Advanced Digital Signal Processing - Wavelets and Multirate

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Lecture No. # 41
Tutorial - Session 1

A warm welcome to the first tutorial session on the subject of Wavelets and Multirate signal processing, in which we begin the exercise of working out a few examples, which would illustrate the ideas that we have discussed conceptually in the course. It is important in a course like this, firstly to understand concepts in depth and secondly understand how to apply them in the case of specific examples or situations. Keeping that objective in mind, we begin today by taking a tutorial set of examples on some of the basic concepts, which we built up towards the earlier part of this course. Namely, the idea of piecewise constant approximation, the idea of the spaces l 2 r, l 1 r and so on, the idea of successive approximation across the ladder of subspaces of l 2 r, the idea of incremental information and how it relates to sequences apply to filter banks. So, let me specifically put before you the problem that I shall solve before you today to illustrate some of these ideas.

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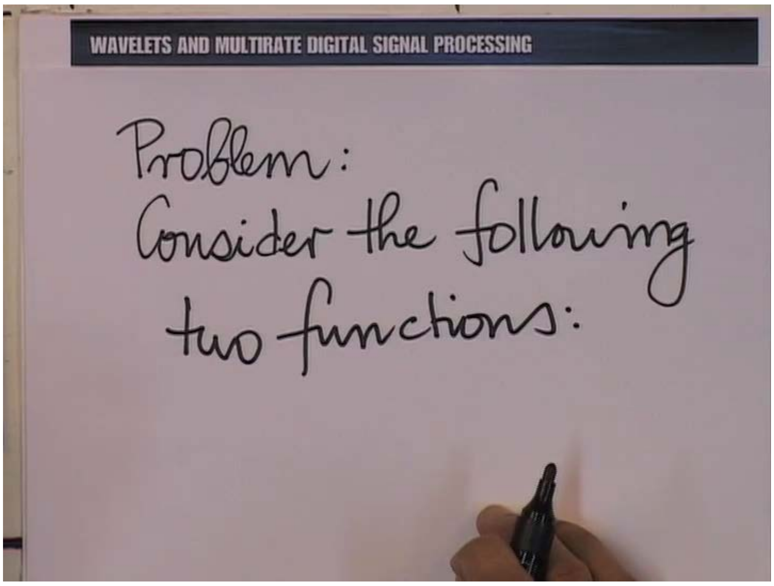

Now, in this session here, we are going to consider the following problem. Consider the functions, the following two functions and I am going to draw the two functions before you.

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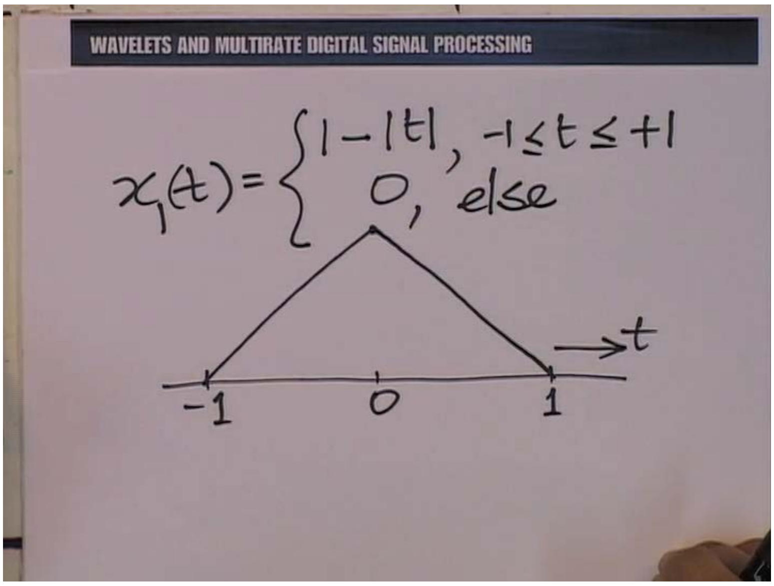

Function 1, we shall call x 1 t and it looks like this. It is a straight line between 0 and 1 and 0 and minus 1 and symmetric. So, it is easy to see that this function is described by 1 minus modulus of t, for t between minus 1 and 1 and 0, else. We will crescent this function and we will call it x 1 t.

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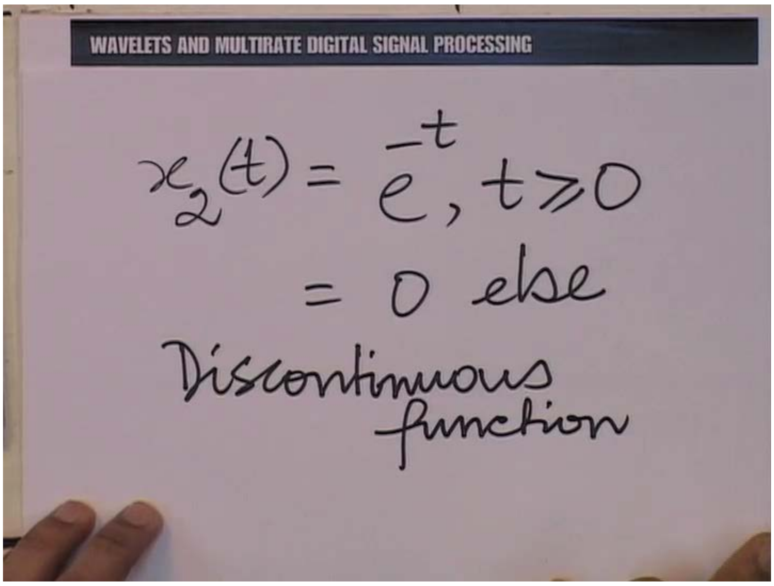

Similarly, we will consider another function for variety. We will call it x 2 t and we will define it to lie only on one side of 0. So, we have x 2 of t is e rise the power minus t, for t greater than equal to 0 and 0 else.

So, we take two examples, essentially an example of a discontinuous function and an example of a continuous function. This one, the one which we saw earlier was a continuous function, although not differentiable. We shall do the ladder analysis, the ladder subspace analysis on each of them.

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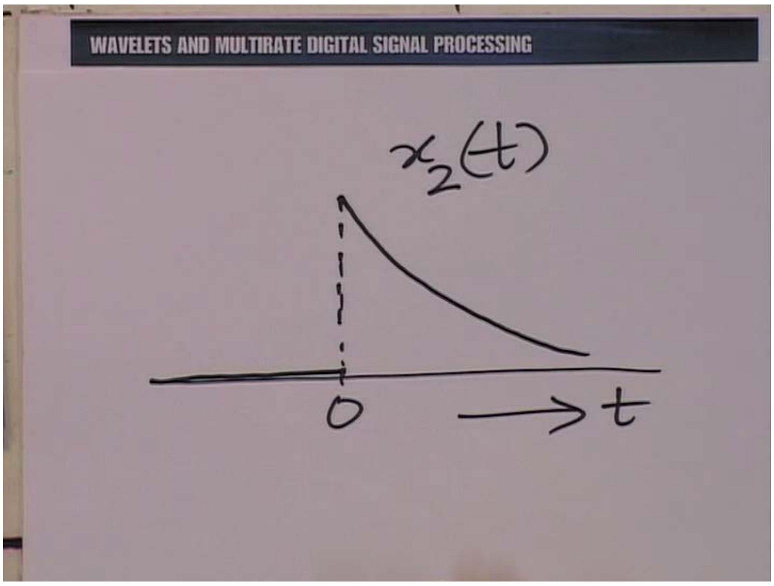

So, let us sketch x to t also for completeness. x to t would appear like this. That is 0 until this point and then it rises to 1 and then drops exponentially. Now, the first thing that we would like to verify as a tutorial exercise is that, each of these functions belong to l 2 r, belongs to the space of square integrable functions. Although trivial, we should complete this exercise to understand the problem completely.

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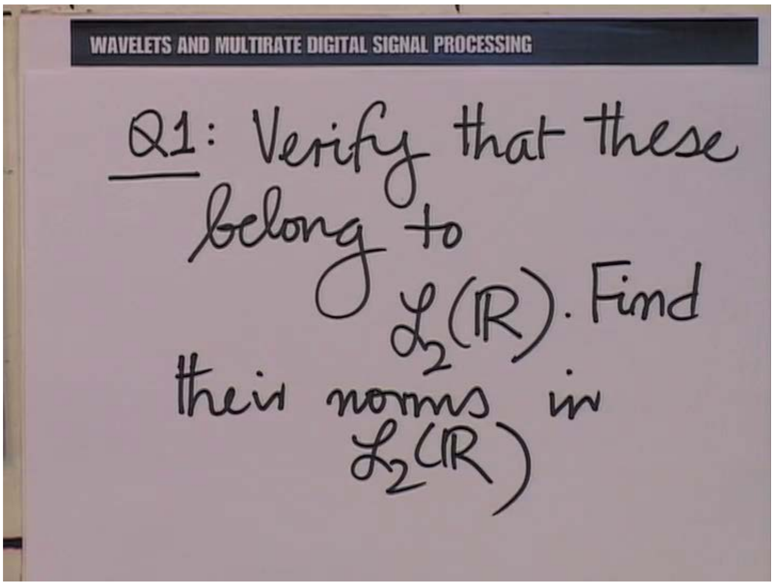

So, of course, the first is very easy you see. So, the first question or question one is, if you would like it to be is verify that these belong to L 2 R or the space of square integrable functions. It is very easy but let us complete it. In fact, let us make the question a little more in terms of what it asks and let us say, find their norms in L 2 R. At least, that is a little bit of work to do. So, two problems can be taken together. There is no difficulty here.

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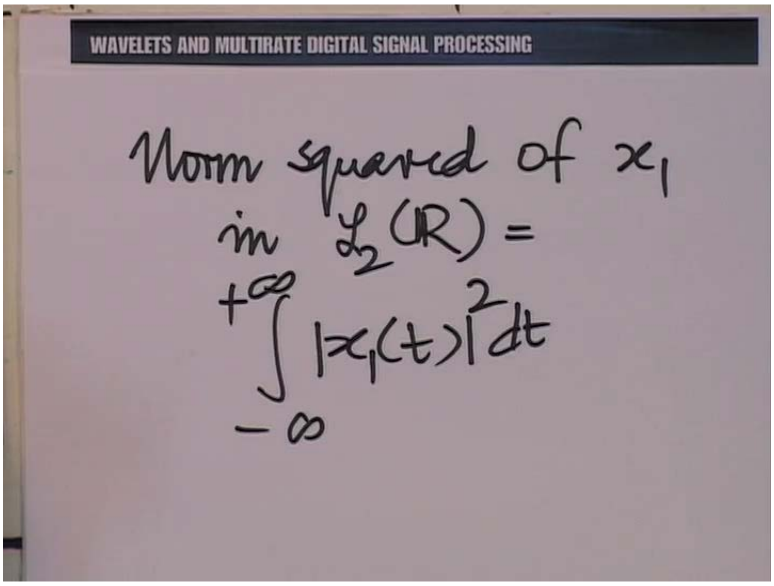

In fact, we can see that the norm squared actually of x 1 in L 2 R, would be given as the integral over all t mod x 1 t square d t. Now, this is easy to calculate.

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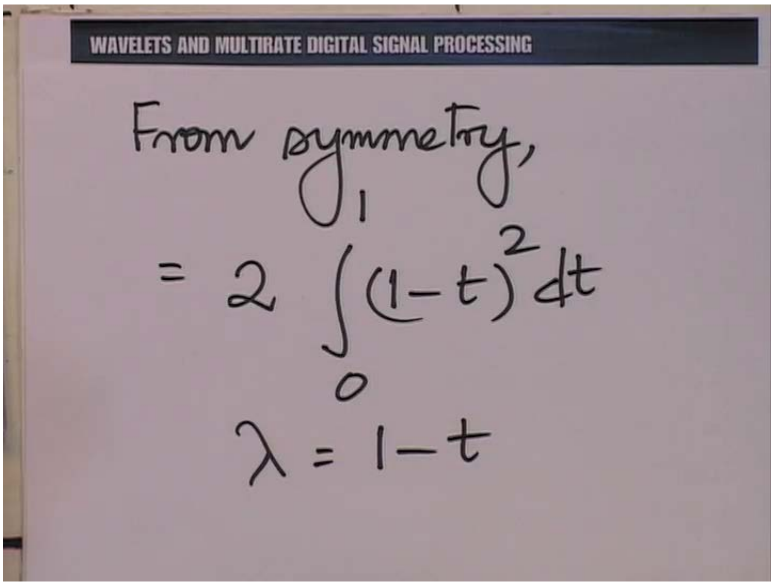

In fact, looking up the symmetry, it is clear that this is 2 times the integral from 0 to 1 1 minus t the whole squared d t.

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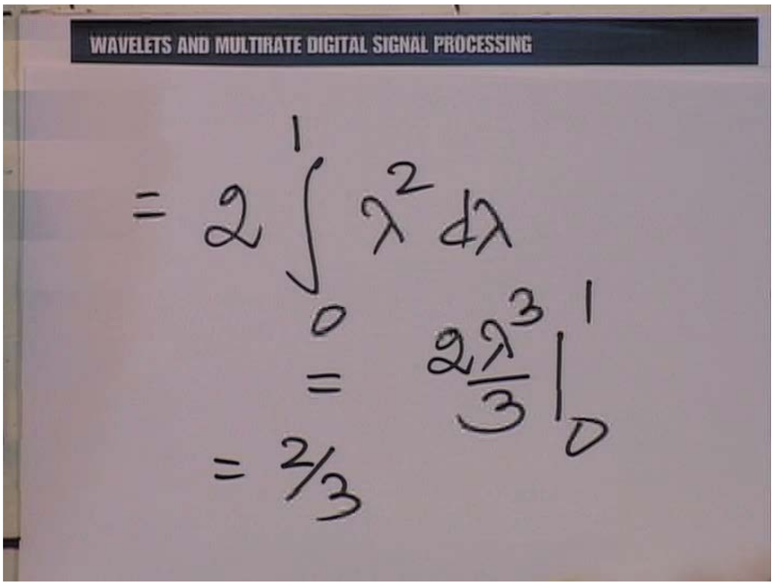

But then if we make the replacement of variable, lambda is 1 minus t, we have this is twice lambda squared minus lambda minus d lambda. Again, lambda would go from 1 to 0. But because we absorb the minus sign, it goes from 0 to 1.

So, it is a simple transformation. I am skipping one step. This is very easy integral to evaluate. So, this is 2 by 3. This is the norm squared.

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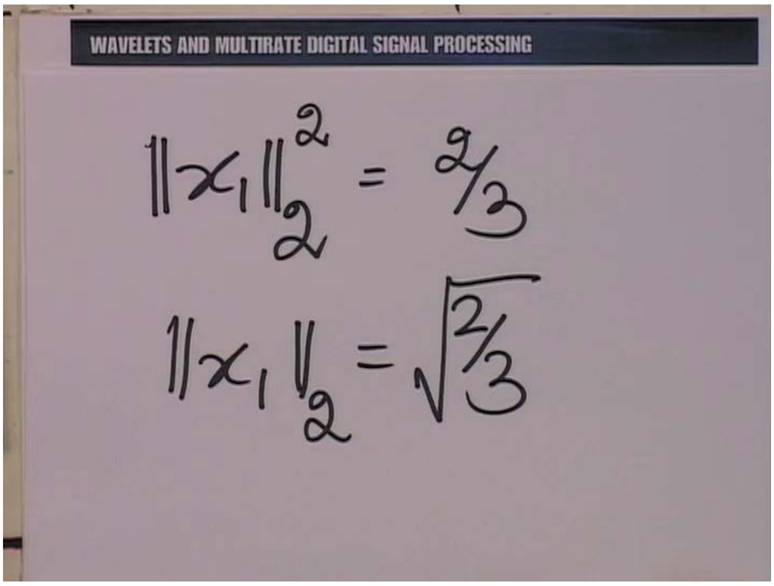

So of course, just to complete the discussion, we will realize that the norm of this function, I must make it clear, the norm of x 1 in L 2 R, which we write like this and squared is 2 by 3. Therefore, the norm itself is square root of 2 by 3, and that is positive square root, simple enough. Now, we take the second function.

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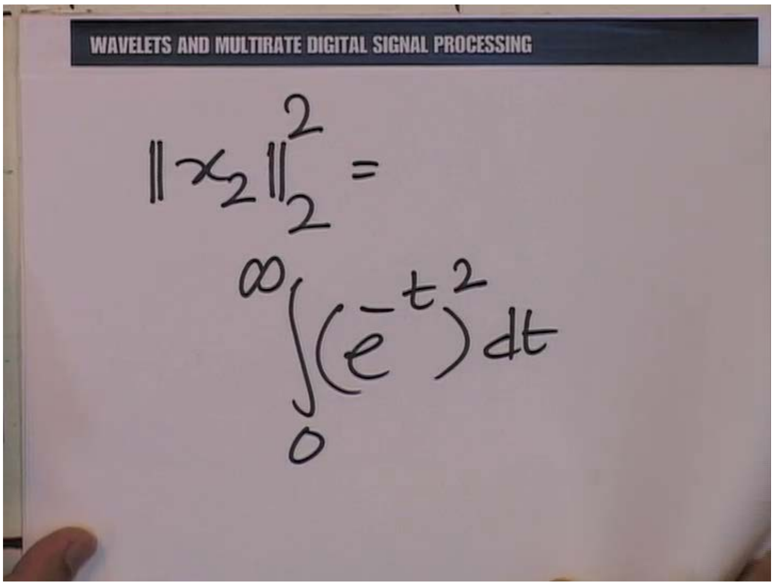

We find norm similarly of x 2 squared and that is easy to calculate. It is e rise the power minus t squared d t integrated from 0 to infinity. That is an easy integral to evaluate.

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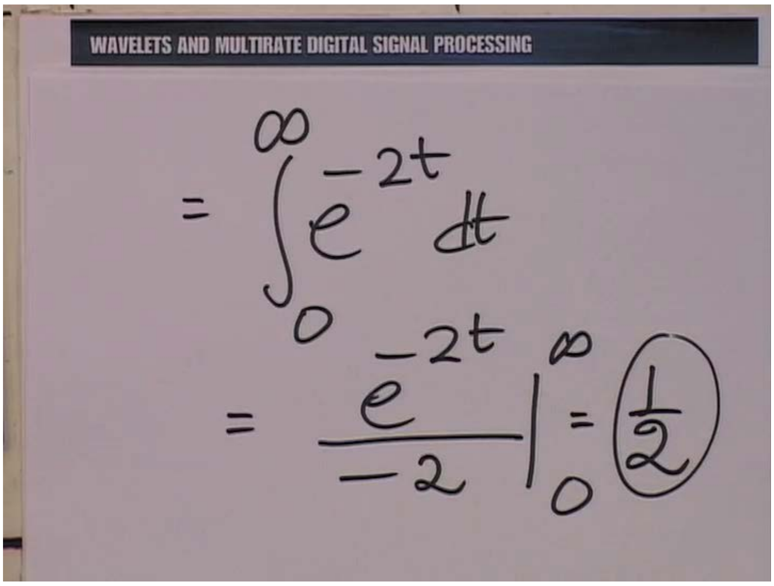

It is essentially e rise the power minus 2 t d t integrated from 0 to infinity and that is e rise the power minus 2 t by minus 2 from 0 to infinity and that is half.

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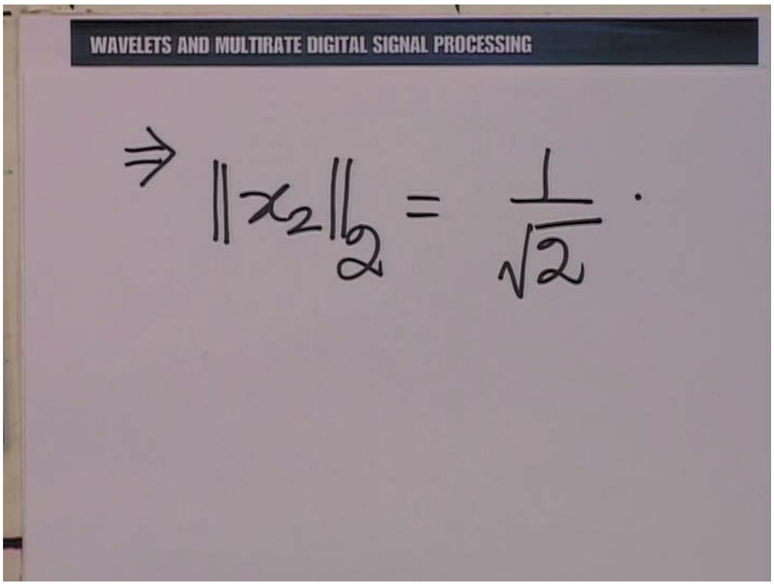

Therefore, it is very clear that the norm of x 2 in l 2 r is 1 by square root of 2; positive square root of 2. This was the easy part of the exercise. So, of course, you know in calculating these norms, we have already verified that the functions belong to l 2 r.

Now, we would like to apply the idea of piecewise constant of approximation on them. So, let us find the projection of each of these functions on the space v 0. As we know it, in the Haar multi resolution analysis. So, what we shall do is to look at these two functions from the point of view of the Haar multi resolution analysis.

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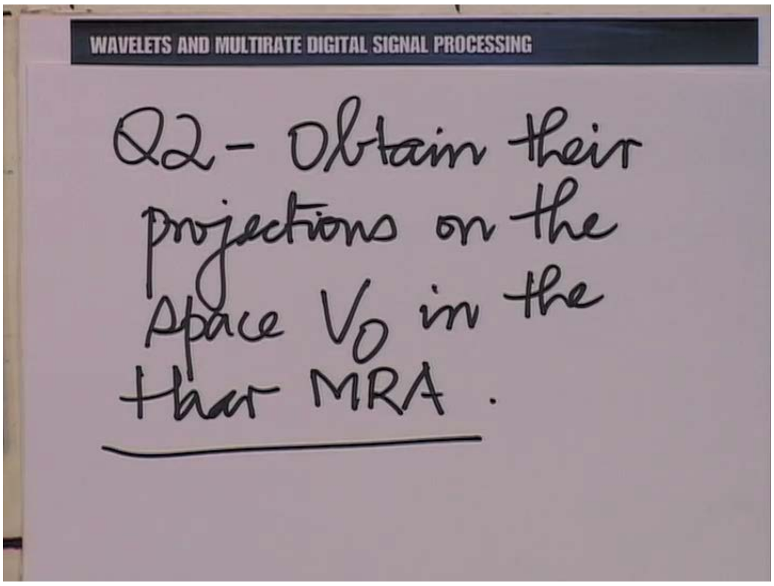

So, question two before us is, obtain their projections on the space v 0 in the Haar multi resolution analysis. What does it mean to obtain the projections in the space v 0? It means, make a piecewise constant approximation on intervals of length 1, in fact the standard intervals of length 1. The standard intervals of length 1 are the interval 0 to 1, 1 to 2, 2 to 3 and so on. Of course, in the negative side, minus 1 to 0, minus 2 to minus 1, minus 3 to minus 2 and so on and so forth.

Now, if we go back to the function x 1 t, we shall see that it is extremely easy to see the nature of the piecewise constant approximation that we would get. In another words, the projection on v 0.

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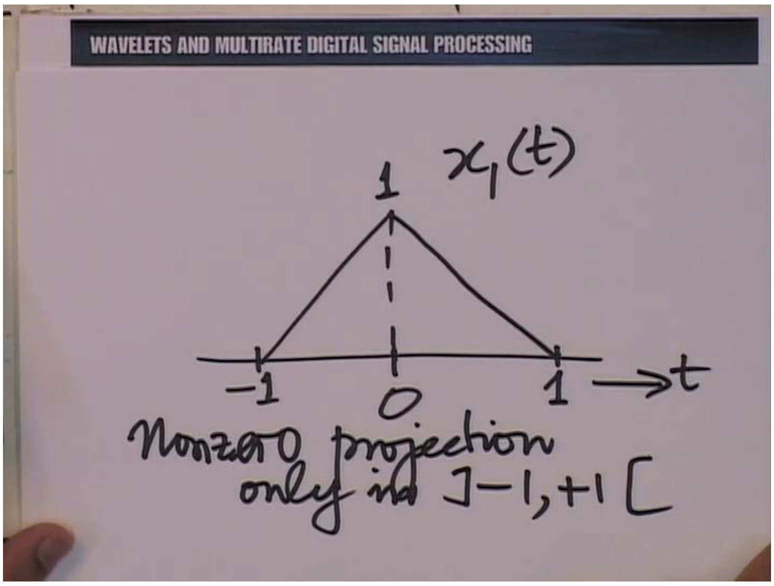

So, x 1 t looks like this. It is all laid out for us. You see, it is very easy to see that if I take the standard intervals beyond 1, so 1 to 2 or 2 to 3 or 3 to 4 and so on and infinitum there and minus 2 to minus 1 minus 3 to minus 2 and so on and infinitum back words. The piecewise constant approximation on all of those is 0. So, there is a nonzero piecewise constant approximation only on two of the intervals, namely minus 1 to 0 and 0 to 1. So, so much so for x 1 t; non zero projection only in the interval minus 1 to plus 1. So, it is very easy to calculate this constant here.

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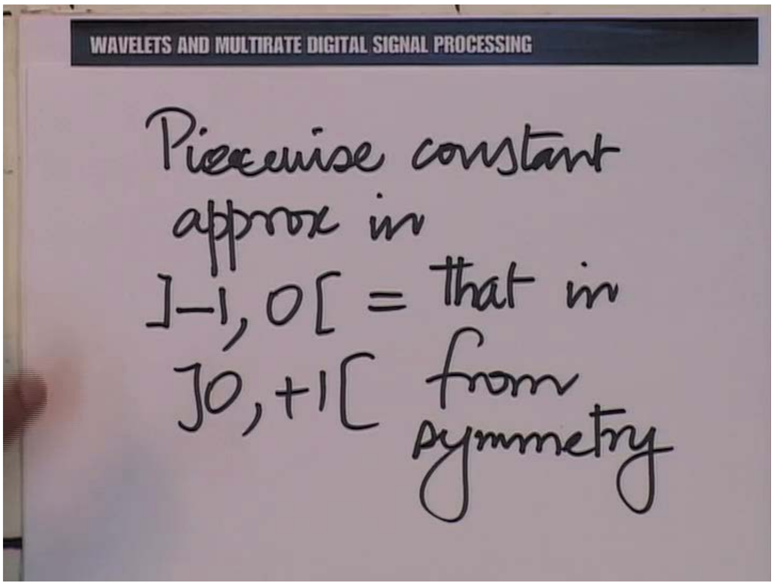

So therefore, the piecewise constant approximation in minus 1 to 0 is equal to that in 0 to plus 1 from the symmetry. It is further very easy to calculate and it is simply the average of the function.

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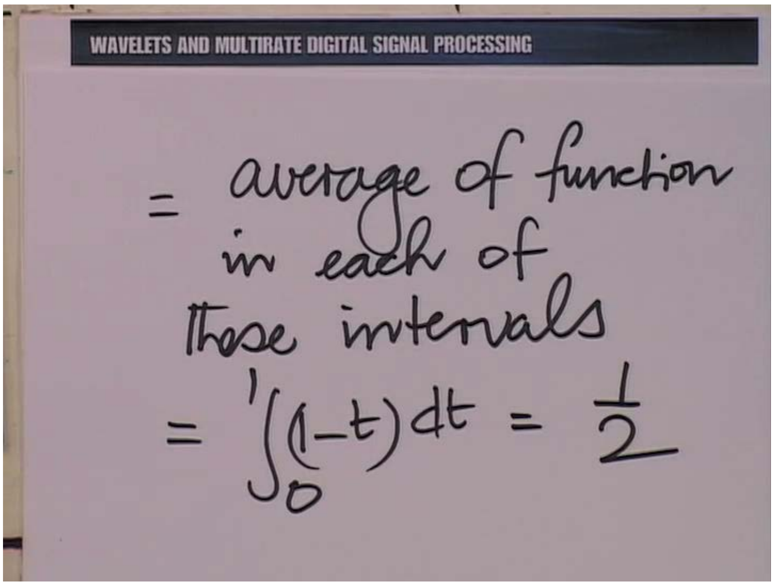

It is the average of the function in each of these intervals. It is easy to calculate. It is essentially again 1 minus t d t integrated from 0 to 1. It is not at all difficult to see that this is equal to half. So, the average of the function in each of these intervals is half. Therefore, we have the following as the projection of the function on the space v 0.

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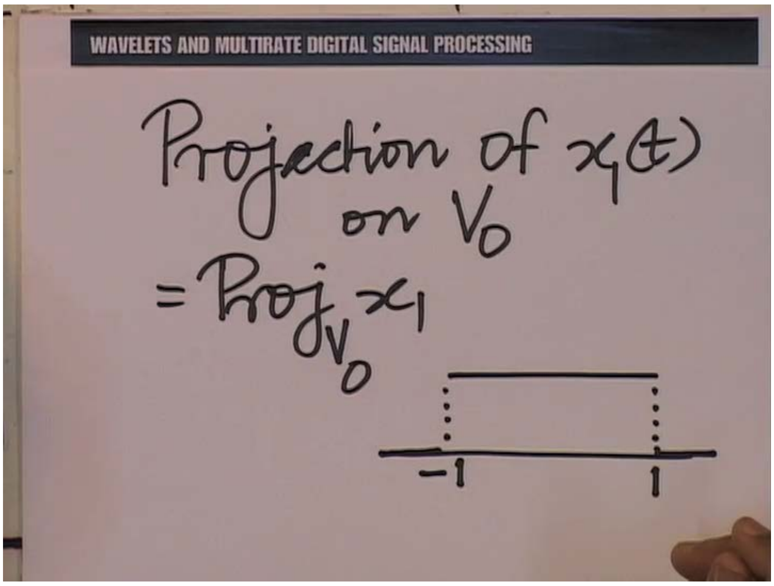

So projection, now we use this notation. Projection of x 1 t on v 0 looks like this. We will denote it by proj on v 0 of x 1 and it would look like this, essentially, a constant n equal to half between minus 1 and 1 and 0 else.

Now, we shall similarly find the projection of the function x 2 t on the space v 0. Here of course, it is going to be a semi-infinite length function. We are going to have nonzero projection values on the positive side of t and 0 projection values on the negative side of t. That is easy to see. I do not need to repeat the argument for that or the explanation for that.

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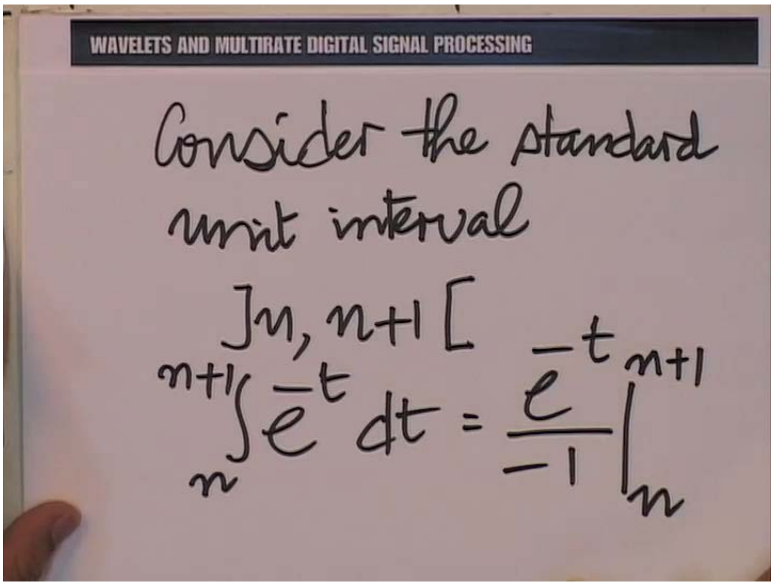

So you see, consider the interval n to n plus 1. Consider the standard unit interval. So, the average of the function, e rise the power minus t on this interval is easy to calculate. It is essentially just e rise the power minus t d t integrated from n to n plus 1. That is simply e rise the power minus t divided by minus 1, taken from n to n plus 1 and we can easily write this down.

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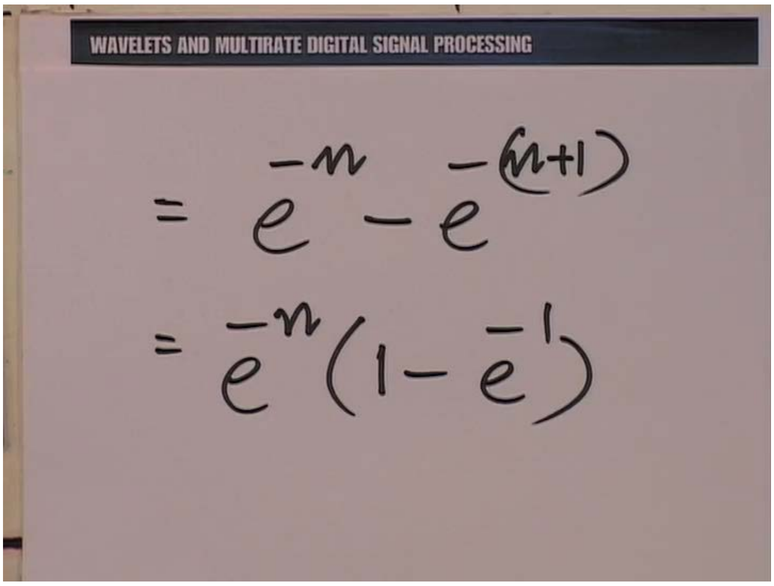

It is e rise the power minus n minus e rise the power minus n plus 1. Whereupon, we could extract e rise the power minus n common and there we are. Now, to understand this piecewise constant approximation, let us sketch the constants that we get. The approximation would look something like this.

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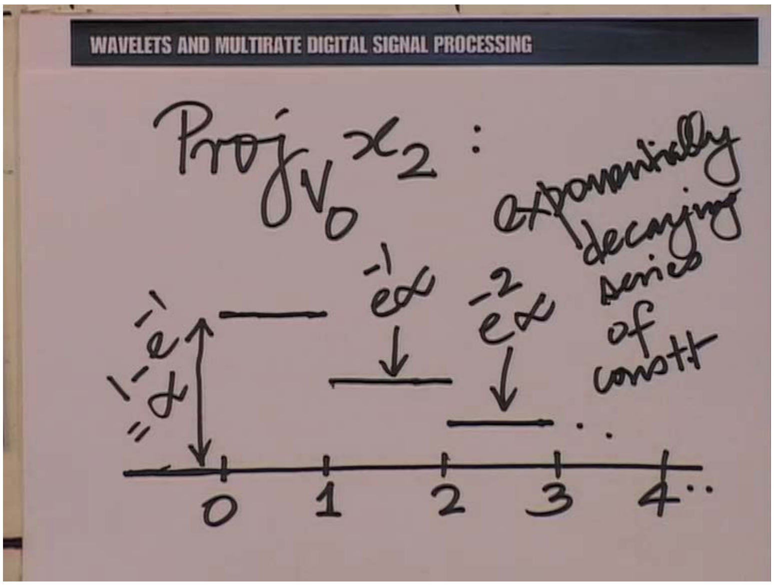

So, what I am drawing now is the projection on v 0 of the function x 2. So, I will show a few intervals here. Of course, the projection is 0 until 0. Between 0 and 1, it takes the value 1 minus e rise the power minus 1. So, this height here is 1 minus e inverse and you can get a rough feel of how much that is. That is about 1 minus 1 divided by 2.7 or something of the kind. So, one can get a good feel of how much it should be.

Now after that, the constants in the standard intervals decrease exponentially. So, let me call this value alpha. Then this is going to be e rise the power minus 1 times alpha here. So, about 1 by 2.7 times alpha. This is going to be e rise the power minus 2 times alpha. So, each time you are going to multiply by an additional e inverse. So, it is going to be an exponentially decaying series of constants here and this indeed is the projection.

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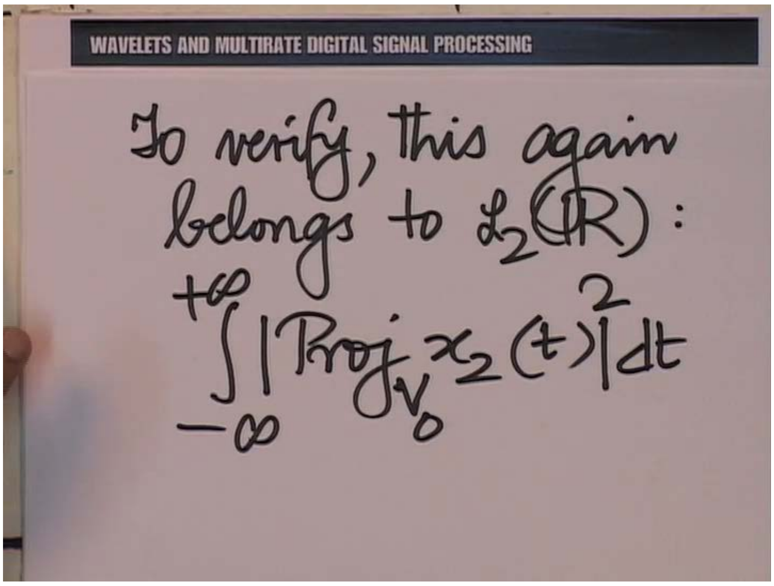

Now, just for completeness, we must verify that this indeed belongs to l 2 r as well. Though it is not surprising, it must. But just to fix our ideas, just to understand that we are doing things correctly, let us verify that this again belongs to l 2 r.

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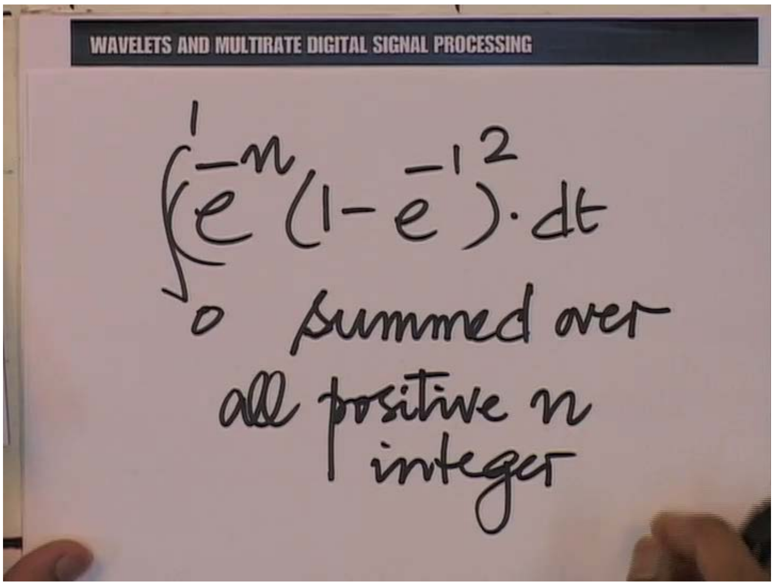

Indeed, the integral, over all t of this projection, magnitude squared d t is going to be e rise the power minus n 1 minus e inverse the whole squared integrated with respect to t from 0 to 1, let say or from n to n plus 1, essentially over unit interval and note that, this is nothing to do with t here. So, integrated over unit time could be 0 to 1 or anywhere else. This is summed over all n; positive n. So, it is very easy to see what this is.

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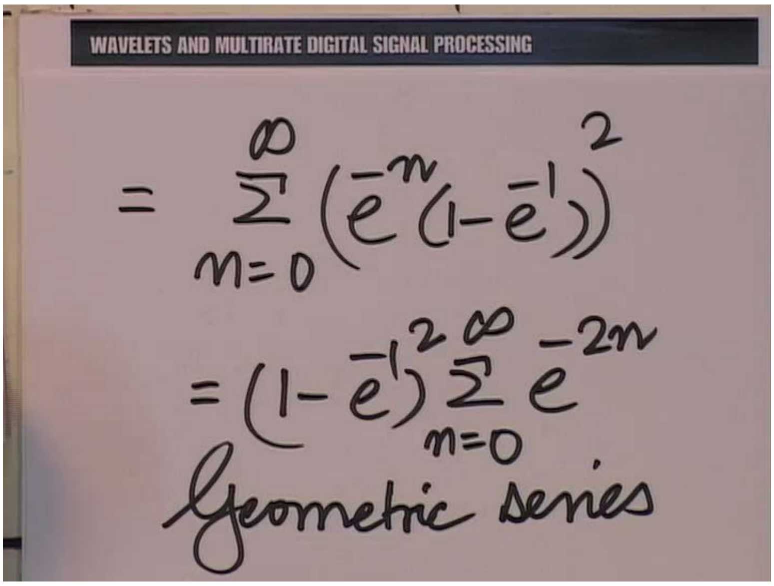

This is essentially summation over n going from 0 to infinity e rise the power minus n 1 minus e inverse the whole square multiplied by 1. That is a very easy integral and very easy summation to evaluate. It is very easy to see that this is a geometric series here. That is what I meant when I said that if you look at the constants, they form an exponential sequence on their own.

Infinitely, this is something that I wish to draw one's attention to in the context of exponentials. Even on projection of an exponential on the space's v m for all integer m, we get exponential sequences. So, exponentials in going from continuous to discrete remain exponential in nature. We are illustrating this with this example anyway.

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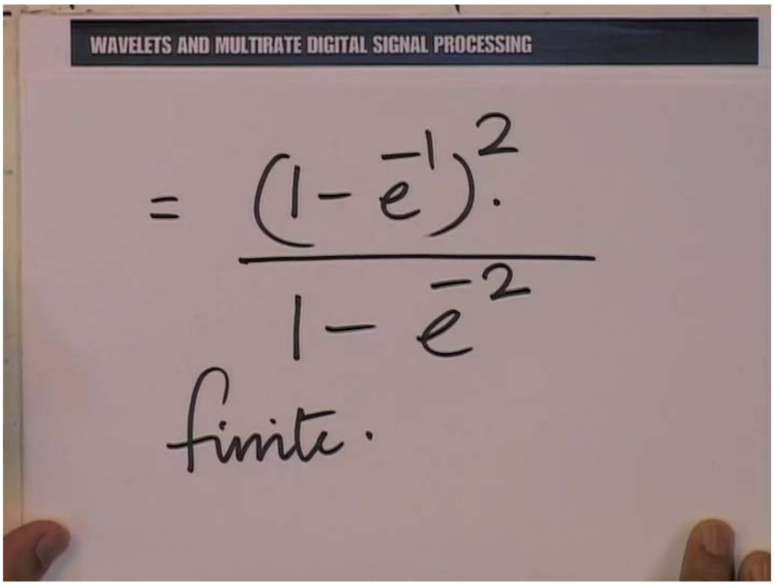

So, it is very easy to evaluate this. This is essentially nothing but 1 minus inverse the whole squared. Now, the first term in the series is 1. The common ratio is e to the power minus 2. Therefore, this is the sum and of course, this is finite. Therefore, the projection also belongs to l 2 r. Nothing that surprises us, but it is our job to verify. Now, let us go to the next question. The next question is to project these functions on the space v 1.

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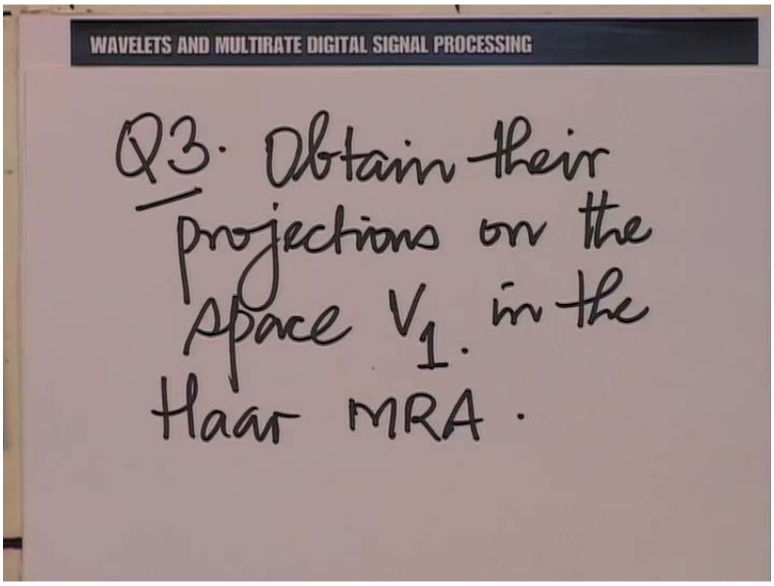

So question three is, obtain their projections on the space v 1. Of course, in the Haar multi resolution analysis. Now, I wish to draw your attention to some important issues here. You see, what we have done for v 0 and v 1 can later be generalized to any v m. But it helps to start with a specific example. That is why we are looking at v 0 and we are looking at v 1. We are intentionally looking at two successive subspaces in the ladder to understand the idea of incremental information.

So, let us not forget where we are going when we are doing this example. Now v 1, of course, you will recall is the space of piecewise constant functions on intervals of length half 2 rise the power minus 1. Of course, we have the standard intervals there. The standard intervals are the intervals 0 to half, half to 1, 1 to 1 and half, 3 by 2 to 2 and so on and so forth on the positive side and then minus half to 0, minus 1 to minus half, minus 3 by 2 to minus 1 and so on, on the negative side.

So, again I do not need to go through any discussion to convince you that before minus 1 and after 1, the projection would be 0 for the function x 1 t. For the function x 2 t, of course, the projection is going to be 0 for t less than 0. What is of interest to us is only the projection for t greater than 0 in the context of x 2 t and the projection between minus 1 and plus 1 for the function x 1 t.

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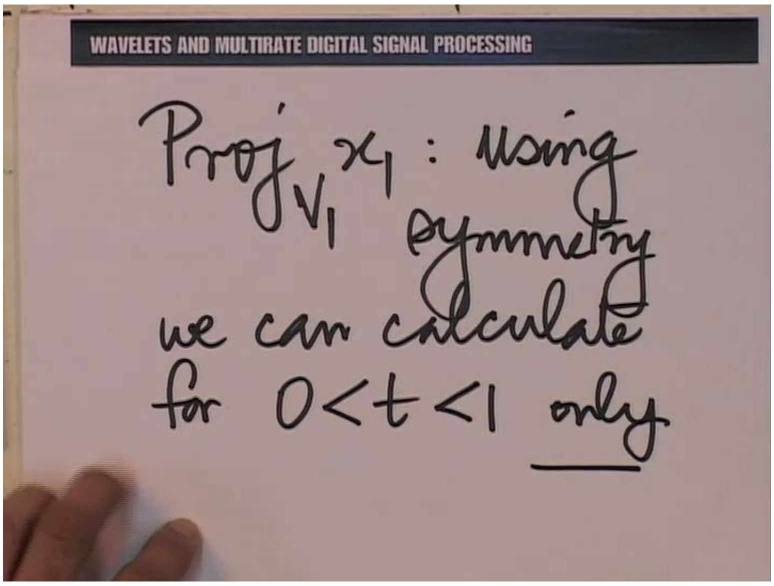

So, with that remark, let us find out the projection of x 1 t from this space v 1. Now essentially, we could use symmetry again. We can calculate only for t greater than 0. In this interval between 0 and 1, we must consider two half intervals.

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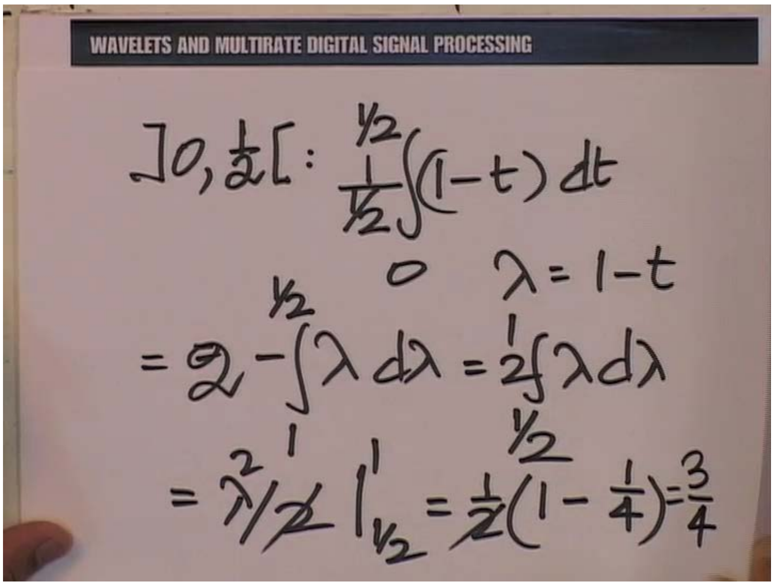

So, we consider the interval from 0 to half. That of course gives us, 1 minus t d t from 0 to half, which of course, I can simplify by using the lambda replacement. So, it is my lambda minus d lambda where lambda is 1 minus t. But remembering that you have a minus t lambda there, let me write it. So, it is minus t lambda. But then when t is 0, then lambda is of course 1 and when t is half, lambda is half too. So, it is half to 1 lambda d lambda. It is a very easy integral to evaluate. So essentially, lambda squared by 2 taken from half to 1 and that is, half 1 squared minus half square. There we are.

So now, this is the integral that we require. But remember, when we take the average, we must also divide by the interval length. So everywhere, I need to make this little correction. I need to multiply it by 2 or divide by half. So, this 2 goes away and therefore, the piecewise constant value on the interval 1 or 0 to half, the first interval is 3 by 4.

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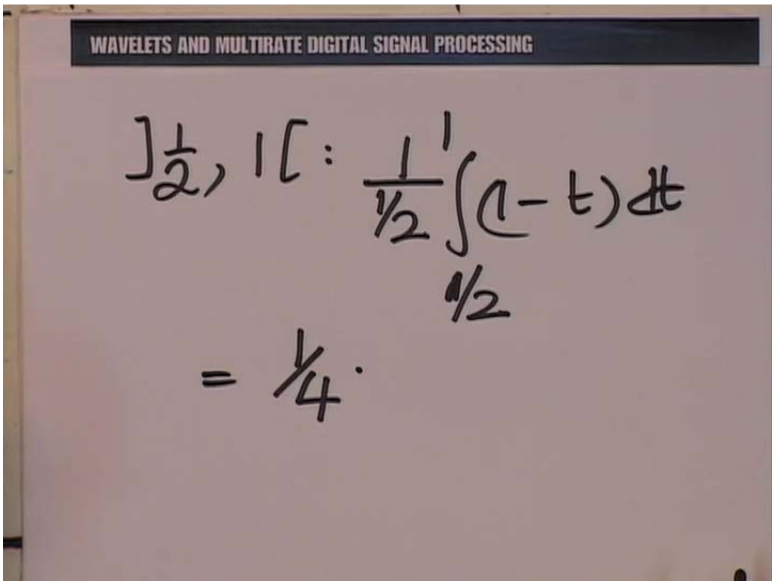

Similarly, we can calculate the piecewise constant approximation on the interval half to 1. I shall not repeat all the discussion. I will simply write down the value. It is essentially 1 by half integral half to 1 1 minus t d t, which is easily calculated to be 1 by 4. We could sketch this projection in toto now.

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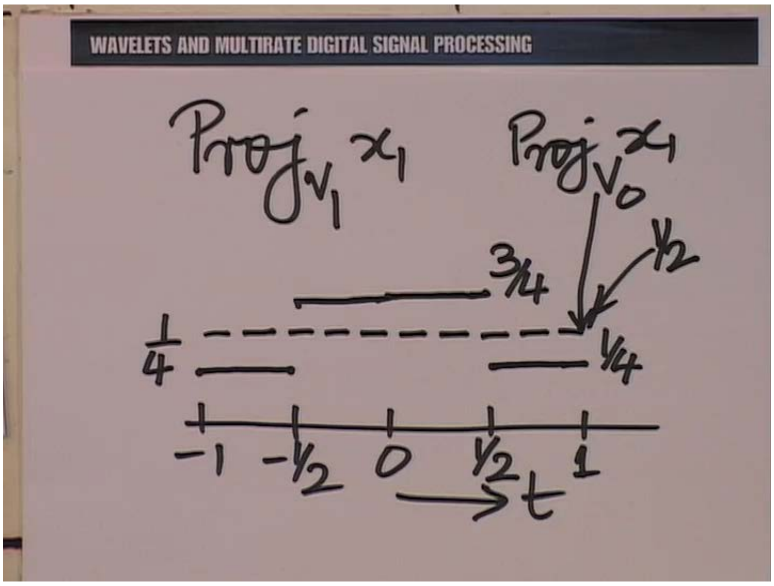

So, it is 3 by 4 there and 1 by 4 here. Some symmetry could also be 3 by 4 between minus half and 0 and 1 by 4 between minus 1 and minus half. Now, we would also like to get our ideas clear about incremental information, about taking dot products, using dot products to calculate coefficients and so on. For example, let us look at this projection here. Let us draw on the same graph the projection of x 1 on the space v 0, which I will show in dotted here. You will recall that this value is half. Now obviously, if I were to take the difference of these two functions, it would give me the projection on the space w 0, and that is the standard space w 0 in the Haar multi resolution analysis.

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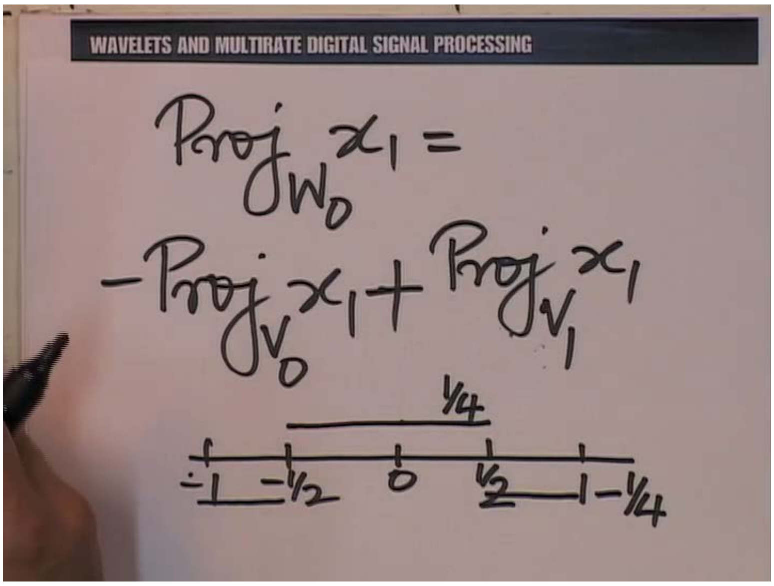

So, the projection of x 1 on the space w 0 can be obtained in two different ways. We are going to show both those ways here. It can be obtained as projection on v 0 of x 1 minus projection on v 1 of x 1. Of course, to project on w 0, you would take the negative. So, you would subtract the projection of x 1 on v 0 from the projection on v 1 of x. I can easily draw that graphically. If I were to subtract this dotted line from the solid line, it is very easy to see, I would get a function that looks like this. Let me sketch it. So, it would be positive in this part of the axis of t and negative here. Of course, the value here is 3 by 4 minus 2 by 4. So, we have, you see the value here is the difference here.

So, what is noteworthy here? Of course, we expect that to happen. Is that this difference and this difference are the same? This height and this height as expected are the same and each of them is equal to 1 by 4. So, we have 1 by 4 there and minus 1 by 4 here. It is very easy to express this in terms of psi t. That also illustrates to us towards the idea of a basis.

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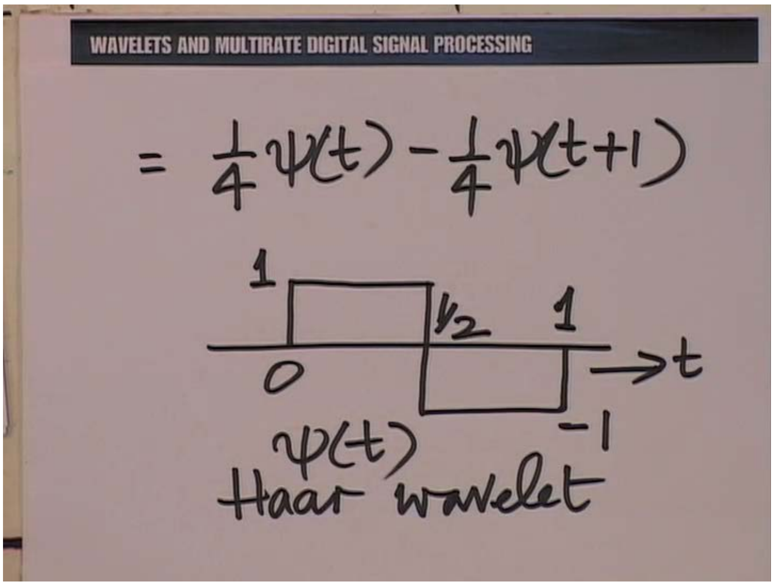

This is essentially 1 by 4 psi t minus 1 by 4 psi t plus 1. Recall that psi t plus 1 would essentially be the standard Haar wavelets. So, let me draw that. This is the standard Haar wavelet. Now, it is easy to visualize that if I take this Haar wavelet shifted, so that it occupies the interval minus 1 to 0, which essentially means writing down psi of t plus 1 and multiplied by minus 1 by 4. You get this part of the function. So, that is this contribution.

So, this term essentially contributes this part of the function here and this term here, contributes to this part of the function. So here, we illustrated the idea of the incremental subspace; that is the subspace w 0 in the Haar multi resolution analysis. Now, let us do the same exercise for the function x 2 t.

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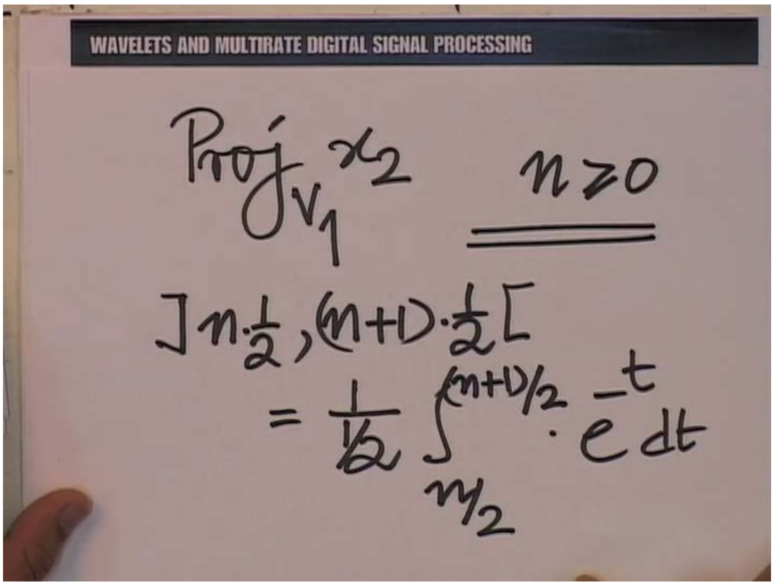

So, we will calculate the projection on v 1 of the function x 2. For this again, we need to consider the standard interval, n into half up to n plus 1 into half. So, from this interval, the piecewise constant approximation is 1 divided by half integrated from n into half to n plus 1 into half e rise the power minus t d t. Of course, needless to say, here we have n greater than or equal to 0. It is easy to calculate.

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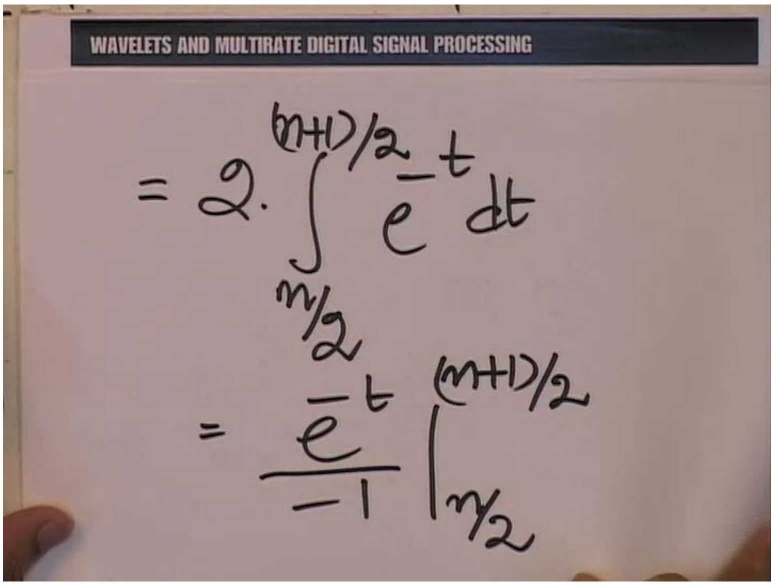

This is essentially two times integral n by 2 to n plus 1 by 2 e rise the power minus t d t and that is e rise the power minus t again by minus 1 from n by 2 to n plus 1 by 2, a very easy to integral to evaluate and a very easy substitution to make.

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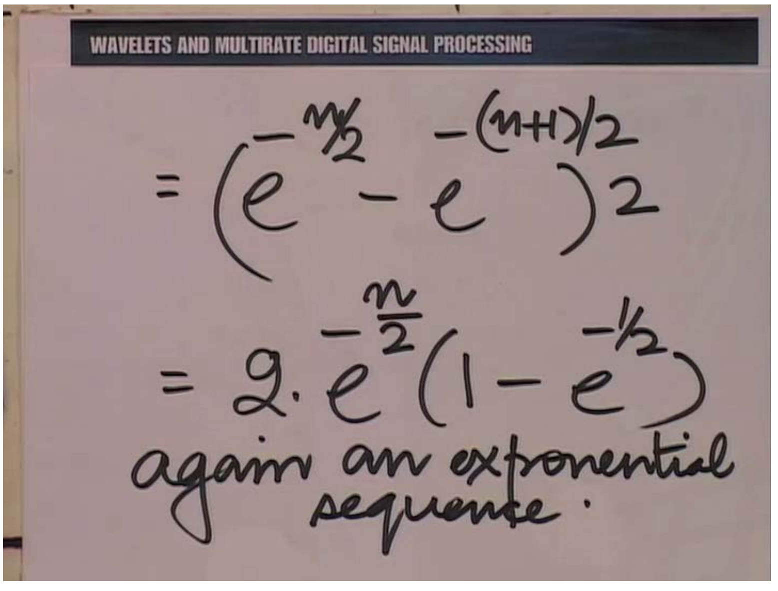

That is, e rise the power minus n by 2 minus e rise the power minus n plus 1 by 2. Of course, multiplied by 2. So, we need to keep the 2 intact, must not forget the 2. Now again, of course, one notices the exponential nature of this 2. So, this is two times e rise the power minus n by 2 taken common 1 minus e rise the power minus half, so again an exponential sequence. This further illustrates the idea that, when we project the exponential function on any of these subspaces v m in the Haar multi resolution analysis, we get an exponential sequence 2. So, exponential replicate themselves in discretization.

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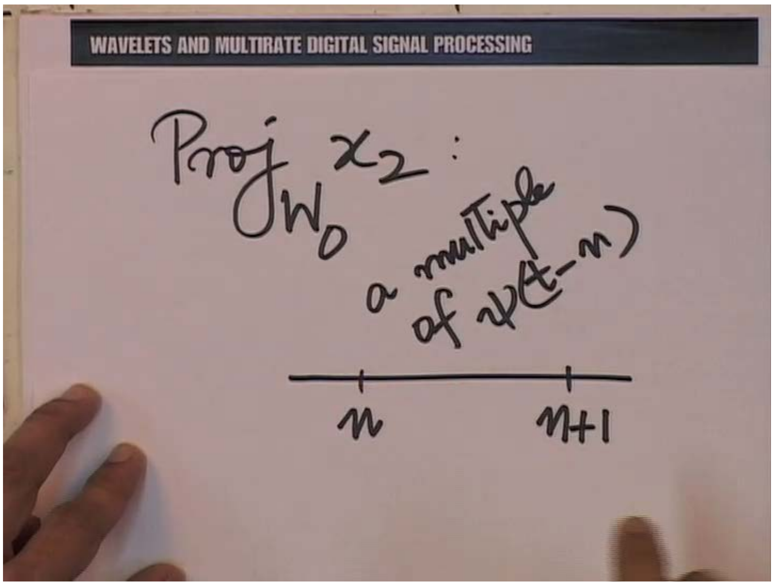

Anyway, on a similar note, we could find the projection of this function x 2 in the space w 0. Now, we need to do a little work here. Rather than just blindly calculating integrals and putting down constants, let us instead use the knowledge that we gained from the calculation in x 1 to calculate this projection in x 2. So, we need to consider the specific interval n to n plus 1. You see, we know that the projection of x 2 in the space w 0 would essentially require a multiple of psi t in the standard unit interval n to n plus 1.

So, we have to put here a multiple of psi t, in fact psi t minus n. So, we would need to translate psi t to occupy this interval here, and we would need to multiply by suitable constant to get the specific value of the projection or the function in this relation.

Now, how would we do that? You see, if you know the projection between n and n plus half and the projection between n plus half and n plus 1, we could essentially subtract the projection in v 0 from the projection in v 1. That is, the piecewise constant approximation on each of these half intervals to get the projection on w 0. So essentially, the factor by which we must multiply psi t minus n is the piecewise constant approximation between n and n plus half minus the piecewise constant approximation between n and n plus 1. So, let us write that down.

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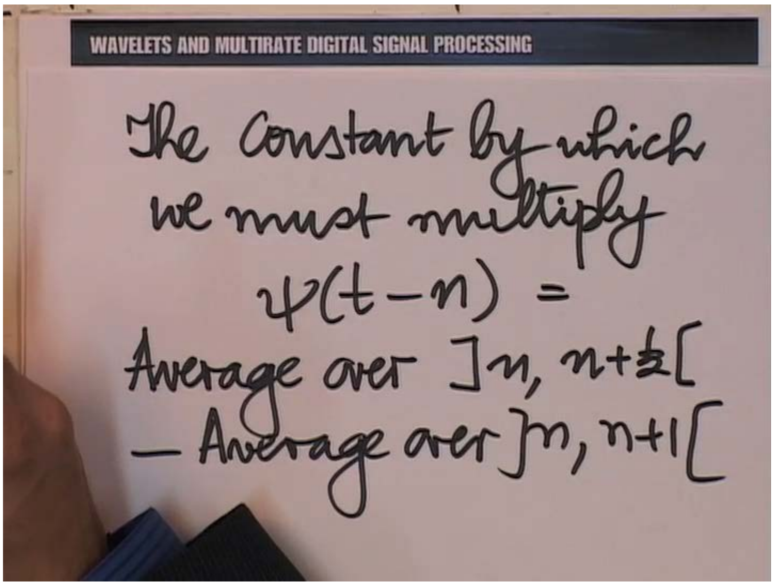

What we are saying in effect is that, the constant by which we must multiply psi t minus n is essentially the average over n to n plus half minus the average over n to n plus 1. That is easy to calculate too.

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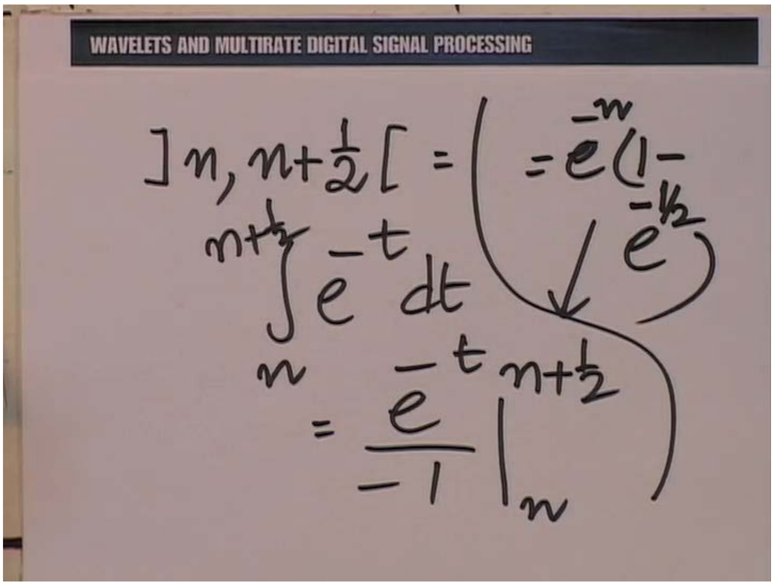

Indeed it is n to n plus half, when we average the function. It is essentially integral e rise the power minus t d t integrated from n to n plus half. That is, e to the power minus t by minus 1 from n to n plus half and we can easily calculate this. It is e rise the power minus n into 1 minus e rise the power minus half. We of course know what we average is over n to n plus 1.

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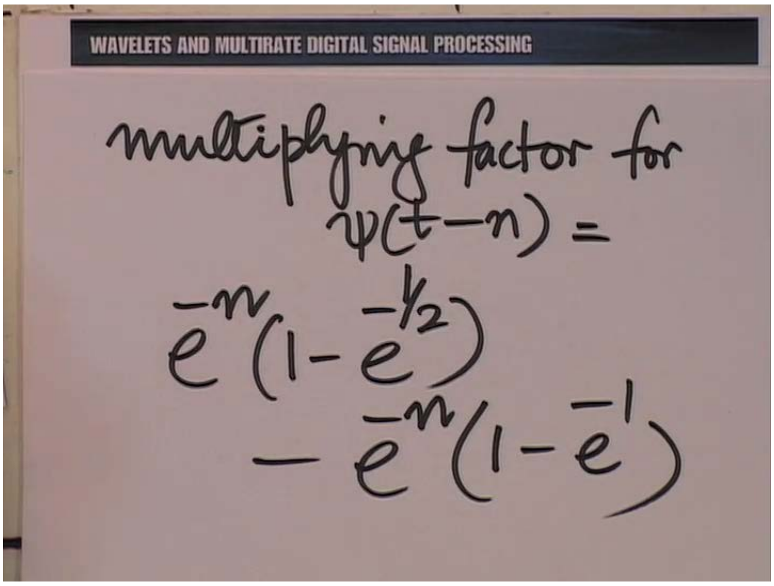

Therefore, the multiplying factor for psi t minus n is essentially e rise the power minus n into 1 minus e rise the power minus half minus e rise the power minus n 1 minus e rise the power minus 1.

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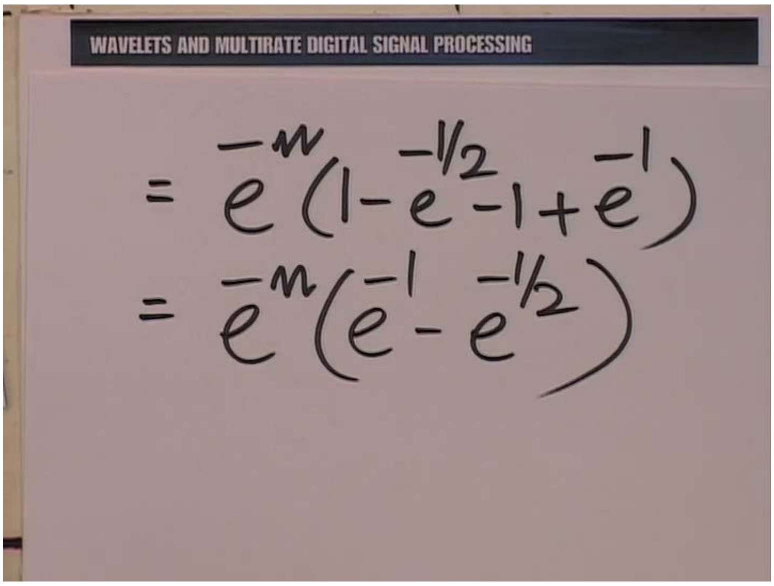

Of course, we can simplify that. It is e rise the power minus n. Now, what is important here is that, once again we get an exponential sequence.

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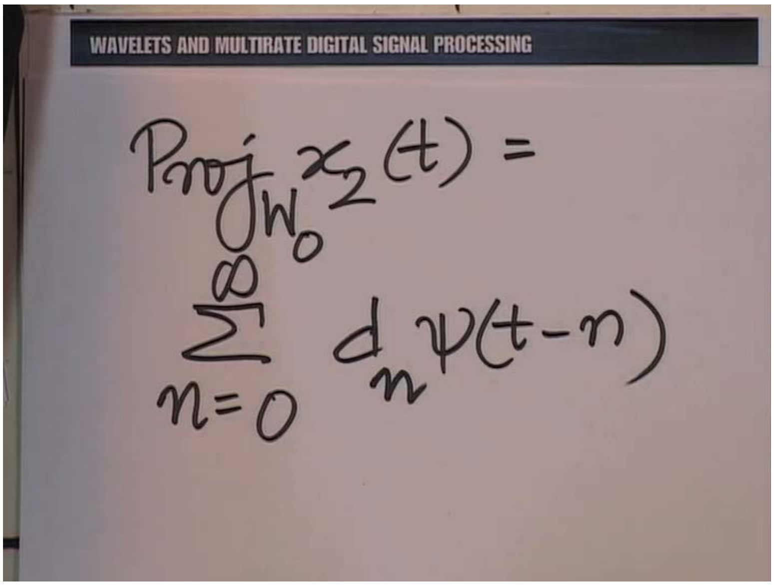

So in fact, the projection on w 0 of the function x 2 in terms of t is summation, n going from 0 to infinity psi of t minus n multiplied by this constant. Let us call this constant d n. Now interestingly, d n is again exponential in nature. In fact, again it is a decaying exponential sequence. Though not specifically important from the point of view of wavelet analysis, this is another observation once again.

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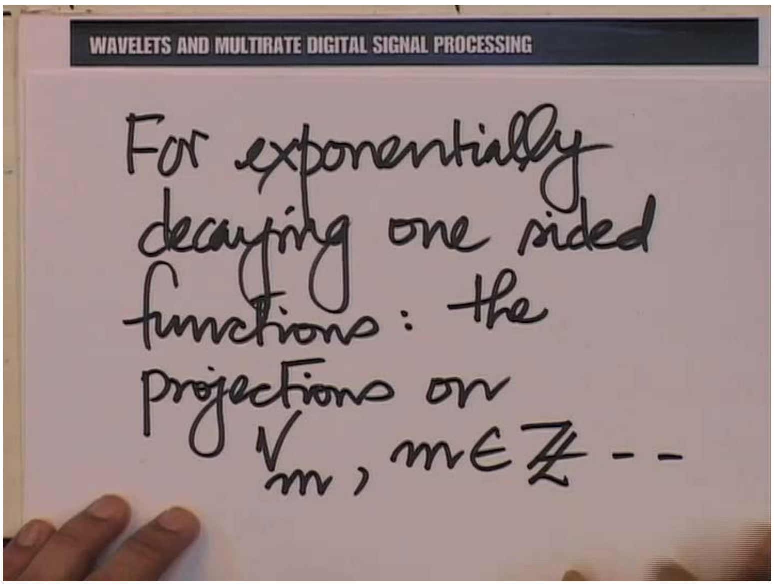

For exponential sequences, interestingly now for exponential functions or exponentially decaying functions to be more precise, we observe something interesting. The projections on all of these v ms in the Haar m r a, and the projection on the w ms are all exponentially decaying piecewise constants.

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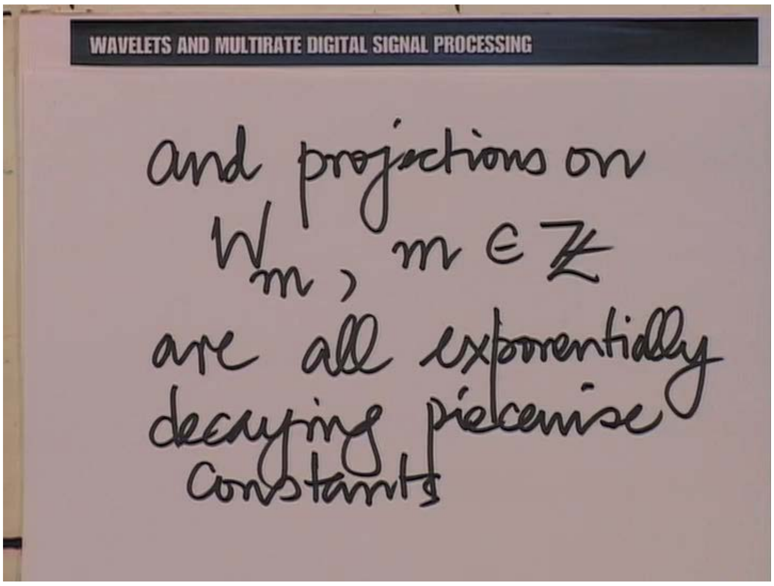

I mean, this is an informal way of saying it. What I am saying is, they are all piecewise constant as expected, but the constants decay exponentially. The projections on the vs' are actually exponentially decaying and they retain the same sign. So here, you notice that they all have the same sign. They are all nonnegative. As far as the projection on w 0 is concerned, here of course, the alternate. After all the projections, I mean the piecewise constants are positive for 1 half interval and the negative for the other half interval. That is a minor matter. But there is an exponential character there.

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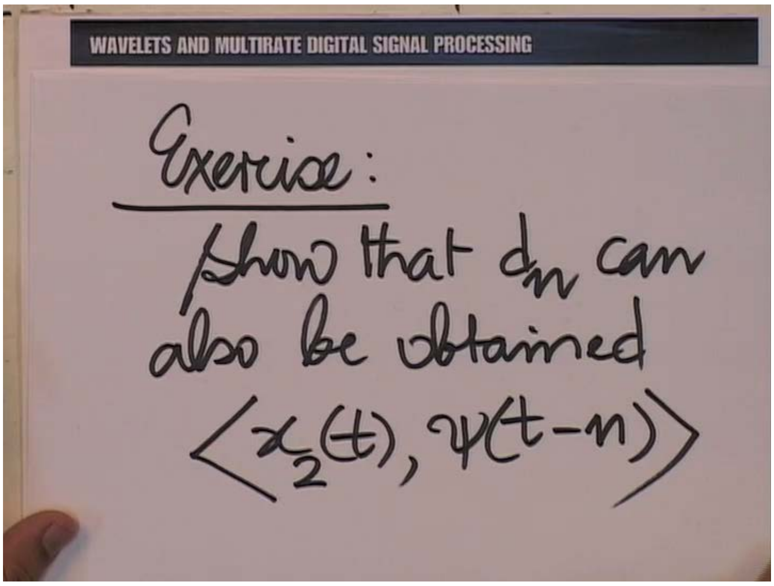

Now, I need it in an exercise. This is something that we can do very easily. Show that d n can also be obtained by taking the inner product. Essentially, the inner product of x 2 t with psi t minus n is also obtainable as an inner product. In fact, we could probably do that.

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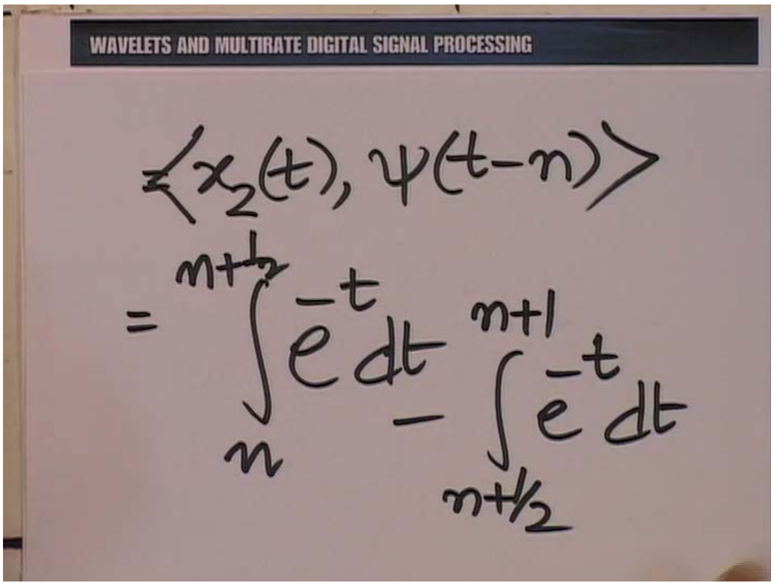

This inner product x 2 t with psi of t minus n would essentially be the integral of e rise the power minus t d t from n to n plus half minus the integral of e rise the power minus t d t from n plus half to n plus 1.

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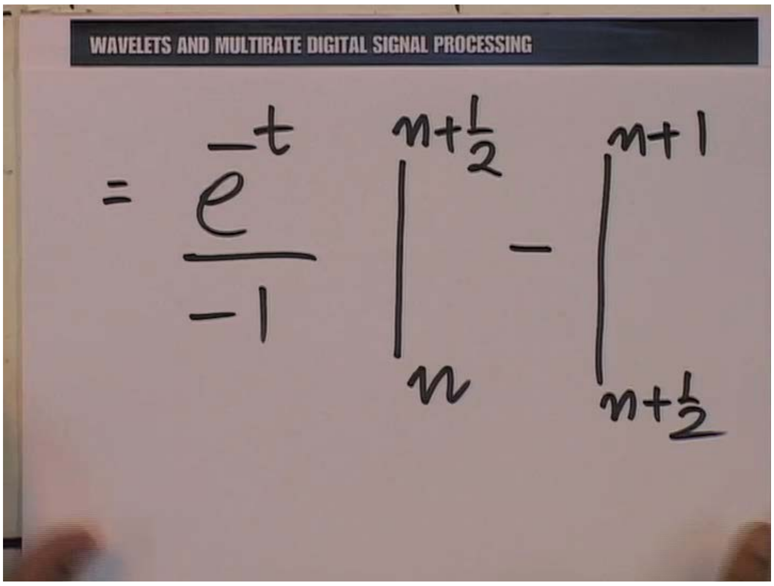

That is easily calculated to be e rise the power minus t by minus 1 integrated from n to n plus half and subtract from it the integral from n plus half to n plus 1.

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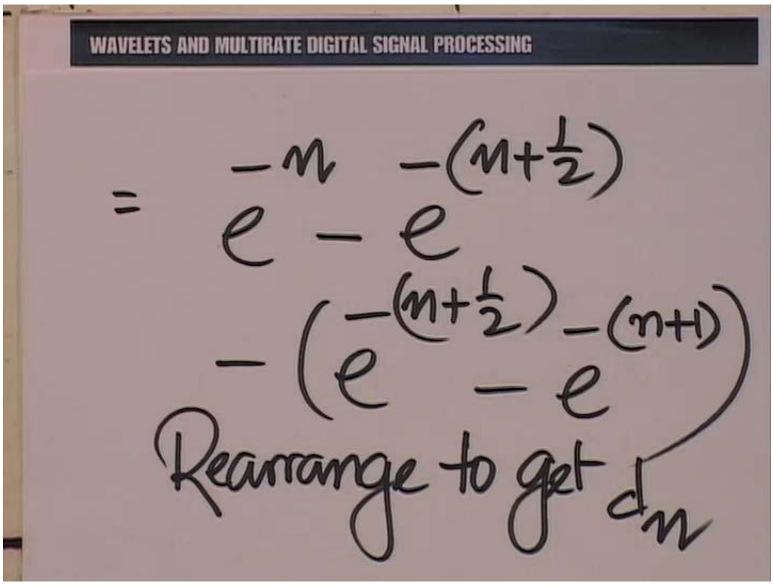

That is easily seem to be e rise the power minus n minus e rise the power minus n plus half minus e rise the power minus n plus half minus e rise the power minus n plus 1. We can rearrange this to get d n. I leave that rearrangement as an exercise. Now, I continue this process. I would like to find the projection of x 2 t on the other spaces v m and other spaces w m for both these functions.

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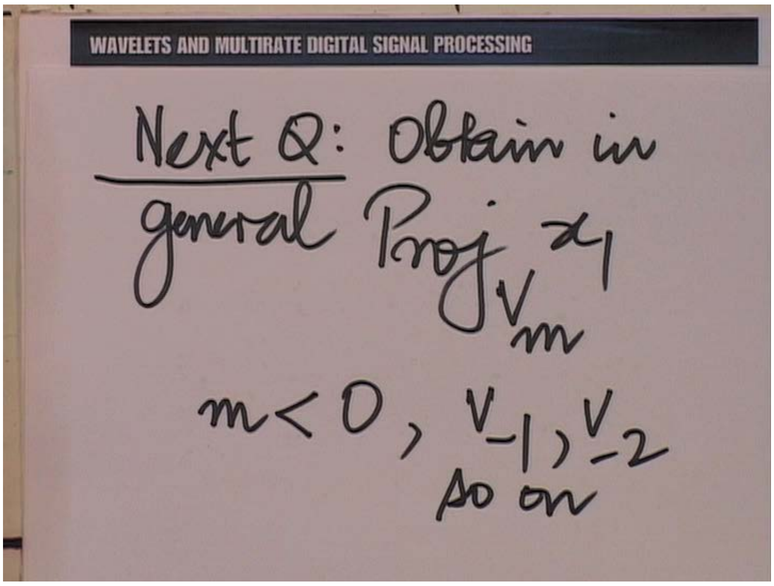

So, let us take the next exercise or the next question. Obtain in general the projection on v m of the function x 1 first for m less than 0, so for example, v minus 1, v minus 2 and so on. So, let us take v minus 1. For example, v minus 1 is essentially this space of piecewise constants on intervals of length 2. Now here, we need to take the standard intervals of length 2.

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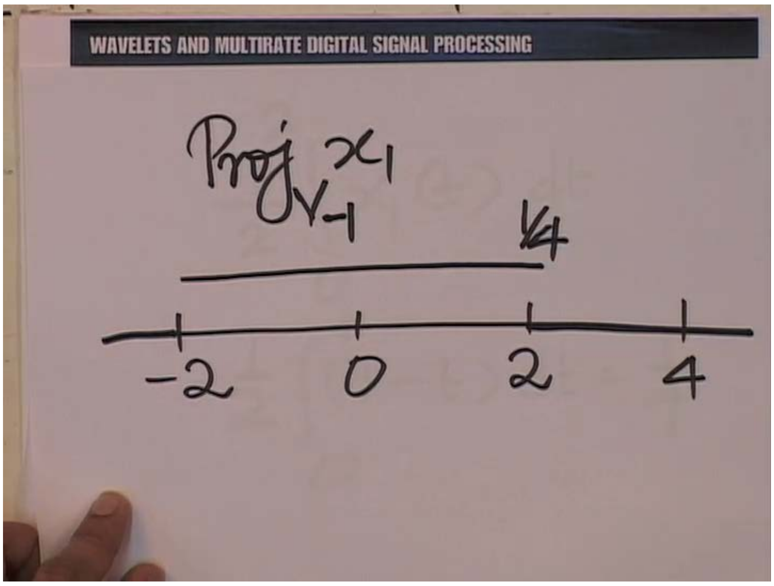

So for example, we would take the intervals 0 to 2, 2 to 4 and so on. Of course minus 2 to 0 and so on and so forth.

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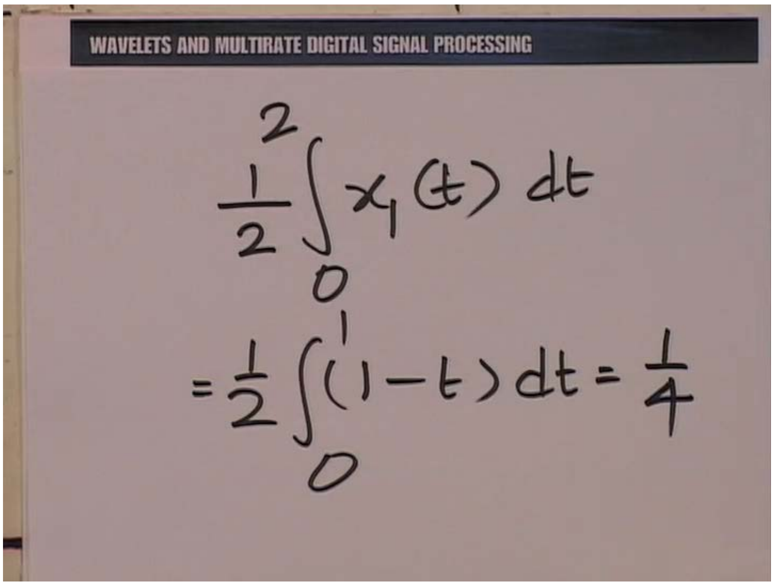

So, you see on 0 to 2, when we average the function x 1 t, what we get essentially is integral x 1 t from 0 to 2 d t divided by 2. That is half integral 0 to 1 1 minus t d t. We have calculated this before and that is half, but this would be one forth in all. So, in other words, it is very easy to see that if I take the piecewise constant approximation of the function x 1 t on the standard intervals of length 2, I would have one forth in this region and 0 outside. This is the projection of x 1 t on the space v minus 1. In fact, I could draw the most general projection here.

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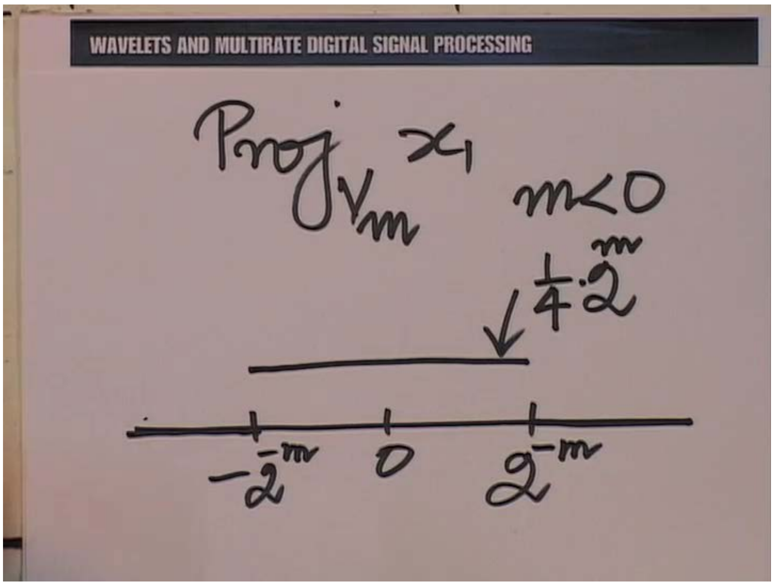

I could consider projection of the function x 1 on the space v m for any m less than 0 and we expect that that projection would be nonzero. So, let us take for example, m is equal to minus 2. It would be nonzero up to 2 square, and that is from 0 to 4. So, 2 rise the power minus m here and minus 2 rise the power minus m there. The piecewise constant value on this interval would be one forth into 2 rise the power of m and it would be 0 else.

Now, what is clear is that, one could find the projection on the w m's for m less than 0 by taking a difference. So for example, the projection on w minus 1 is the projection on v 0 minus the projection on v minus 1. So, one can go on calculating that and I leave that to you as an exercise.

Similarly, one could find the projection on v 2, v 3, v 4 and so on by taking averages on smaller and smaller intervals. I leave that to you as an exercise two. So, I shall conclude this session by putting down couple of exercises that you must do to understand this problem better. The exercises are as follows.

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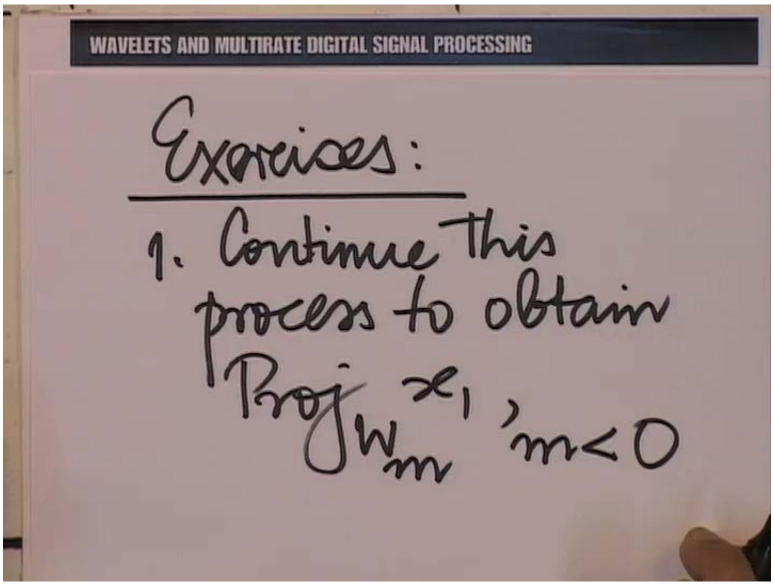

Continue this process to obtain projection on w m of x 1 for m less than 0.

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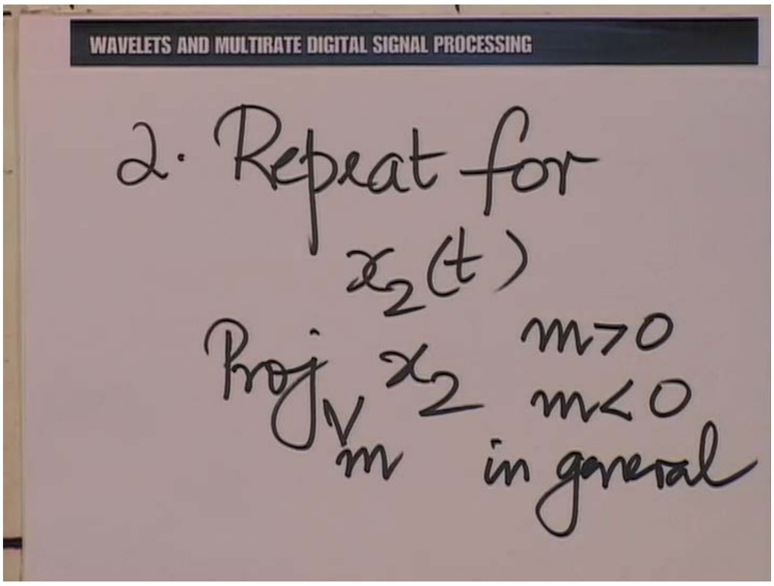

Secondly, repeat for x 2 t. In other words, find the projection of x 2 t on the space v m, where m greater than 0 and m less than 0 in general.

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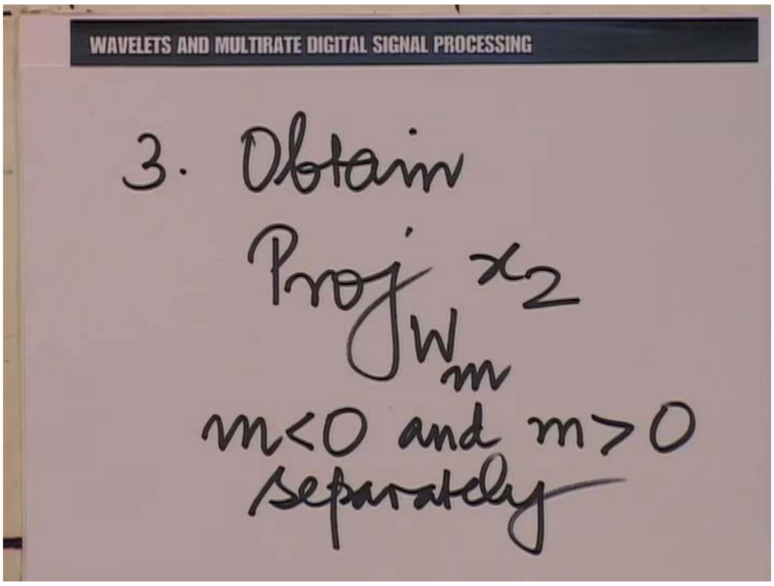

Finally, obtain the projection on the space w m of the function x 2, again for m less than 0 and m greater than 0 separately. To really understand the Haar multi resolution analysis and to grasp completely the idea of projection on successive larger subspaces, it helps to take a few examples.

I have held your hand through a couple of examples here. I do hope that you will complete these exercises to fix your understanding of this ladder. With that, we conclude this tutorial session and we hope to meet again in other tutorial sessions. So, we will have a few expositions and it is essentially sessions, where we expose the student to efforts by other students, and attempts to solve the problem by selecting a few illustrated examples based on wavelets and filter banks, so that, not only does one learn concepts, but one also learns how to apply them. We conclude the tutorial here. Thank you.