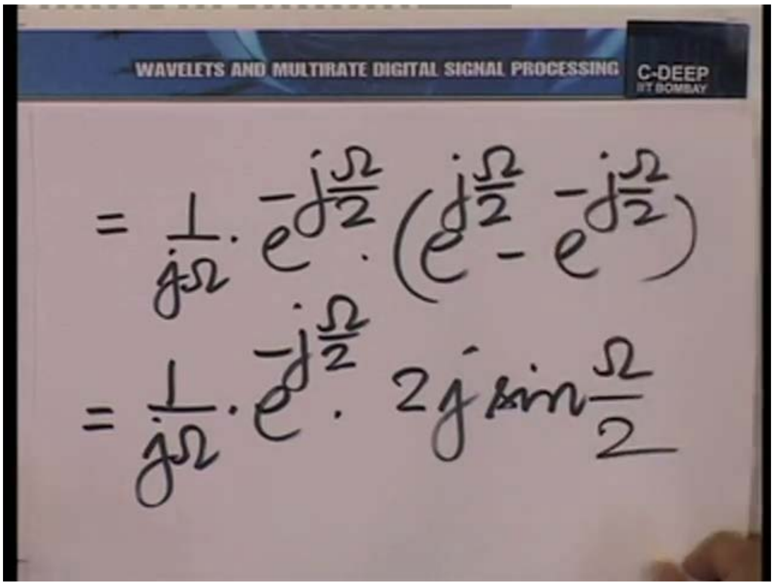

So, as I said if you ignore the high note, the heights are equal here. Then when you convolve this h of 4 t with this, you could treat it as the convolution of this, with this impulse plus the convolution of this with just this impulse and you could sum these two independent convolutions.

When you convolve this with this impulse, you simply relocate this at the position 0 and in fact, that gives you back h of 4 t. When you convolve h of 4 t with this impulse located at half, it simply shifts this function to lie at half. So, in effect when you have h of 2 t convolve with h of 4 t, you get something like this.

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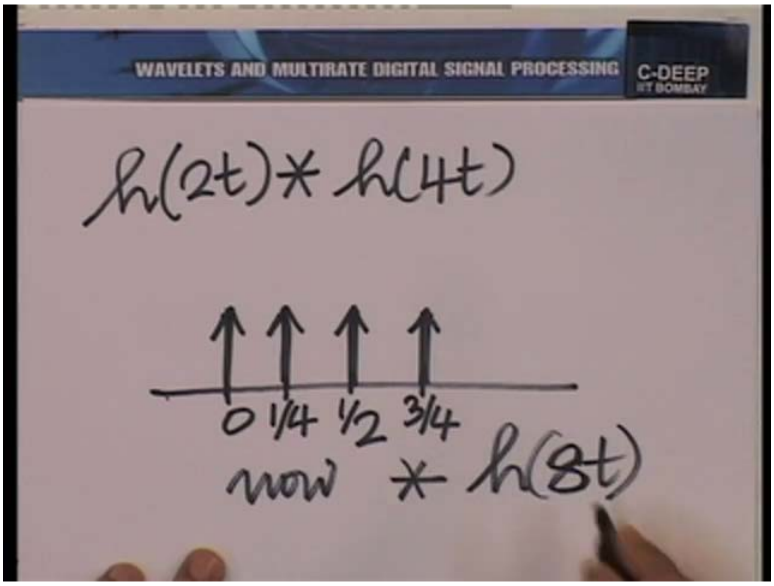

You get an impulse located at 0, 1 at one fourth, 1 at half at 1 at half plus one fourth which is 3 by 4. So, you get impulses here. Now, convolve this again to take the next term with h of 8 t as that infinite product asks you to do. Say, if you take three terms, then you would be convolving this with h of 8 t.

How will h of 8 t look? h of 8 t looks like this, become even closer together 0 and 1 by 8. I am convolving h 2 t convolved with h 4 t and then the whole convolved with h 8 t. What will you have? Essentially, this has to be located here, here, here and here and all these relocated h of 8 t should be added together. Now, when you locate h of 8 t here, you get an impulse at 0 and 1 by 8 here in the middle. When you relocate h of 8 t here, you will get an impulse here and at one fourth plus one eighth. That is one eighth plus one eighth, three eighth. So, let me straight away now draw. This convolution results in impulse at each of these places. ((Audio not available: 46:32-47:20))

Now, you know we seem to be getting where we want to. What is happening if you think about it? Each time you bring in one more term, you are getting a train of impulses where the train has doubled the size but it lies at the same support. H of t lay on the support 0 to 1, H of 2 t lies on the support 0 to half.

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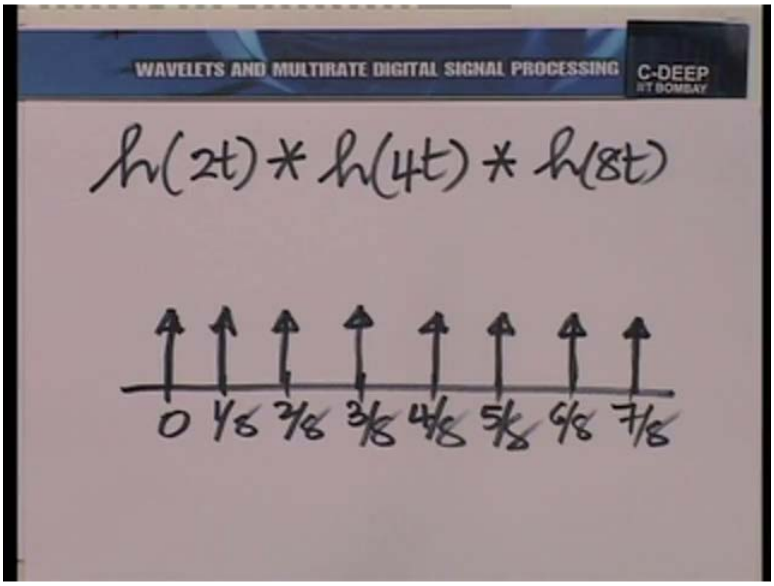

Of course, I would not really say 0 to half, you know there is an impulse at 0 and an impulse at half but then when you go to h 2 t convolve with h 4 t, you get an impulse at 0 at one fourth at two fourth and at three fourth. When you go and bring in one more term, you get 8 impulses. When you bring in one more term next time, you are going to get 16 impulses and then 32 impulses and the last impulse comes to closer and closer to 1.

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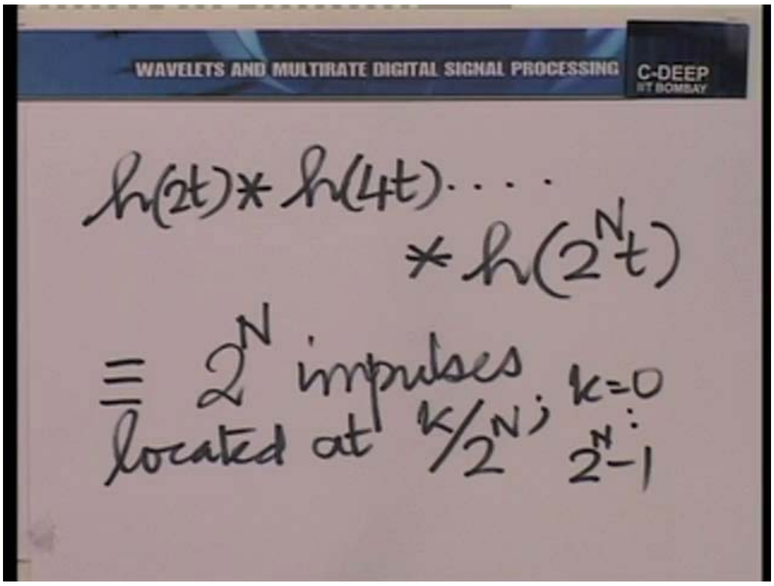

So, what we have here effectively is h 2 t convolve with h 4 t and so on and so forth up to h 2 raise the power of N t is essentially, how many impulses? You see which you reach h of 8 t you have 8 impulses. So, when you reach h 2 raise the power of N t, you have 2 raise the power N impulses located at k divided by 2 raise the power of N, k going from 0 to 2 raise the power N minus 1.

So, you know the last impulse as you can see the last impulse is located at 2 raise the power of N minus 1 divided by 2 raise the power of n. So, last impulse goes closer and closer and closer to 1.

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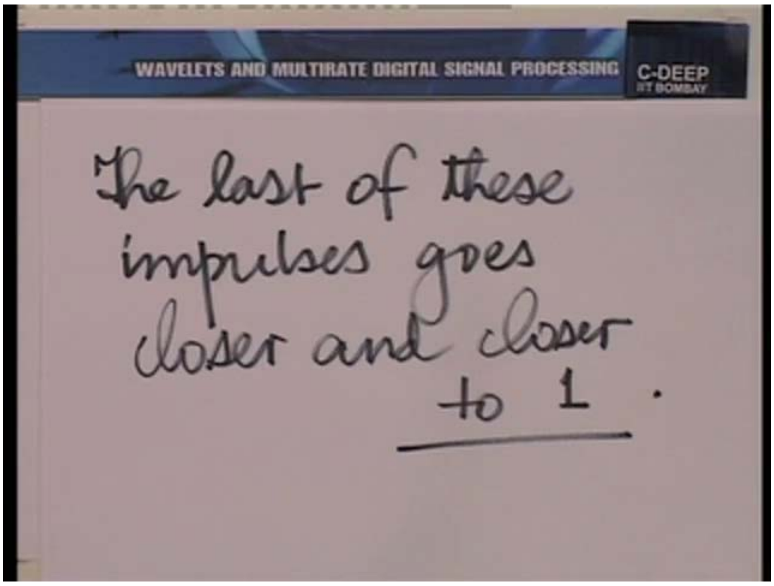

The last of these impulses goes closer and closer to 1. So, you know when you have impulses located closer and closer and closer together, you are ultimately coming to a continuous function. You remember that idea of expressing a continuous function in terms of impulses essentially captures this. When you say x of t is a conglomeration of impulses located every point t with strength equal to the value of x at a point t. That is exactly what you are saying.

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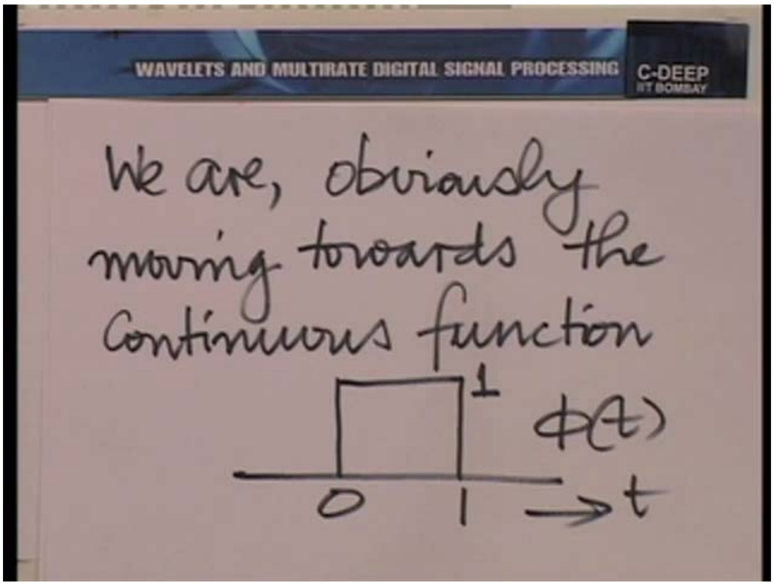

When you bring impulses closer and closer together, they fuse together to form a continuous function and it is very easy to see here what continuous function we are moving towards it is flat and indeed, it is very clear that we are moving towards ((Audio not available: 50:49-51:20)) which is essentially phi t low and be hold a very beautiful relationship. We have started from the Haar low pass filter. We have repeatedly convolved a train of impulses.

So, for a first time it is a train of impulses located at 0 and half and then 0 and one fourth and you have repeatedly convolved these iterated the filter bank. Repeatedly convolved these strange of impulses and you moving towards the continuous time function which is indeed, phi t as you can see when you put those impulses closer and closer and closer together.

So, now we see the connection between iterating the filter bank and producing phi. We now need to complete a little detail. How do we get psi but that is very easy. We already got phi and we know the dilation equation for psi.

So, we have phi now. How will psi look psi t is essentially summation n going from minus to plus infinity gn phi of 2 t minus n and for the Haar case, we know what gn is g n is essentially 1 and minus 1. So, we can write down psi t in terms of phi t and construct from there, phi 2 t minus phi 2 t minus 1. This is phi 2 t this is minus phi 2 t minus 1 when we put these two together, we get psi t.

There we have completed this iteration and building phi t and psi t starting form h n and g n. Now, we have a convincing reason to conclude that there is an intimate relationship between the low pass filter and the high pass filter in the two band filter bank and the scaling function and the wavelet function in the multi-resolution analysis.

In fact, we have constructively established that relation. We have shown a procedure by which we can construct phi t and psi t from these impulse responses and therefore, we are now convinced that if we understand how to design two band filter banks and if this iteration is going to converge each time we design a properly designed two band filter bank which allows this iteration; we get a new multi-resolution analysis.

With that back ground, we shall conclude the lecture today and proceed in the next lecture. Therefore, to explore the two band filter bank more deeply.

Thank you.