Advanced Digital Signal Processing – Wavelets and Multirate

Prof. V.M. Gadre
Department of Electrical Engineering
Indian Institute of Technology, Bombay

Lecture - 05
Equivalence: Functions and Sequences

A warm welcome to the fifth lecture, on the subject of wavelets and multirate digital signal processing. To put the discussion of the current lecture in perspective, let us recapitulate what we did in the previous lecture. In the previous lecture, we had looked at the equivalence between functions and vectors in depth. In fact, in that equivalence we saw that by bringing in the motions of the inner product between functions and of course, the inner product between the sequences and then noticing that Parseval's theorem is a version of a similar statement on the inner product.

We now, we understood that this analogy between functions and vectors, leaves as much to gain because it helps us picturize very well, it helps us visualize very well; what we were talking about, when we talked about the ladder of sub spaces. In general, in functional analysis; this is a very serious analogy. So, it is not just a stimuli, so to speak; it is a metaphor. We can actually think of functions as generalized vectors and gain a lot from that kind of analogy rather that kind of an equivalence or a generalization.

And, now we need to bring in another dimension to this discussion, which we had briefly begun in the previous lecture. The dimension of replacing work with functions, with work with sequences. Sequences are easier for us to deal with; in fact sequences can be dealt with using a computer. You could store the samples of a sequence in memory point after point; you could process them in discrete time step by step and produce an output which is again a sequence. And, low and behold if whatever you are doing with the sequence, maps exactly will what you wish to do with the original continuous time functions; then this is an added advantage.

In fact, this is true for the spaces V 0 contained in V 1, contained in V 2 and V minus 1 contained in V 0 and so on; as we saw briefly in the previous lecture, but which we shall now go into greater depth today. So, let us come back to that discussion; how could we think of a function in V 0 as a sequence, as an equivalent sequence? Well, very simple; essentially, what we are saying is look at the coefficients in the expansion of that function with respect to the basis of the space V 0. So, there you are.

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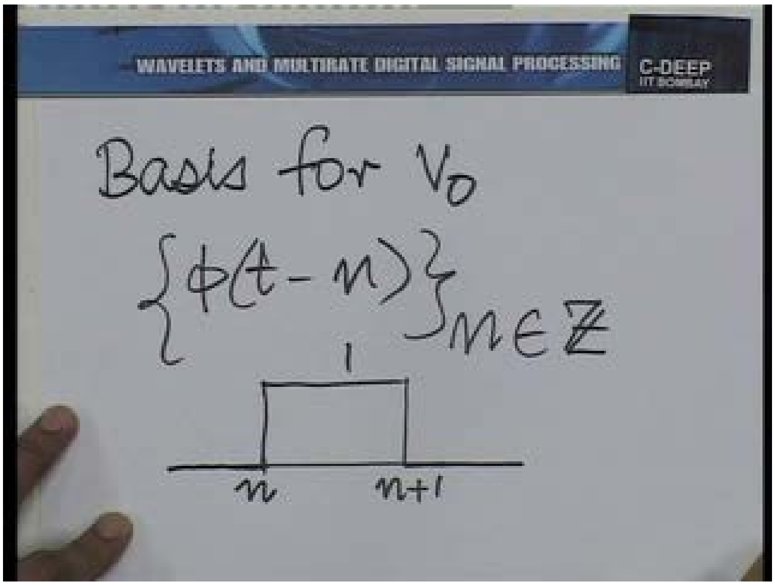

We have the following standard basis for V 0 given by phi (t minus n), n over all the integers. And, let us sketch a typical phi (t minus n), it is 1 between n and n plus 1 and 0 else; 0 everywhere else. This is how phi (t minus n) looks. Now, in fact this is also an orthonormal basis; I introduced that word, orthonormal basis.

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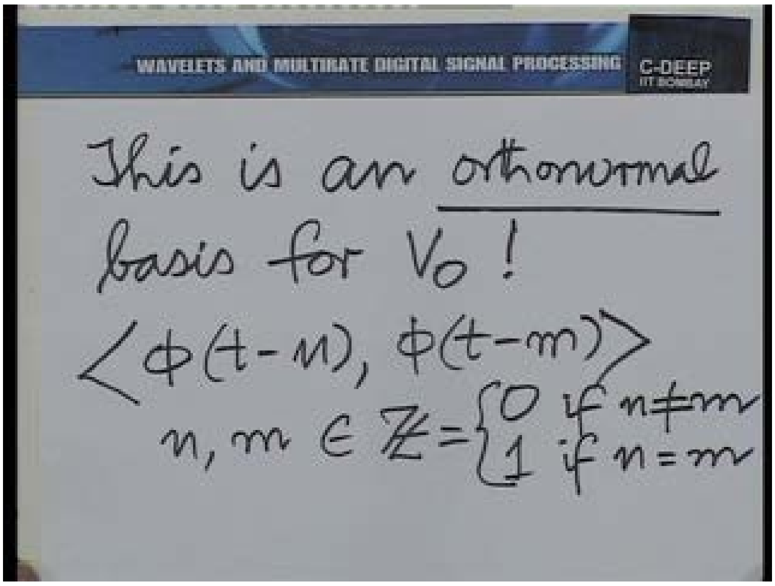

What do you mean by that? If I take the dot product of 2 of them, phi (t minus n) and phi (t minus m), for any 2 integers n and m; n and m belong to a set of integers. Then, this dot product is equal to 0, if n is not equal to m and 1 if n is equal to m; very easy to check. It does not required too much of working to prove this and I leave it as an exercise for you to show; all that you need to do is to calculate the integral in the product, after all phi (t minus n) and phi (t minus m) are in fact non-overlapping, when n is not equal to m. And, if n is equal to m, they overlap completely and then of course, the integral is just the integral of a rectangle of unit height over unit length; so, it is 1. Whatever it be, let us consider a function in V 0 again to fix our ideas of the connection between functions and sequences. So, let us take this example again little bit of repetition from the previous lecture but let us fix our ideas with it.

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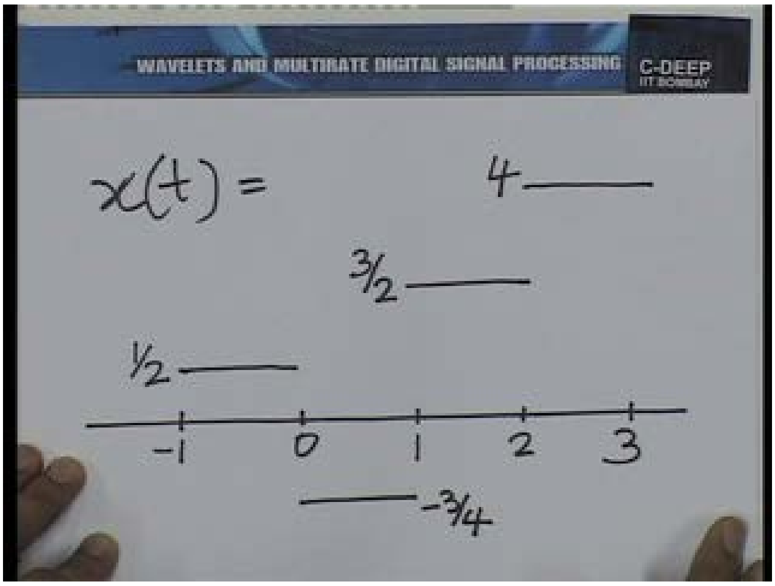

So, we have this function; let us say x (t) given by the following graphical representation. Let it take the value let us say half for some variety between minus 1 and 0, minus 3 by 4 in the region from 0 to 1, 3 by 2 in the region 1 to 2 and let us say 4 in the region 2 to 3.

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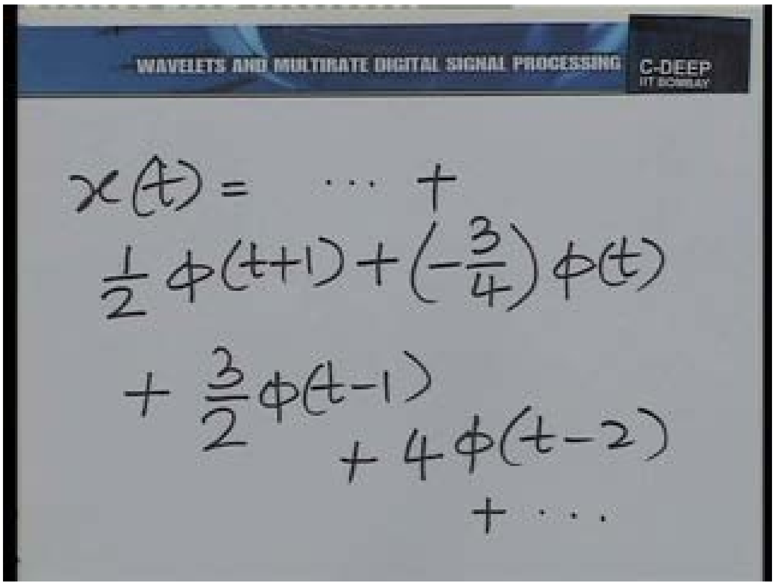

So, it is very easy to see that x (t) can be written as half phi (t plus 1) plus (minus 3 by 4) times phi (t) plus 3 by 2 times phi (t minus 1) plus 4 phi (t minus 2) and so on. So, you know you could continue this beyond 3 and you could continue it before minus 1. And, what we said in the previous lecture was that equivalent to this continuous function is the sequence constructed by dot, dot, dot and then half at minus 1, minus 3 by 4, at 0, 3 by 2 at 1, 4 at 2 and so on.

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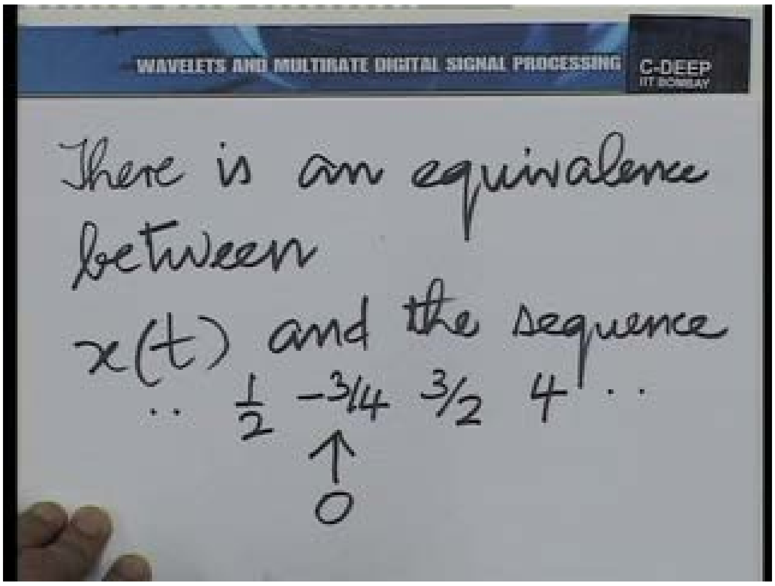

In another words, there is an equivalence between x (t) and the sequence. Now, again just to recapitulate the notation for a sequence we write the 0 point and mark the arrow 0 here; to show that this is the sample at n equal to 0 and then of course, you have 3 by 2 at 1, 4 at 2, half at minus 1 and so on. So, when we write a sequence in this notation, what we mean is that this is the sample at 0, n equal to 0 and then the other samples are arranged in the right order. So, for example, this is the sample at n equal to 1, the sample at n equal to 2, sample at n equal to minus 1 and so on; in the correct order around the sample that we had marked. Now, as an alternative we could have as well written the same sequence in the following way this is just to introduce notation properly.

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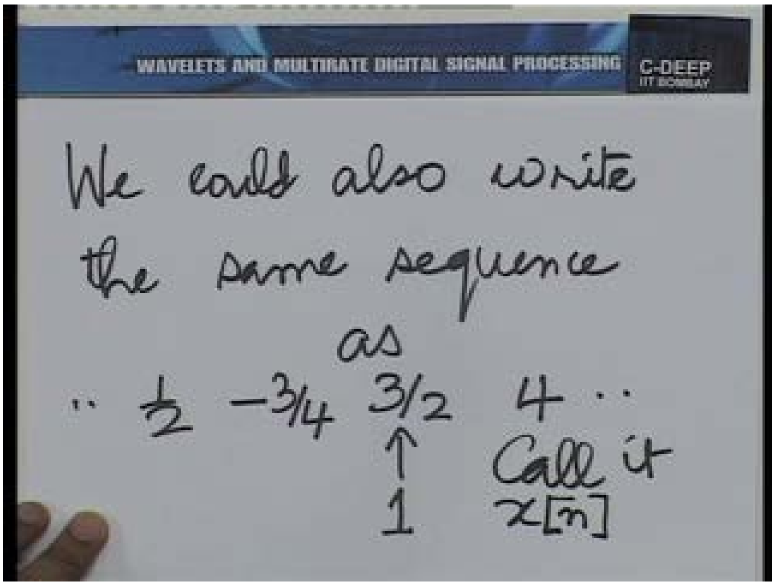

So, we could also have written. So, you could mark 1 for example, the sample at point number 1 and then the sample at point number 2 is 4, the sample at 0 is minus 3 by 4, the sample at minus 1 is half and so on. So, it is a same sequence written differently; just for some variety in notation, sometimes we might prefer to do something like this. Any way coming to the point and stressing it once again there is an equivalence; there is an equivalence between the function and the sequence. This function belongs to V 0 and the sequence, now belongs to what is called the sequence, the set of square of integrable sequences; so, let us call the sequence x [n].

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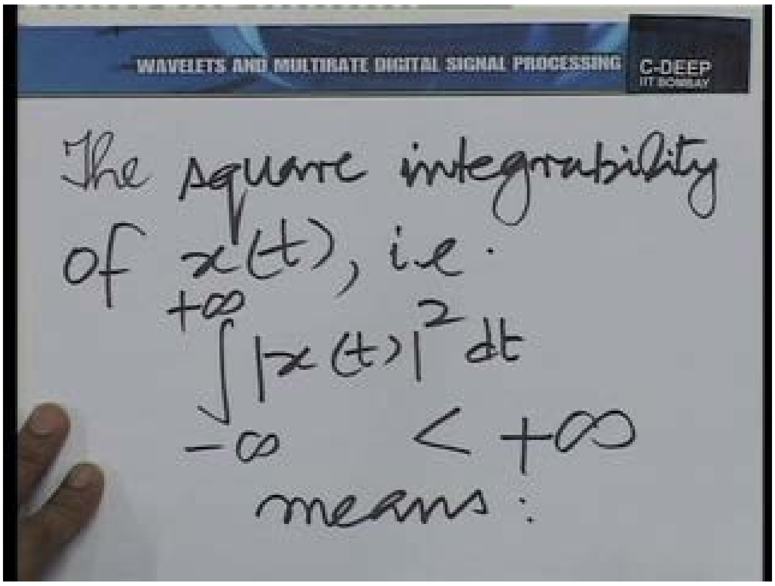

In that case, the square integrability of x (t). In other words, the fact that integral from minus to plus infinity mod x (t) squared dt is finite.

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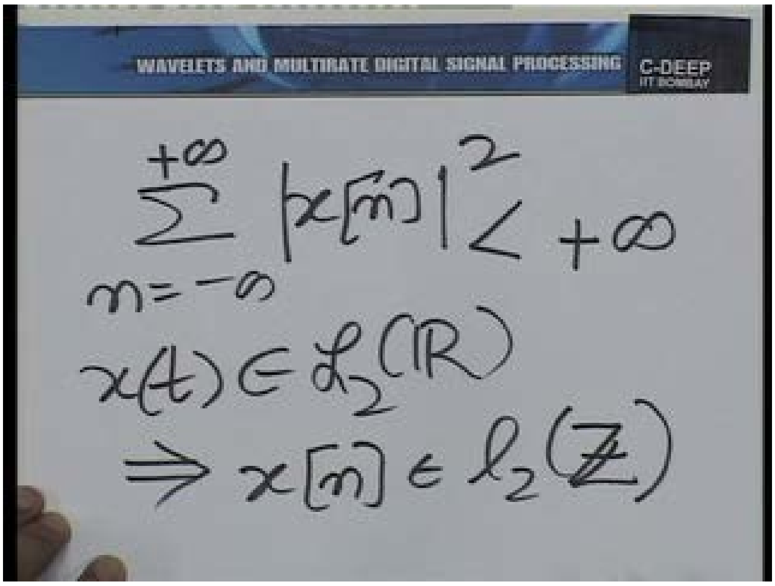

Means, summation from n going minus to plus infinity is also finite with the argument as mod x [n] squared, this is also finite. And, therefore, we use a term for this; we say x (t) belongs to L 2 R, implies x [n] belongs to small l 2 z, so, introduce this notation; this is a new term we have introduced, small l 2 z. So, small l 2, see just as you have capital L to denote spaces with continuous arguments, you have small l to denote spaces with discrete arguments. And, again we would like to define l p Z in general.

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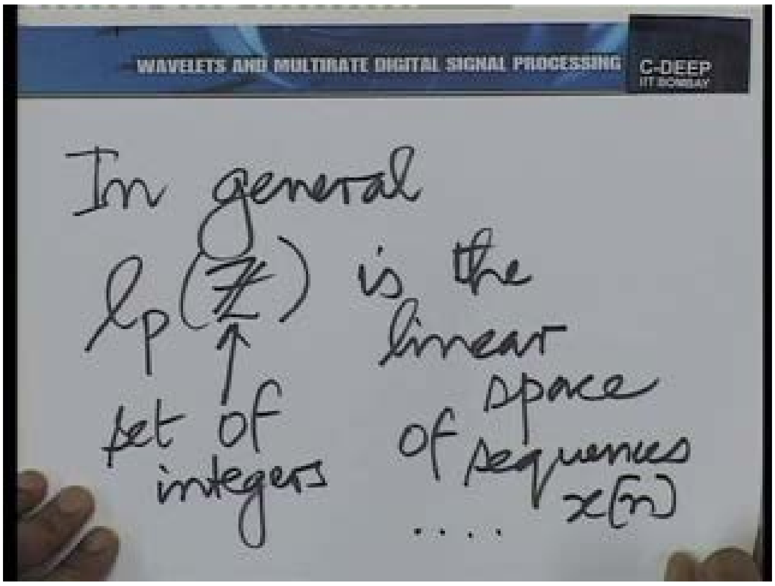

So, in general l p Z, this Z of course the first set of integers. So, l p Z is the set of sequences, in fact it is the linear space of sequences; let the sequences be written as x [n] such that this is continued.

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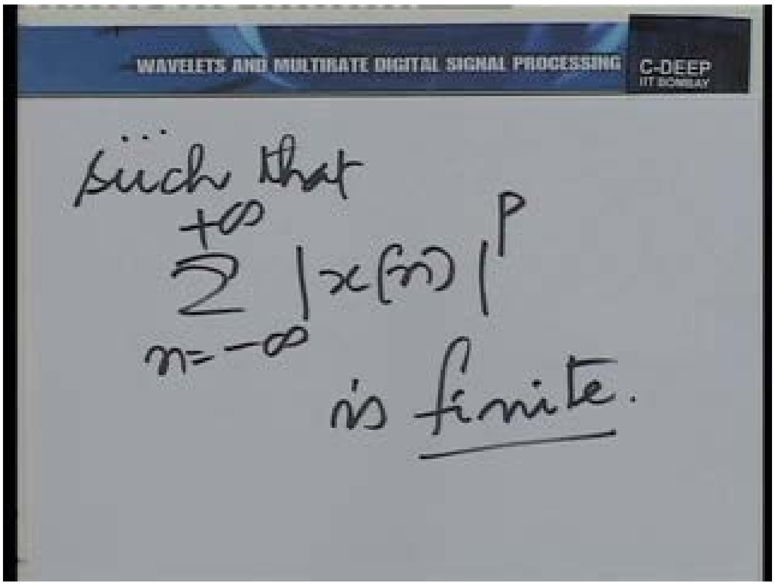

Such that summation n going from minus to plus infinity mod x [n] to the power p is finite. In particular, if you put p equal to 2, you get small l 2 z. Now, what we have just shown is that there is a correspondence.

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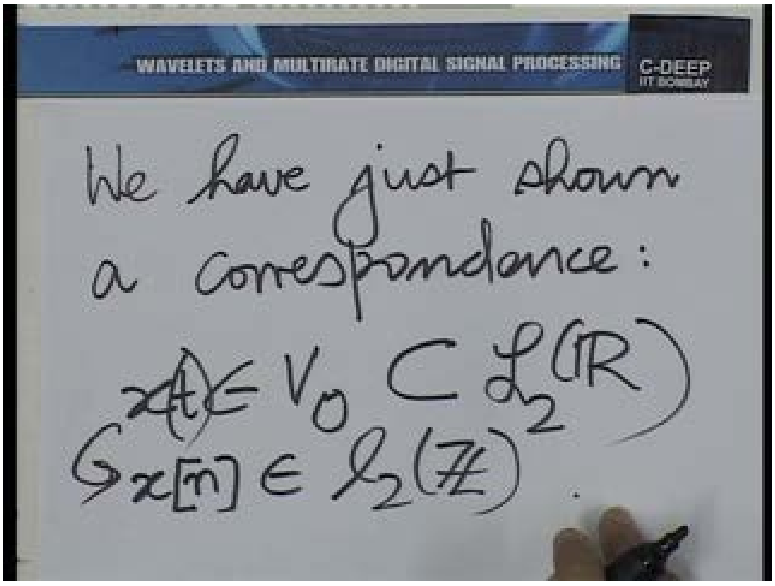

So, we have just shown a correspondence.

If x belongs to V 0 which in turn is a sub space of L 2 r, of course, here we are talking about a continuous time function, x (t); then we have the corresponding x [n] belonging to small l 2 z. Of course, we can make a similar inference for other values of p but that is not of consequence at the moment, so we shall go into it. What is important is, when we talk about inner products, now if you have an orthonormal basis. So, here for an example we have an orthonormal basis.

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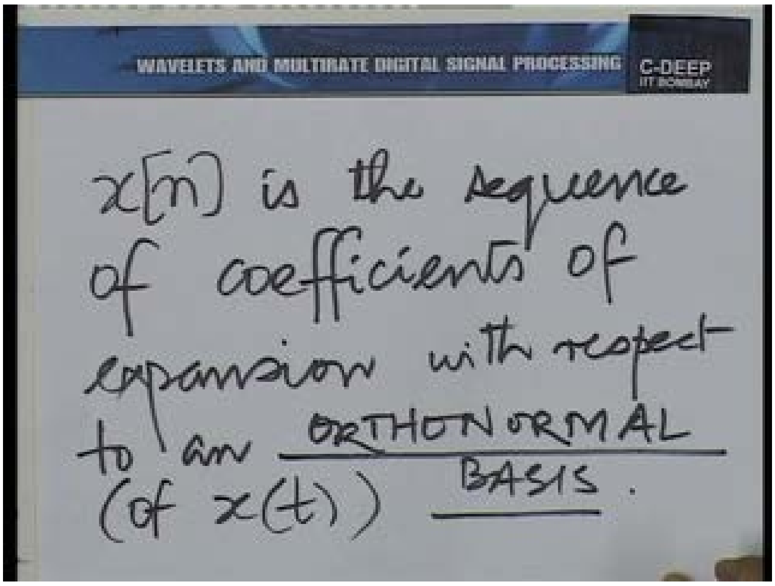

You see, so x of in fact here x [n] is the set of coefficients is the sequence of coefficients of expansion with respect to an orthonormal basis. Now, you know the orthonormal is important here; if the basis is not orthonormal what we are now going to say very soon is not going to be true. So, the orthonormal basis is important here. If x [n] is the sequence of coefficients of expansion with respect to an orthonormal basis of course, of x (t) I mean, then there is also mapping between the inner products; that is what is interesting. So, not only is there just an equivalence, there is also a mapping of the other operations. So, if you have two such function in V 0; suppose, you had x (t) and y (t) both belonging to V 0.

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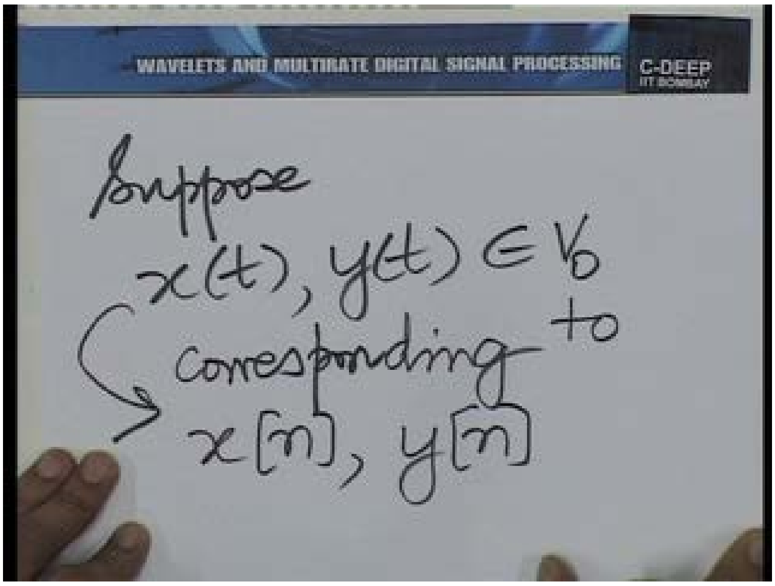

Both of them belong to V 0 which of course belong in turn to L 2 R. And, they correspond to the sequences x [n] and y [n]; square means square bracket. Then, the dot product of x (t) with y (t) in continuous time.

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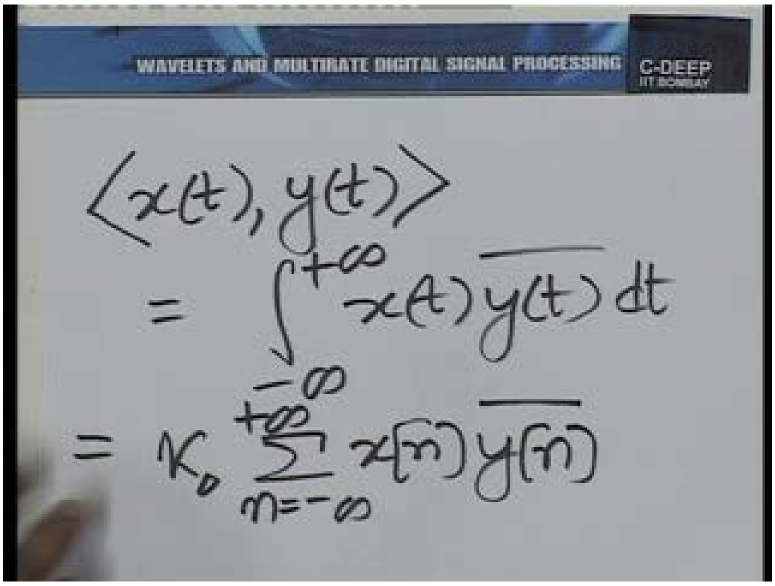

So, inner product of x (t) and y (t) is understood on the continuous time axis which is of course, given by summation minus to plus infinity integral x (t) y bar (t) dt is a multiple, is some constant times; summation n going from minus to plus infinity x [n] y [n] bar, this is important. There is also a carry onward of the equivalence to the domain of inner products. So, whatever we are doing in the context of the continuous functions can be done, can be equivalently done or can be equivalently derived or brought forward to the context of the sequences which are associated. So, we can go to the extent of forgetting about the underlined continuous functions and deal with the sequences directly.

Now, what is the physical or the practical meaning of this? Well, let us go back to the first lecture where we motivated the very idea of piecewise constant representation and we started with the 2 dimensional situation. We said let us take images and we said let us divide those images into very small areas which we represent using picture elements of pixels. On each of the pixels, we put down one number corresponding to the average intensity of the picture in that pixel area. And, we said that is reasonably representative of the image or the picture, if those areas are small.

Now of course, we are representing the original image by a piecewise constant function and equivalently we could now think of the image has been represented by a 2 dimensional sequence. So, a sequence index with 2 integer variables; let us say n 1 and n 2. n 1 going from 0 to 511and n 2 also going from 0 to 511; in case we have a 512 cross

512 picture representation or resolution on the computer screen, good. What we are saying now is easy to understand; whatever we want to do with the picture, we can equivalently do with these 2 2 dimensional sequence. If you have 2 pictures, you want to mix and match or do whatever you want to do; you could do it equivalently with these 2 dimensional sequences.

If you want to gain some inferences, you could gain the same inferences by looking at the same sequences, good. But where does this take us? Now, what is going to be useful to us is to see how we can move from one resolution to the next; that is what is of interest. You see, ultimately our movement in all this discussion is to extract incremental information and incremental information is extracted by going from one subspace of l 2 r to the next in the ladder. So, let it be the function belong to V 1.

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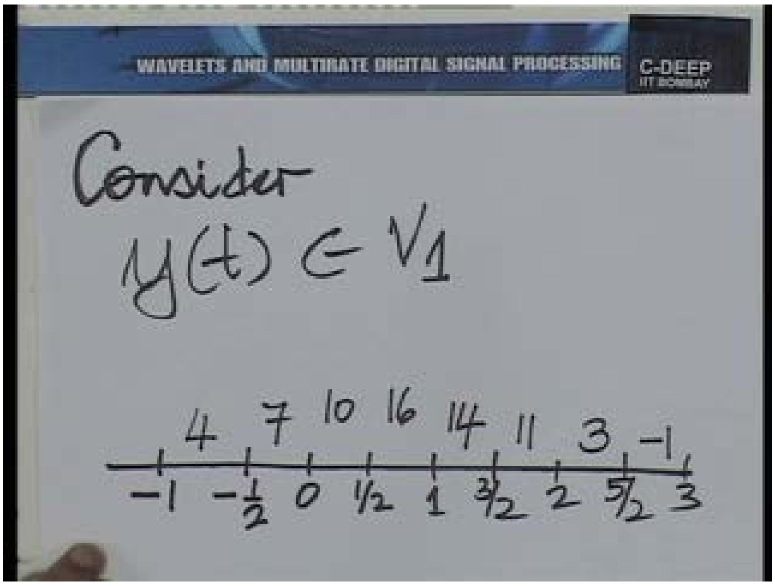

Consider a function; let us say y (t) belonging to V 1. Now, V 1 you will recapitulate is the space of functions piecewise constant on intervals of length half. So, let us take one such; let us take a function from minus 1 to let us say 3. So, minus 1, minus half, 0, half, 1, 3 by 2, 2, 5 by 2 and 3. So, let I will just write down, I will not sketch; I just write down values, the piecewise constant values in each of these intervals. So, in this interval the value can be let us say 4, in this interval the value is 7, in this interval the value is 10, in this interval the value is 16. Let us see in this interval the value is 14, 11; let us say 3 and minus 1 for some variety. These are the piecewise constant values in these respective intervals. So, in fact, we have the sequence here.

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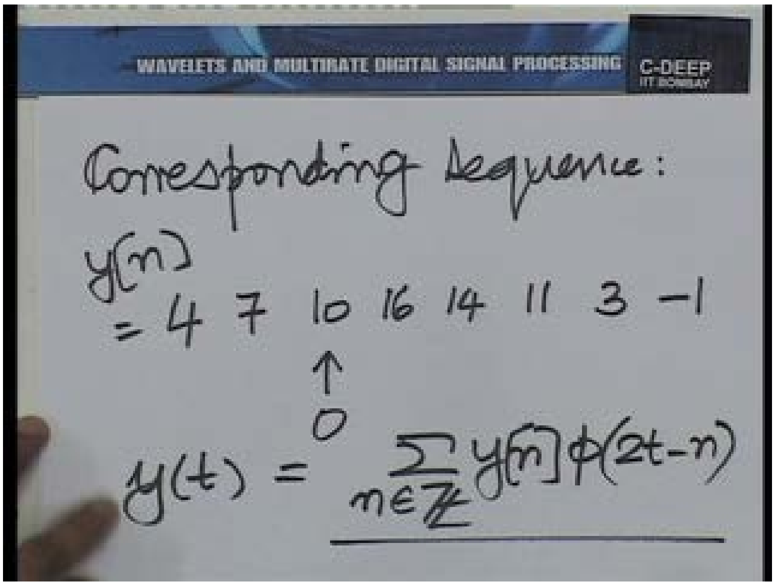

The corresponding sequence. So, I use the notation with the standard marker of 0; so, at 0 I have 10 and then I have in order 16, 14, 11, 3, minus 1, 7 and 4, this is my corresponding sequence. So, in fact what we are saying in effect is y (t), if I call this sequence y [n], what we are saying in effect is y (t) is summation n overall the integers y [n]; note phi (2 t minus n), that is important.

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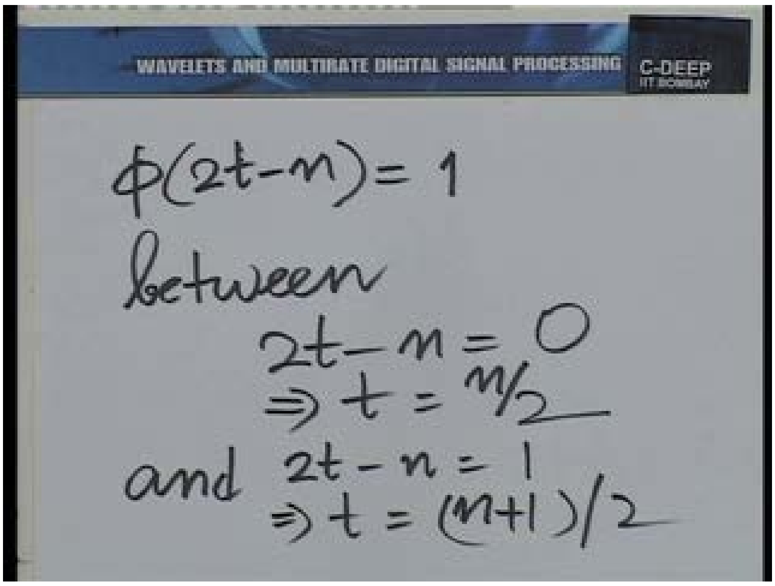

Notice that phi (2 t minus n) is by definition going to be equal to 1 between 2 t minus n equal to 0, that is t equal to n by 2 and 2 t minus n is equal to 1; which means t is (n plus 1 by 2). So, for example, phi (2 t minus 1) is going to be 1 between 1 by 2 and 2 by 2, half and 1 and that agrees with what we just wrote down, good.

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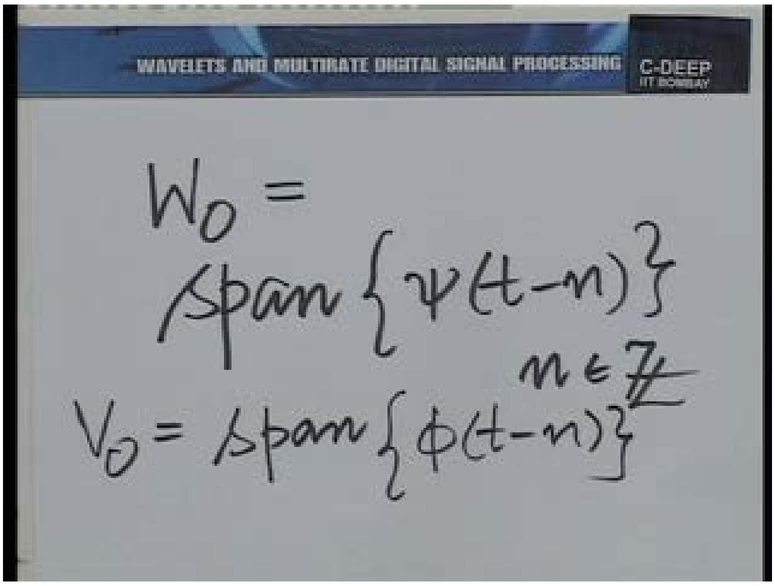

Now, suppose we decompose; so we know that we have this decomposition of V 1 into W 1 and V 0 or rather W 0 and V 0, I am sorry. So, we said that we have this space W 0 given by the span of psi (t minus n), n over all the integers and V 0 of course, is the span of phi (t minus n) for all integer n. And, now we are going to introduce the notion what is called an orthogonal compliment.

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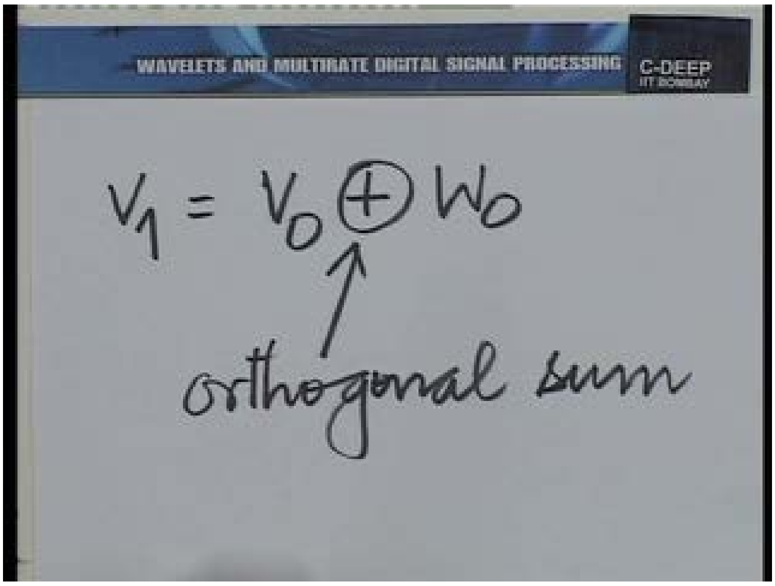

So, we are going to say; well, V 1 is the orthogonal sum; this represents orthogonal sum and we are going to explain this in more detail. We say that space V 1 is the orthogonal sum of the spaces V 0 and W 0. If there is a unique way to take a vector in V 1 and decompose it as a sum of a vector in V 0 and a vector in W 0, with the vectors from V 0 and W 0 being perpendicular, being orthogonal; in other words their inner products being 0. What is this idea of an orthogonal sum? The idea of an orthogonal sum is to decompose a linear space into sub spaces of course of smaller dimension; where you also have mutual perpendicularity or mutual orthogonality between the space, between the vectors in these 2 spaces of the composition.

Now, I shall give you a simple example, from a 3 dimension situation. Let us take the 3 dimensional space of the room that we are in at the moment and let us take the 2 dimension space of the floor. The 3 dimension space of the room is the orthogonal sum of the 2 dimensional space of the floor and a 1 dimensional space comprised of all multiples of a vector perpendicular to the floor. So, as I expected a 3 dimensional space is an orthogonal sum of a 2 dimensional space and a 1 dimensional space. The 1 dimensional space is form by all multiples of a vector perpendicular to the floor. The 2 dimensional space is form by all vectors lying on the floor. So, if you take any vector lying on the floor and any vector in that 1 dimension space of multiples of a unit vector perpendicular to the floor, these two vectors are orthogonal, perpendicular; in 3 dimensions very easy to visualize.

Now, you can always generalize to n dimensions. So, for example, if you had 10 dimensional space, it could be an orthogonal sum of a 4 dimensional space and another 6 dimensional space; where if you take a vector from that 4 dimensional space and vector from the 6 dimensional space, they are perpendicular to one another. Perpendicular as understood by taking the inner product between these two vectors in that 10 dimensional space. Now, I must at this point make a little remark; the inner product allows us to bring in the notion of an angle between functions, a more general version of orthogonality. So, we say two vectors are perpendicular, if their inner product is 0. In general we can also define the angle between two vectors using the inner product and we shall do exactly that in a minute. So, we can talk about the angle.

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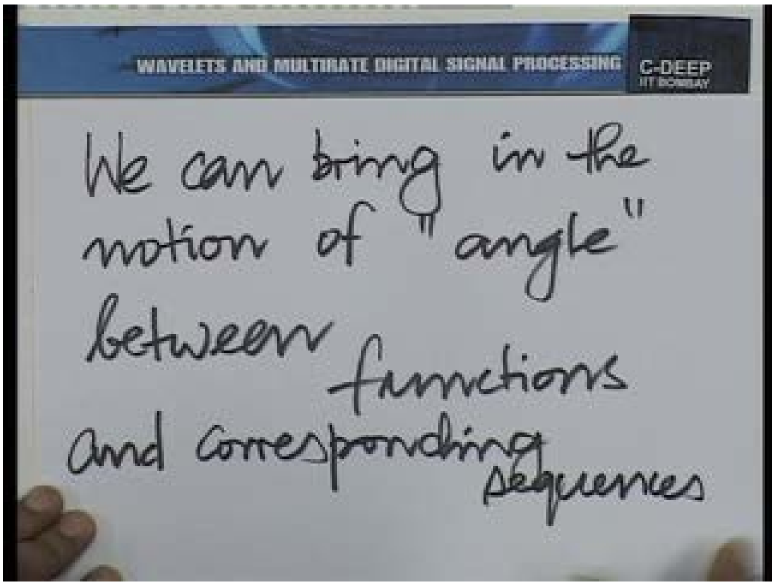

And, in fact once we bring in the notion angle between functions, we can also bring in the notion of angle between the corresponding sequences.

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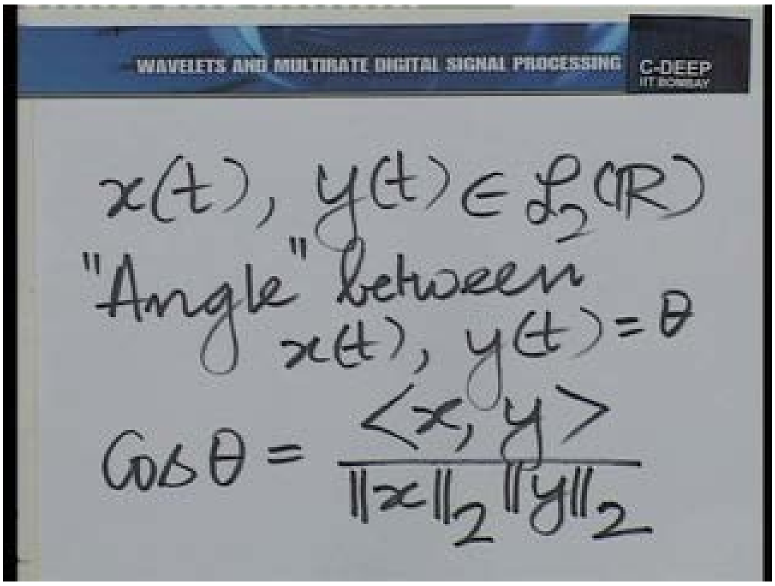

So, if you have two functions x (t) and y (t); of course, belonging to L 2 R, we are going to confine ourselves that space. Then, the angle between x (t) and y (t) is essentially defined by the following. You see, let it be theta, we say cos theta is essentially the inner product of x (t) with y (t) divided by the norm in x 2 of in l 2 of x and the norm of y in l 2 multiplied together. Now, this is very similar to the idea of a dot product between 2 vectors.

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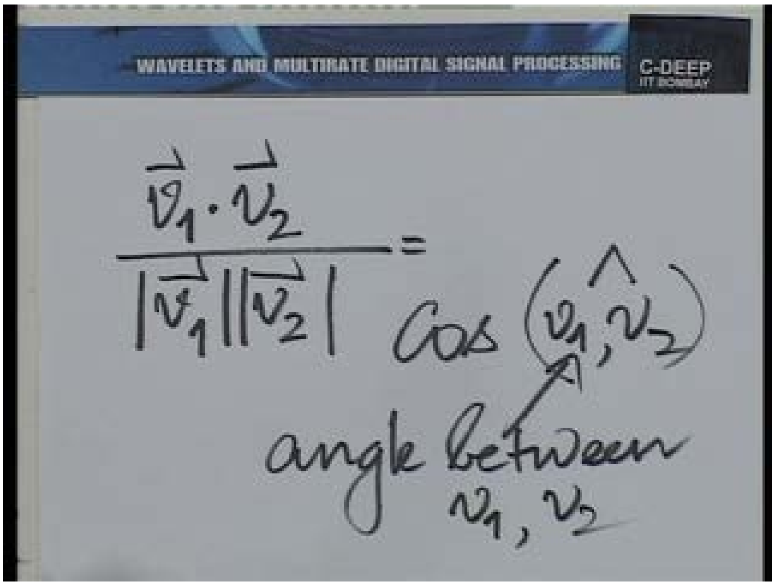

So, if you recall, if you have two vectors let us say V 1 and V 2; then V 1 dot V 2 divided by the magnitude of V 1 and the magnitude of V 2 gives us the cosine of the angle between V 1 and V 2.

So, in a restricted sense you do have the notion of angle between functions. And, whatever you did to construct the angle for the functions can also be done for the corresponding sequences associated with the functions and therefore, you have the notion of the angle even between the corresponding sequences. And, I ask a question and I leave it to you to ponder over the answer; are those two angles the same? Do they actually match? I think they should, should not they? And, I leave it to you as an exercise to actually show that they do.

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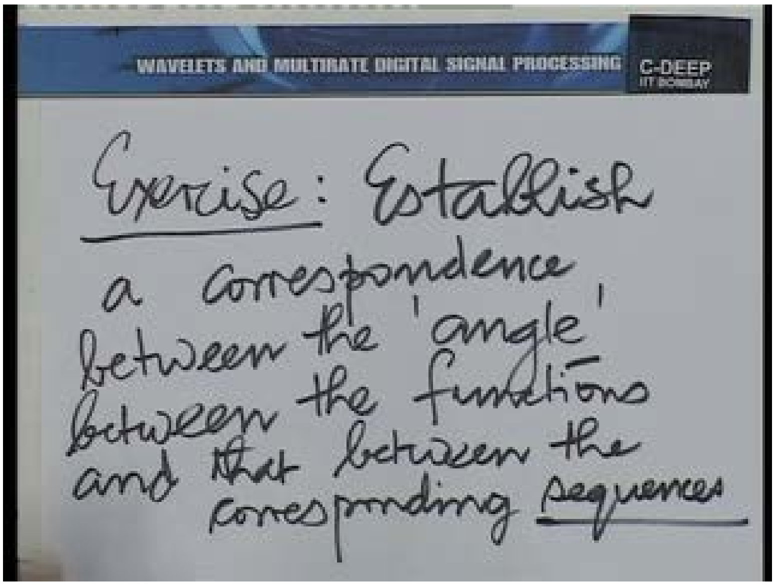

So, exercise. Establish a correspondence between the angle, between the functions and the corresponding sequences.

Anyway, our correspondence has become deeper and deeper and whatever we have been doing with the functions, we discover can now be done with the sequence. Now, the next step is to ask can we also be think of decomposition in terms of decomposition of the sequences. So, for example, let us go back to that function in V 1 that we had, a few minutes ago. We had this function in V 1 and we could then decompose V 1 into V 0, the orthogonal sum of V 0 and W 0. Now, just for a minute let us keep aside the discussion of this particular function. Let us look at the typical function in V 1, a typical function in V 0 and a typical function in W 0.

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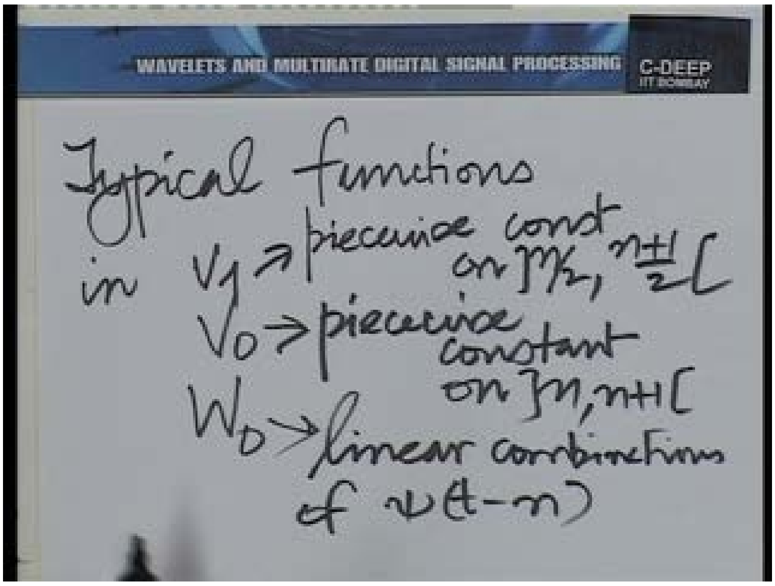

So, let us look at typical functions in V 1, V 0 and W 0. Now, these are piecewise constant on the standard unit intervals. These are linear combinations of psi (t minus n) for integer n. And, these are piecewise constants on n by 2, n plus 1 by 2, for integer n. Suppose, we take a given function in V 0 and a given function in W 0; now, what would the dot product be?

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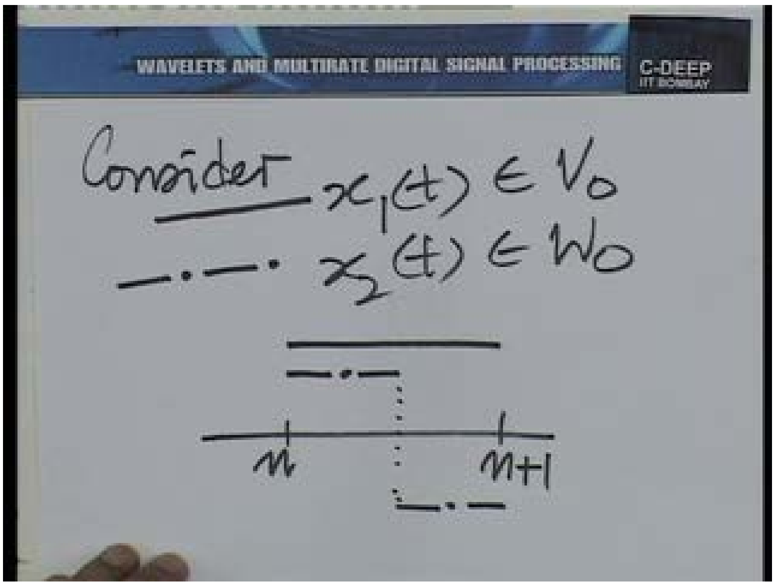

So, what I am saying is, suppose we considered say x 1 (t) belonging to V 0 and x 2 (t) belonging to W 0. Let us focus our attention on a particular interval; let say n to n plus 1. Let us draw the function x 1 (t) by a solid line and this one by a dot dash line. What for the typical function look like? The x 1 (t) function would look like this and the x 2 function might look something like this. If I multiply these functions together and integrate it, you can visualize the integral piece by piece on each of these intervals n to n plus 1. The integral on any one of these pieces is obviously 0, because the positive and the negative areas are equal and this can be true of all the intervals. And, therefore, obviously these two functions are perpendicular, orthogonal because their dot product is 0.

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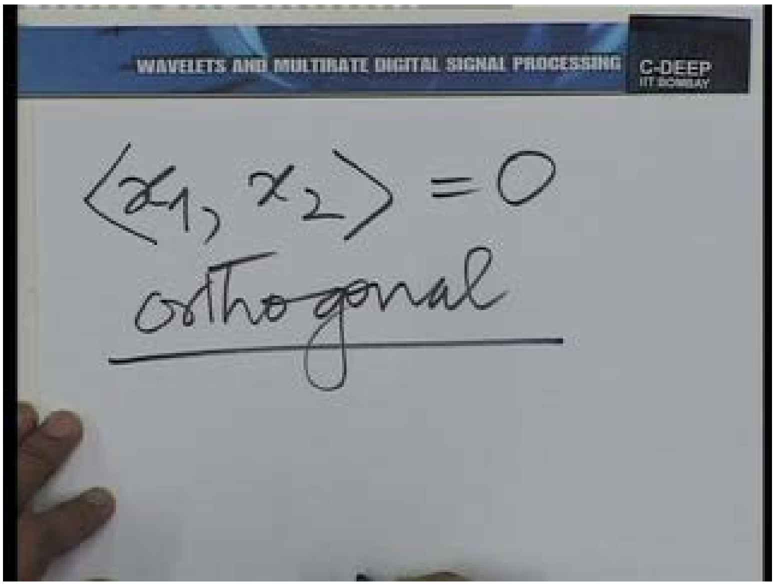

So, the dot product of x 1 and x 2 is 0; they are perpendicular or orthogonal. You know we do not use the word perpendicular anymore; when we talk about functions we should use the word orthogonal. What is more, take any particular function in V 1; let us take the function.

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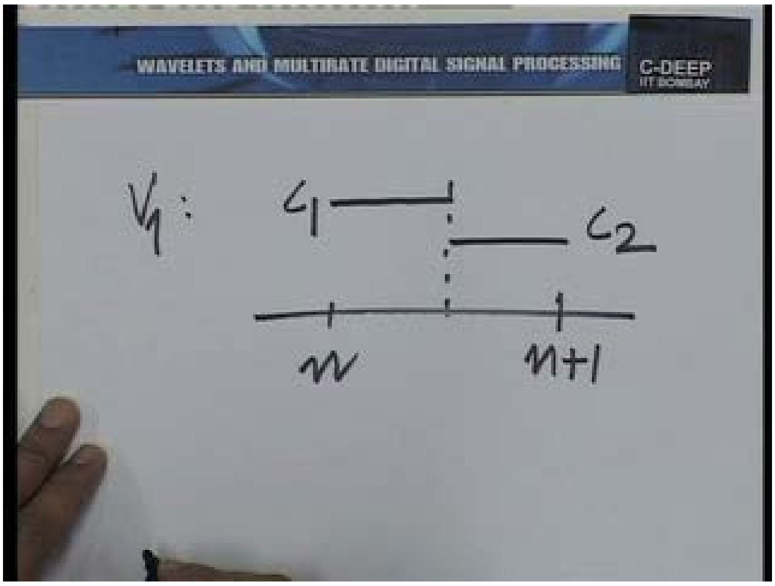

Let us say again focus on any one particular interval a function V 1; let us take the interval n to n plus 1. And, let the function have the value let us say C 1 for the first half interval and C 2 for the second half. It is very easy to see that this can be treated as a function belonging to V 0 plus a function belonging to V 1, where the corresponding function belonging to V 0 looks like this.

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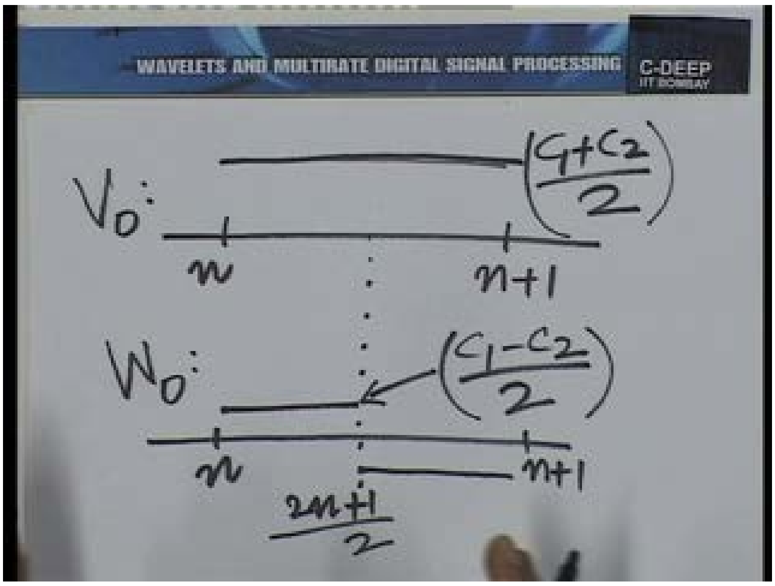

This function is equal to this function, C 1 plus C 2 on the interval(C 1 plus C 2 by 2) over the interval n to n plus 1 plus the function C 1 minus C 2 over the first half interval and the negative of the same thing over the second half interval. So, if you take this point to be n 2 n plus 1 by 2; so, to speak, middle of the interval. Then, this height here is (C 1 minus C 2 by 2). This is you see what I am saying is the function coming from V 0 and the function coming from W 0; I am just showing one segment of each of these function. So, the same thing can be done for each of the intervals n to n plus 1.

What I am saying is, that this function whose segment over n to n plus I have shown here, being C 1 on the first half interval and C 2 on the second half interval. Of course, it belongs to V 1 is equal to the sum of this function (C 1 plus to C 2 by 2) on the entire interval belonging to V 0 plus this function belonging to W 0 which is (C 1 minus C 2 by 2) on the first half interval and the negative of same thing on the second half interval. So, it is very easy to see that we can in general decompose a function in V 1 into a function in V 0 plus a function in W 0 in a unique way. And, therefore, the orthogonal decomposition of V 1 into V 0 and W 0 is easy to construct.

Now, can we also make a corresponding construction on the sequences and in fact to some extent we have already answered the question. If you look at it, here C 1 would be the value of the sequence at 2 n; the sequence corresponding to the function in V 1 and C 2 would be the value of the sequence at 2 n plus 1. Interestingly, the value of the sequence corresponding to the function in V 1 at the point 2 n and 2 n plus 1 relate to the values of the sequences corresponding to the functions in V 0 and W 0 but at the points n and not 2 n. So, you have value (C 1 plus C 2 by 2) for the sequence corresponding to this function, at the point n not 2 n. and, you have the value (C 1 minus C 2 by 2) corresponding to the function in W 0 but at the point n not 2 n. What I am saying is, now if you think in terms of sequences let us do that. So, you had this function in V 1.

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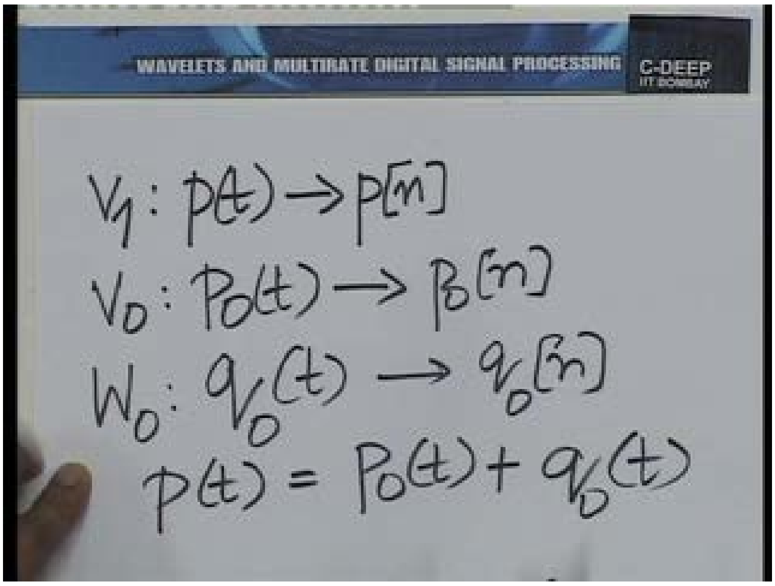

So, let us say the function is now you know for the variety use P (t) belonging to V 1 and the corresponding sequence P (n). You have the corresponding function P 0 (t), let us call it and the corresponding sequence P 0 [n], where p 0 is the component in V 0 so to speak and you have q 0, let us say as the component in W 0 and the corresponding sequence is q 0 [n]. What I am saying is P (t) is of course, equal to P 0 (t) plus q 0 (t). But P [n] is not equal to P 0 [n] plus q 0 [n]; that is not correct. That is because orthonormal basis are different. So, now we need to establish a relation between P 0 [n] q 0 [n] and P [n]; that is the next task that we would like to undertake.

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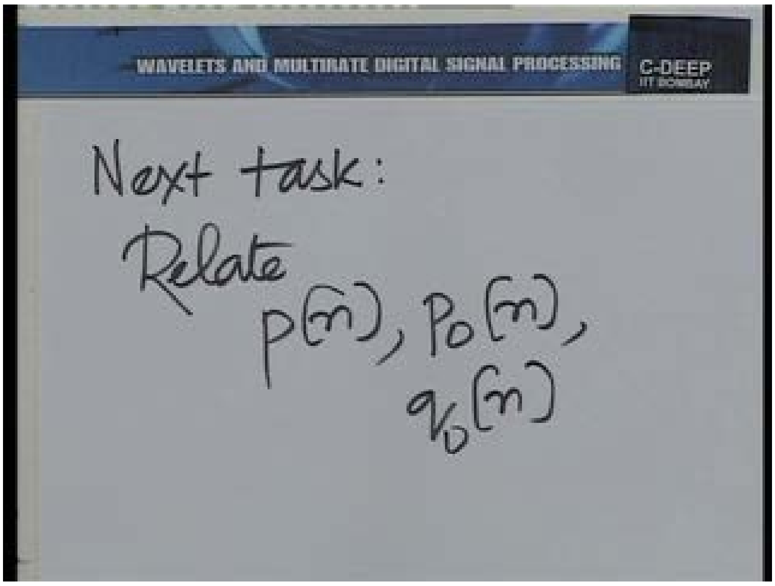

So, our next job is relate P [n] P 0 [n] and q 0 [n] and in fact we already have an answer to that question, let me just go back in one step here. So, we have the answer here. You see P at the value 2 n is C 1, P at the value 2 n plus 1 is C 2. V 0 at the point n is (C 1 plus C 2 by 2); q 0 at the point n is (C 1 minus C 2 by 2). Let me write down all these formally; so, I have a relationship there, I have almost done my job.

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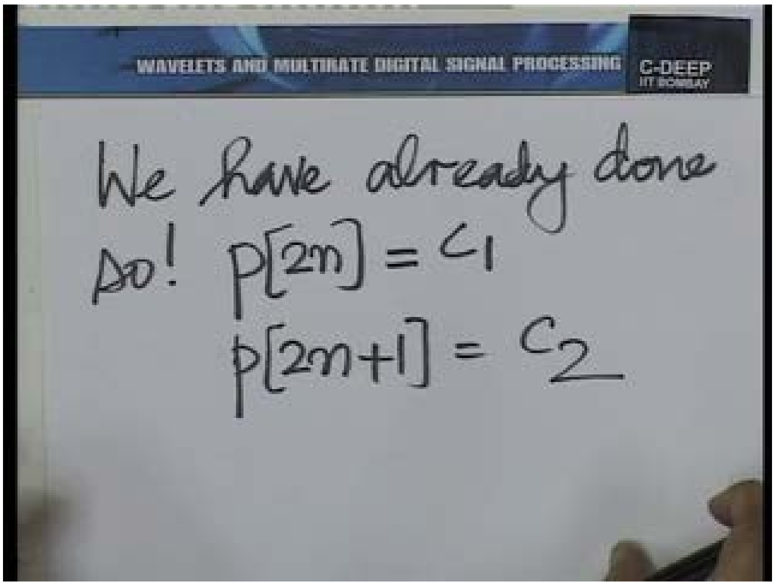

What have we said? We said P [2 n] is C 1; P [2 n plus 1] is C 2.

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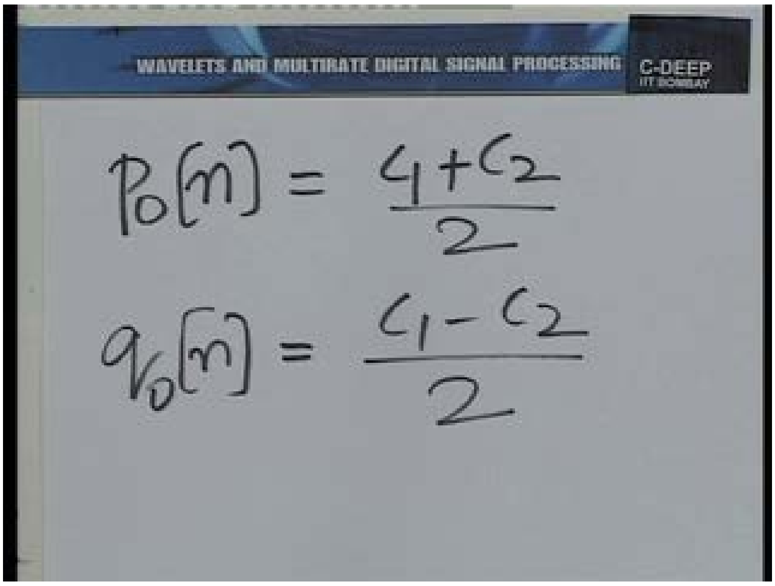

P 0 [n] is C 1 plus C 2 by 2 and q 0 [n] is C 1 minus C 2 by 2. Now, let us combine these equations.

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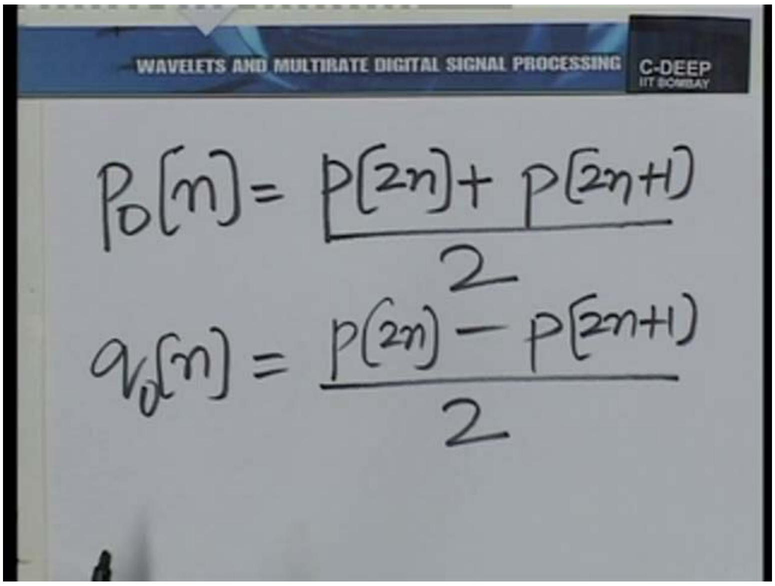

We have P 0 [n] is P [2 n] plus P [2 n plus 1] by 2 and q 0 [n] is P [2 n] minus P [2 n plus 1] by 2. And, now this brings before us a very beautiful perspective. When we talk about sequences, we can also extend that context to talk about discrete time filters acting on sequences. Can we visualize what we have done here as discrete time filters acting on the sequences?

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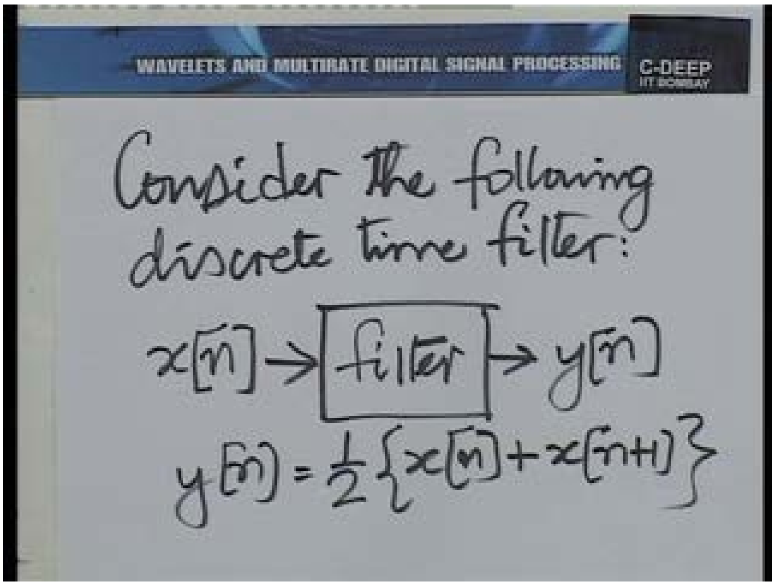

So, suppose you had the following discrete time filter.

Let the input be x [n] and the output be y [n] and y [n] is half {x [n] plus x [n plus 1]. You know it is a non causal filter; let us not worry too much about it for the moment, let us accept it even it is non causal.

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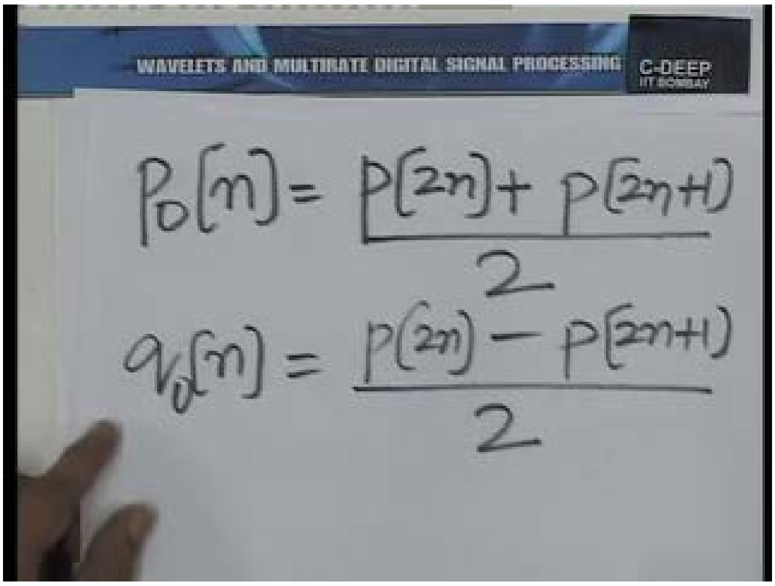

What are we done here in this in this relationship? Let us reflect on the connection between for example, this relationship here and the filter we just construct it. If you think of 2 n as a variable, let us call it let say it l; so, this is P [l] plus P [l plus 1] by 2. Then, in fact this is essentially the filter acting on the sequence P. So, what we are saying is, use this filter that we have here and put P in here, but then a little bit of work needs to be done at this point because if you put in P [n]; let us do that, let us put in P [n].

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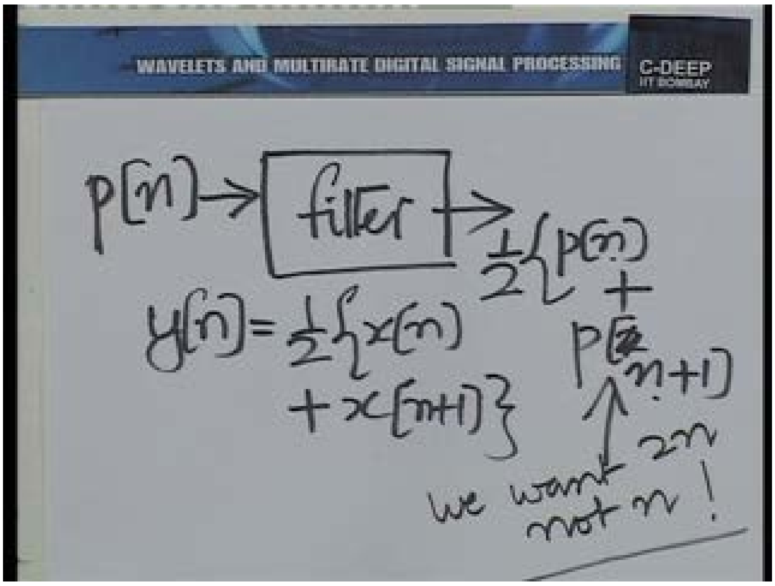

If you put in P [n] there, if you put into this filter that we just wrote down here y [n] is half {x [n] plus x [n plus 1]. What we would get here is half {P [n] plus P [n plus 1], am sorry n plus 1, that is correct. But we do not want P [n] and P [n plus 1] we want to replace here. So, what should we do?

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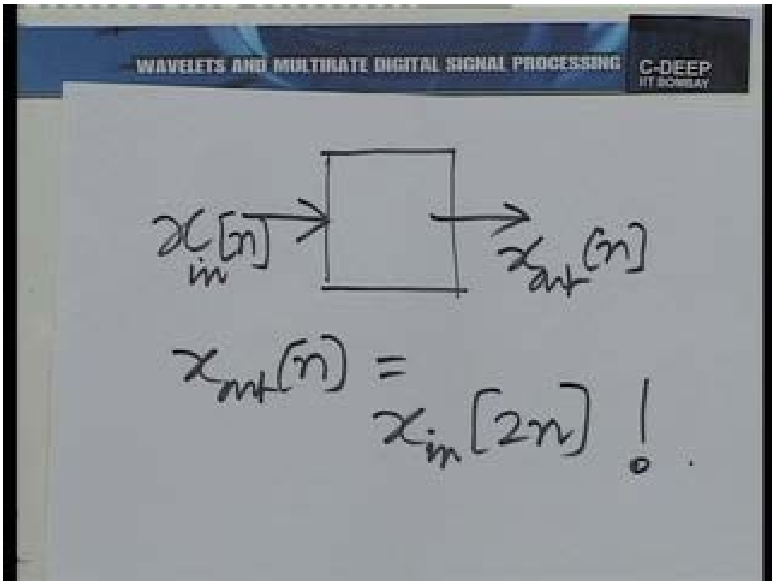

We should have another system now following this, where put in x in, put out x out where x out [n] is x in [2 n]; you want a system like this. Let us interpret this system; what we are saying in the system is x in [n].

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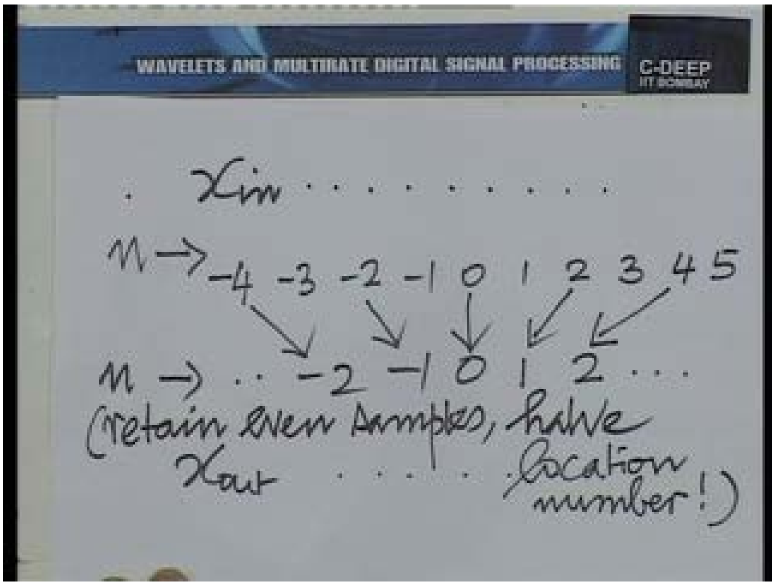

So, let us write down n minus 4, minus 3, minus 2, minus 1, 0, 1, 2, 3, 4 and so on and the sequence x in here, put at these points. So, as for as x out goes you have the index for x out, as far as x out goes, 0 comes from 0, 1 here comes from 2, 2 here comes from 4, minus 1 comes from minus 2, minus 2 comes from minus 4 and so on. So, in other words what are you doing? You are retaining the samples at the even locations and throwing away the samples at the odd locations. Not only that, after retaining the samples at the even locations, you are putting at those samples at half the location number. Let us summarize this; retain even samples and have the location number. Now, this system is a new system as far as a basic codes on discrete time signal processing is concern. We need to christen it; we need to give it a name. In fact, let us go back to that system and let us give it both the symbol and name.

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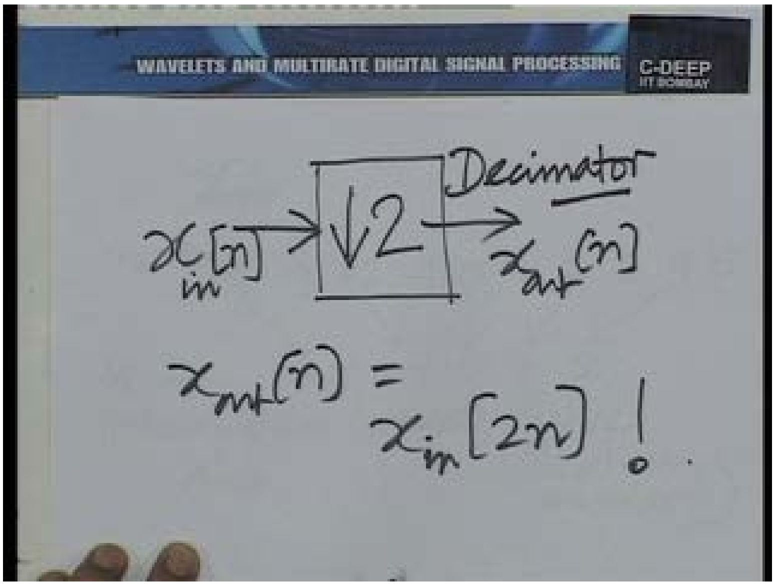

The symbol that we shall give it a down arrow that followed by 2 and we shall call it a decimator. You know the word decimate actually has a very cruel meaning and told that in the days of wars during the roman empire a very cruel thing that warriors use to do was to kill one out of 10 or may be 9 out of 10; I do not remember and that was what was called decimation. Take 10 of them and eliminate from each group of 10, that was a cruel way to deal with people. But the word decimate has also percolated down to the literature on digital signal processing.

Here, decimation means retaining 1 out of so many samples; so, in this case decimation by 2 means retaining 1 out of 2 samples. In fact, the first of each pair of 2 samples out of 0 and 1 you retain 0 out of 2 and 3 you retain 2. Not only that after you retain only 1 out of 2 compress so that the sample number is halved or if you retain 1 out of 3 samples, then compress the sample number is multiplied by one-third. So, if you decimating by a factor of 3 for example, the 0 sample will go to 0, the 3 sample will come to 1, the 6 sample will go to 2, the minus 3 sample will come to minus 1 and so on.

If you decimating by a factor of 2, the 0 sample will come to 0, the 2 sample will go to 1, the 4 sample to 2, the minus 2 sample to minus 1 and so on as we just showed. So, what do we have here? We have a filter followed by a decimator and that together helps us construct the sequence P 0 [n] from the sequence P [n]. Now, we shall see in the next lecture, that we can similarly construct the sequence q 0 [n] from the sequence P [n] by using another filter and a decimate. And, we shall build up further from there to do something to reconstruct P [n] from P 0 [n] and q 0 [n]. And, all this shall together lead us to a totally different structure in discrete time signal processing, which we shall call a two band filter bank. With this little trailer for the next lecture, let us conclude the present lecture. Thank you.