Advanced Digital Signal Processing - Wavelets and Multirate
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Lecture No. #40
M-Band Filter Banks
And Looking Ahead

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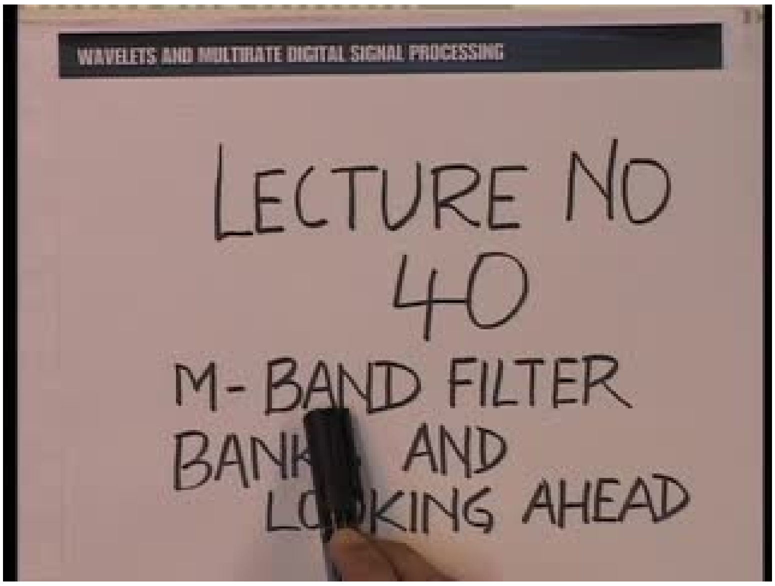

A warm welcome to the fortieth lecture on the subject multirate digital signal processing and wavelets. And in this lecture, we take over once again from an interesting set of presentation, that have taken place in the last few lectures. We intend in this lecture, as the title suggests, to look at what are called the m band filter banks, essentially a generalization of two band filter banks, and subsequently, we intend to look ahead, for apps, more appropriately we intend to look back and then look ahead; look back at where you are in the subject and then look ahead to see what more we can learn.

So, in some sense, this lecture is intended to be a kind of stop a kind of point where we sit down and reflect a point, where we look back at what we have done and we take stock of where we are and see where we can go from here.

Now, somewhere down the line, when we discussed filter banks, we made it clear, that we are actually looking at one particular stream in filter banks, where the down sampling and the up sampling factors were two. In fact, all the while when discussing filter banks, we will be looking at down sampling and up sampling factor of two, safe for one two lectures where we make some kind of generalization.

We establish some general principles, for down sampling by arbitrary factors of integer m, but we did not really analyse in depth. What we intend to do today first, is to take one new case, where the integer m is equal to 3 and analyse in depth the ideal filter bank for the case m equal to 3 or what is called a 3 band filter bank. And I believe that, if we understand the 3 band filter bank in depth, we would understand generalizations to any m. Subsequently, we shall see more and more directions, that we can explore starting from this point in our study. So, let us come to the first issue that we wish to address today, namely, the 3 band filter bank.

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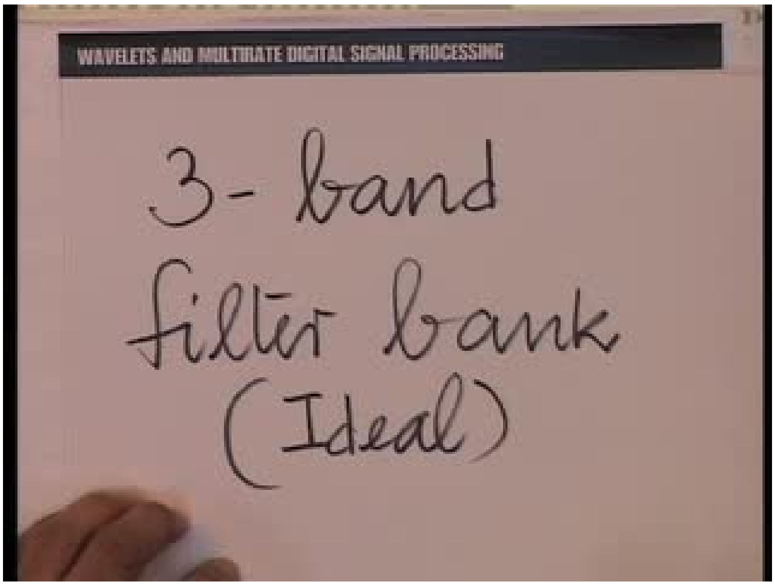

What we intend to do here is, to look at the ideal 3 band filter bank, for which I shall first put down a structure and then proceed from there. So, the ideal 3 band filter bank, of that matter any 3 band filter bank would be a structure that looks like this on the analysis side. So, first let us look at the analysis side.

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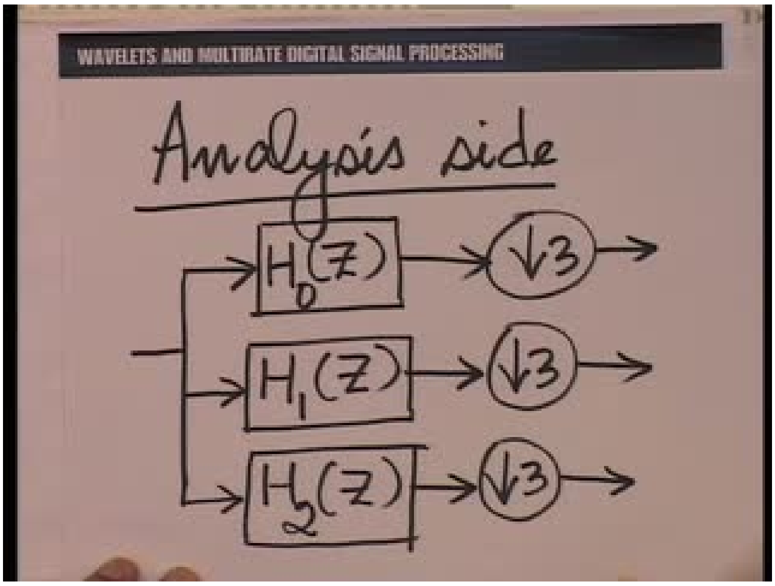

Essentially we have the same situation; we have 3 filters, that is called an H 0 Z, H 1 Z and H 2 Z, followed by a down sampling of 3, but this is true for any uniform 3 band filter bank; uniform actually refers the fact that, we have the same band, a band length of all the filters here.

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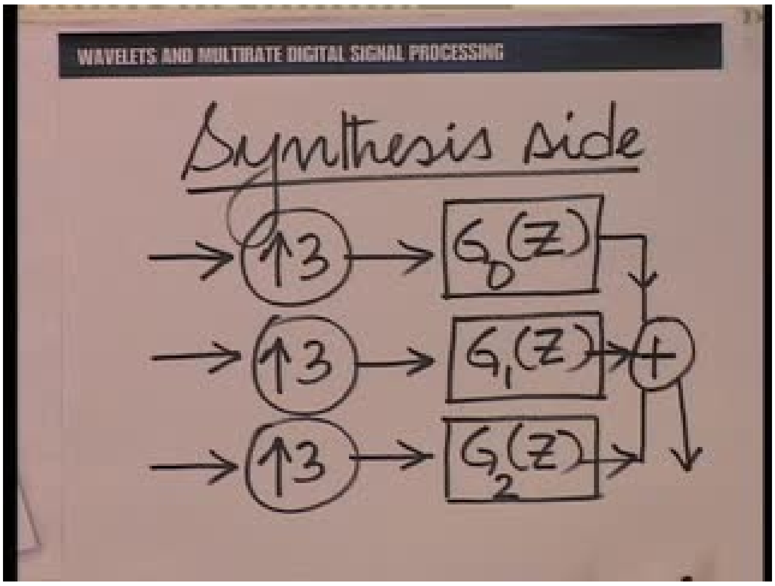

And on the synthesis side, we have in some sense, the transpose of the structure which would look like this; we have an up sampling by 3, followed by a filter and this is true for each of the branches.

The outputs are all summed here. And if on the analysis side, we call the input X, what we expect to get here, and we call this output Y is the same as X, if the system is a perfect reconstruction system; so, let us write that down.

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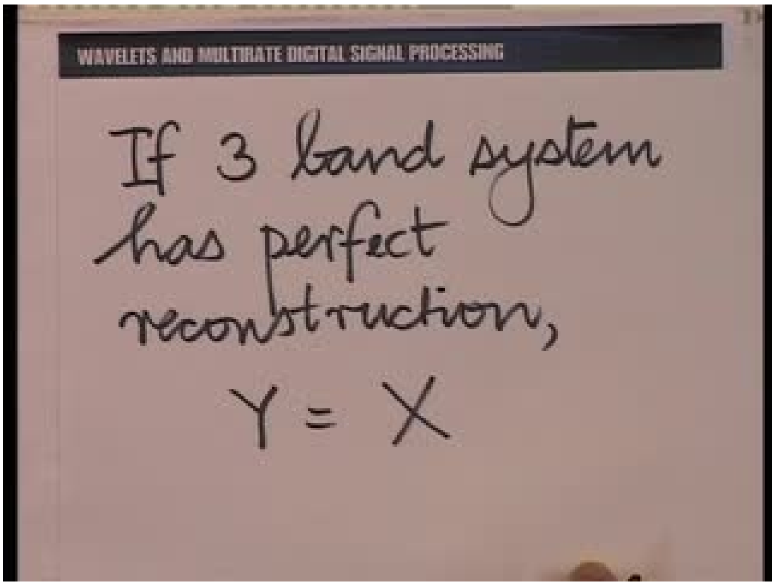

If the 3 band system is a perfect reconstruction system, Y is equal to X. When we do admit a few modifications of X and we should just recall them for a couple of minutes, we do not mind if X is scale by a constant, we do not mind if X is delayed by a constant delay; of course, what we mean is an integer delay, which operates uniformly at all times.

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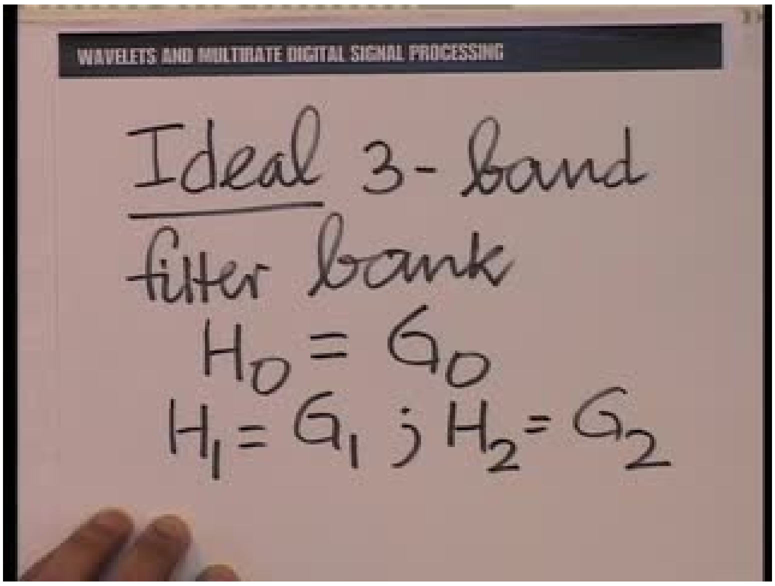

We also do not mind actually, if there is an easily revertible modification, that is not perfect reconstruction, but we can do away with an invertible modification, if it occurs with that is the subject of a different lecture. But the moment we talk about the ideal system, which is in general a perfect reconstruction system. So, what would an ideal 3 band filter bank look like? In the ideal 3 band filter bank, the first thing of course is that, H 0 and G 0 are the same and so to for H 1 and G 1, H 2 and G 2. And in fact, we can draw the ideal frequency responses of each of these filters.

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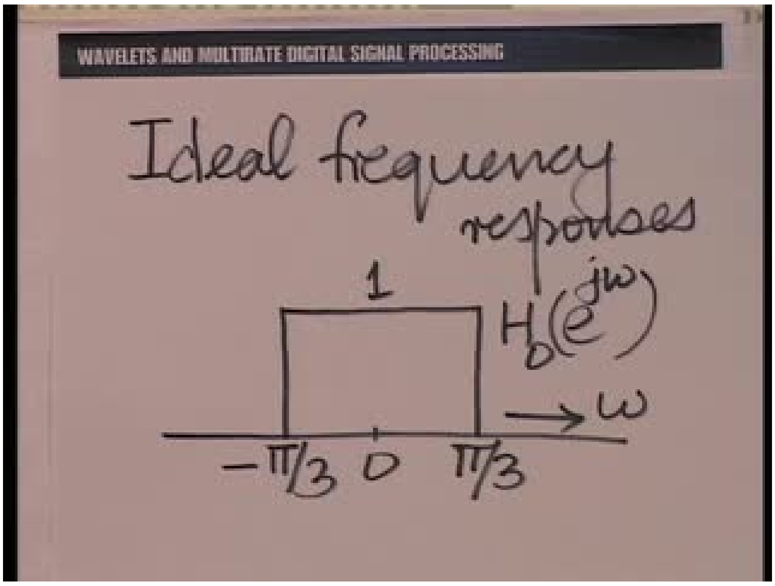

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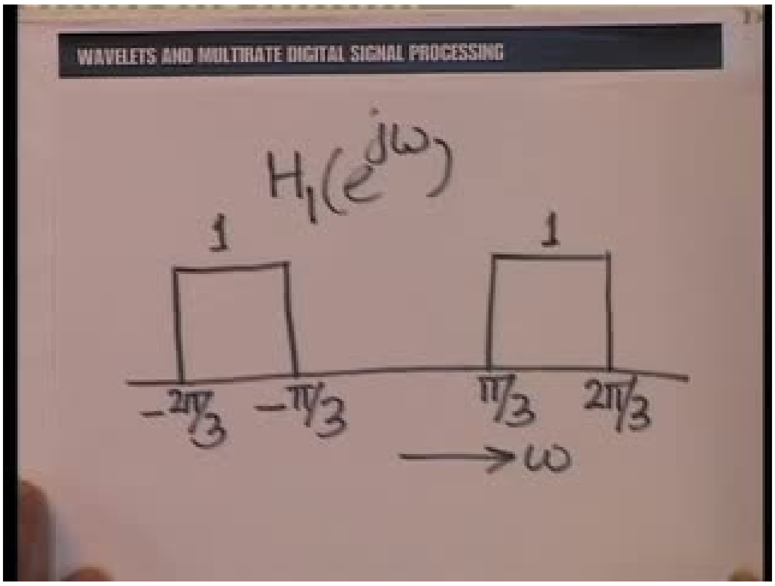

So, X 0 Z as the following ideal frequency responses, such you can see X 0 is an ideal low pass filter with the cut off of pi by 3. Now, H 1 is an ideal band pass filter between pi by 3 and 2 pi by 3.

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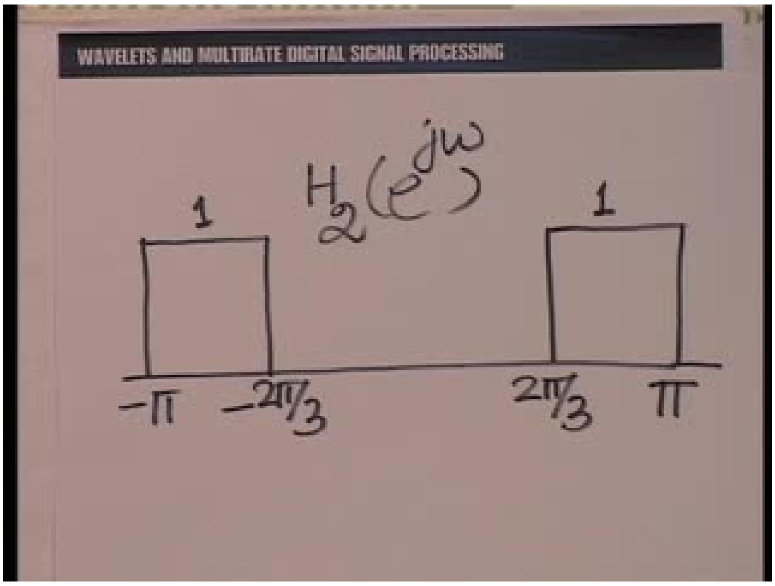

So, you could have a frequency response that looks like this. And finally, H 2 would be an ideally high pass filter with the cut off of 2 pi by 3. So, of course as we expect the 3 filters together H 0, H 1 and H 2, in some sense cover all of the frequency access from 0 to pi, the first of them is a low pass filter, the second a band pass filter and the third a high pass filter.

Essentially, we have divided the frequency access between 0 and pi into 3 parts; a generalization from the 2 band filter bank. Now, as the case of the ideal 2 band filter bank, it will help us, if we analyse this ideal filter bank with the so called prototype sequence or a prototype discrete time Fourier transform. So, we get our ideas very clear as to what happens.

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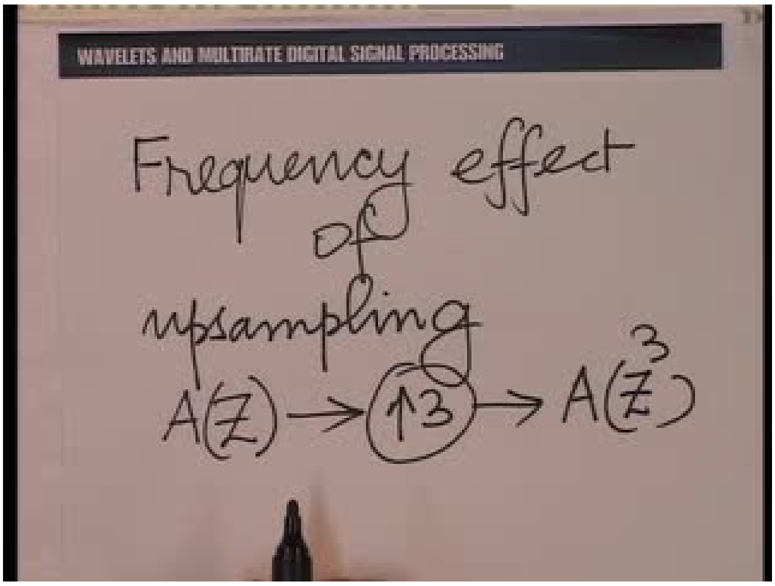

But before that, we need to see the effect of down sampling and up sampling on the frequency response. So, up sampling is easy to deal with; let us recall what happens when we up sample. So, you will recall that, if you have let us say an A Z, we infet into an up sampler by a factor of 3, what emerges is A Z cubed.

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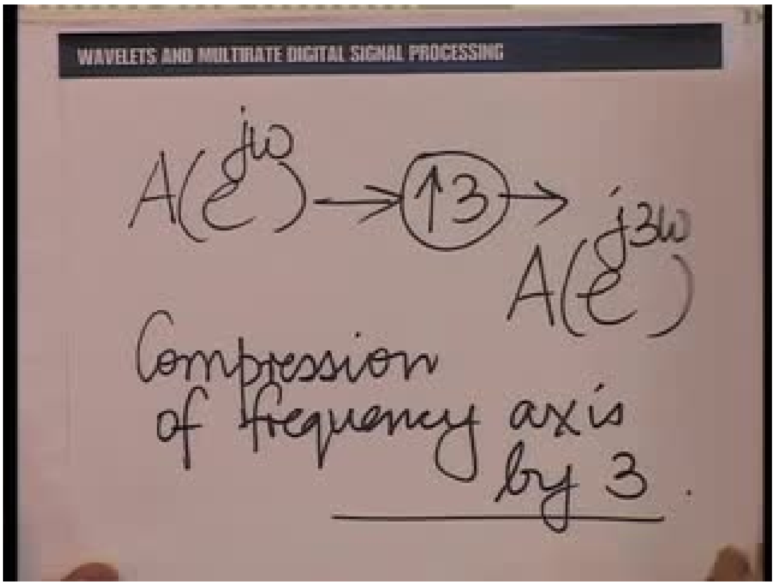

And if we look at the frequency domain for this, A inverse the power j omega is transformed on up sampling into e raise the power j 3 omega.

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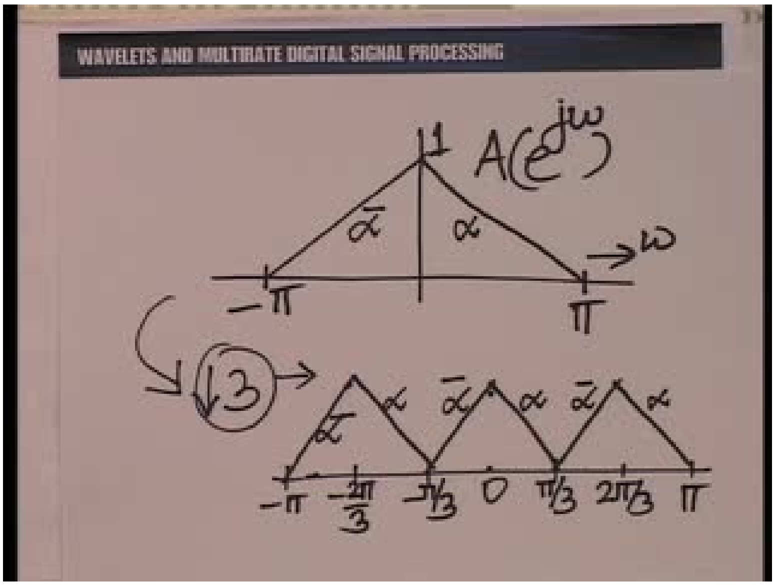

So, essentially there is a compression of the frequency access by a factor of 3; just to take an example, suppose we have the spectrum that looks like this, so this is a typical prototype spectrum as you recall; if you have the spectrum which looks like this, of course we list the same, this would be periodically repeated at every multiple of 2 pi; on up sampling by a factor of 3, we would have a spectrum that looks like this.

So, I need to of course mark out the points pi by 3, 2 pi by 3 and pi. And essentially what happens is, there is a compression by a factor of 3. So, the original spectrum is cruised to occupy the range between minus pi by 3 and pi by 3; and then, of course the same thing is brought to 2 pi by 3 and also to minus 2 pi by 3.

So, if I choose to call this part of the spectrum alpha and this part of the spectrum alpha conjugate, as would be the case when the underline sequence is real, then you have alpha here, alpha conjugate there, alpha here, alpha conjugate here again, and alpha here, alpha conjugate here again. And now, the conjugate relationship is still away, because as you can see this part and this part are complex conjugates, this part and this part are complex conjugates, and this part and this part are also complex conjugates. The complex conjugate relation is still maintained after compression by a factor of 3 so much. So, for what happens, when we up sample by 3?

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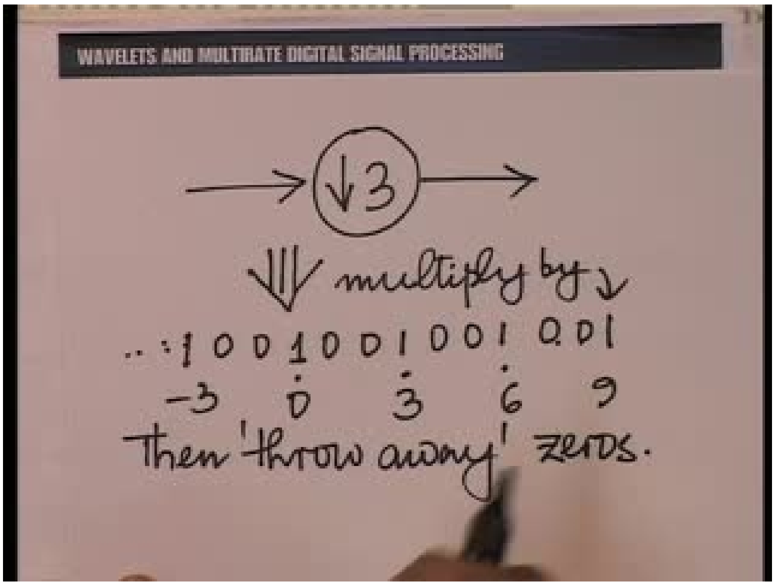

Now, we need to analyse, what happens when we down sample by 3. So, you see effectively down sampling by 3 creates aliases of the original spectrum. What we expect is that, we are going to have the original spectrum shifted and added, and in a way that is true, because you will recall that down sampling by 3 amounts to multiplying first by a sequence, which is 1 at every multiple of 3. So, 0 it is one, at 3 it is 1, at 6 it is 1 and so on, at minus 3 and so on, so forth behind, and 0 else for and so on. So, it demands to multiply first, multiply by this, then essentially throw away the 0.

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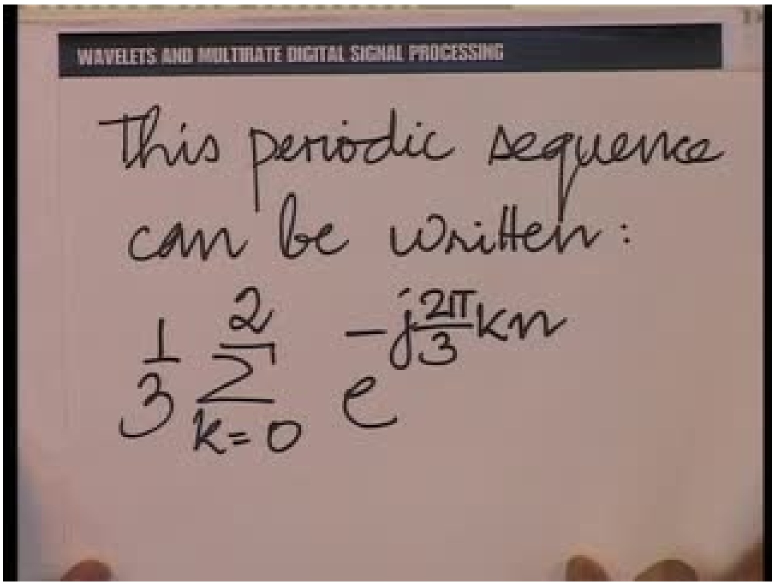

If you recall, since this is a periodic sequence, we could express it in terms of it is inverse discrete Fourier transform. If you take one period of this periodic sequence, we could take it is, we could take it is discrete Fourier transform and bring it back using the inverse, and that would give us a useful representation for the sequence in the form of modulates. So, it is easy to see, that this periodic sequence can be written, summation k going from 0 to 2 e raise the power minus j 2 pi by 3 times k n; then of course divided by 3.

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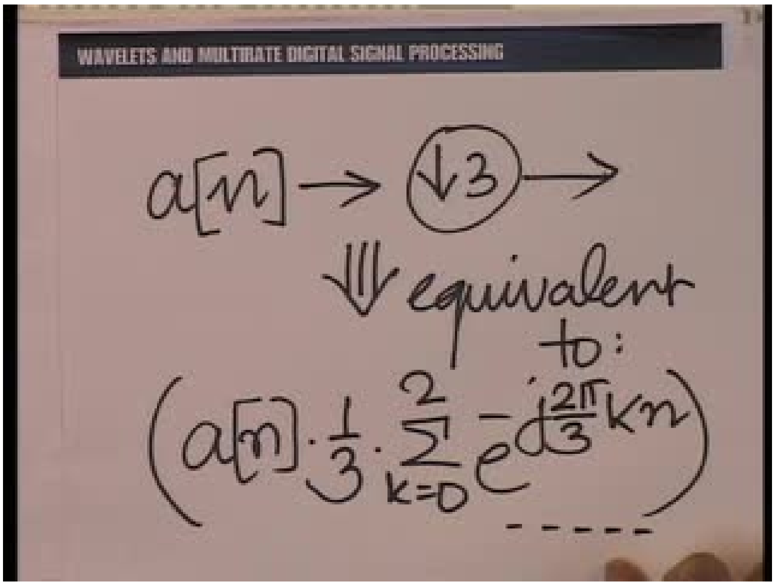

So, effectively what we have done, is to multiply the original sequence by this periodic sequence; subsequently, we reduce the Z cube to Z. So, what we are saying effectives? You have taken, if we down sample a of n by 3, it is equivalent to multiply a n by 1 third summation k going from 0 to 2 e raise the power minus j 2 pi by 3 times k n, that is easy to deal with this in the Z domain. So, we can find the Z transform of this very easily; this is not all, this is not complete.

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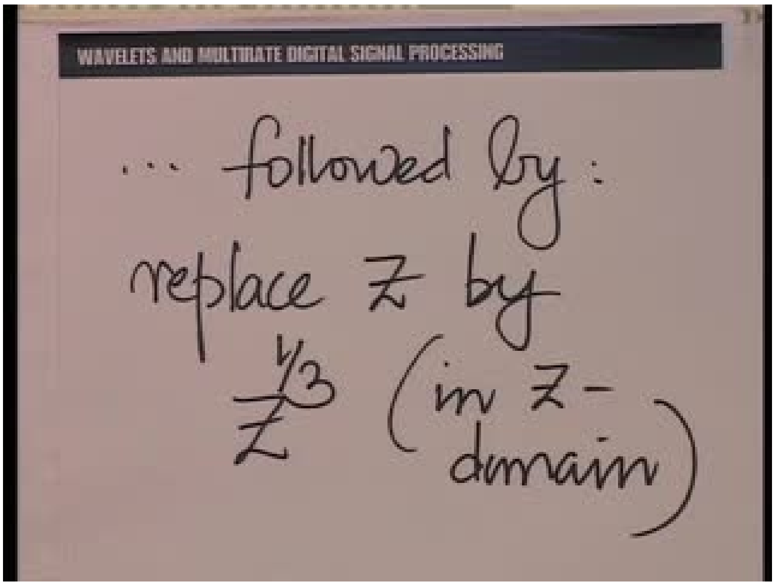

So, equivalent to multiplying by this followed by, replace Z by Z raise the power 1 third in the Z domain of course; what is what is meant by throwing away the zeros?

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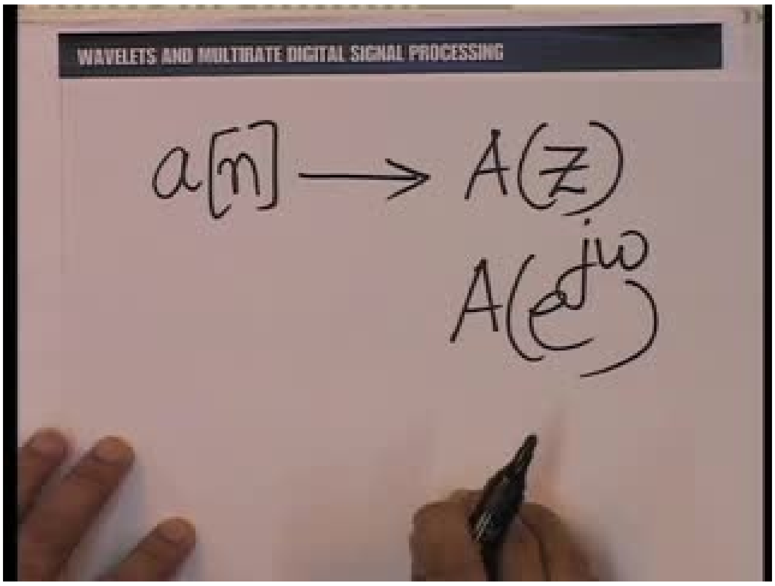

So, we recognize that, after you do this, already have a function of Z cube there. So, let us do this in the Z domain. So, if a n have the Z transform A Z, then the discrete time Fourier transform e a to A e raise the power j omega.

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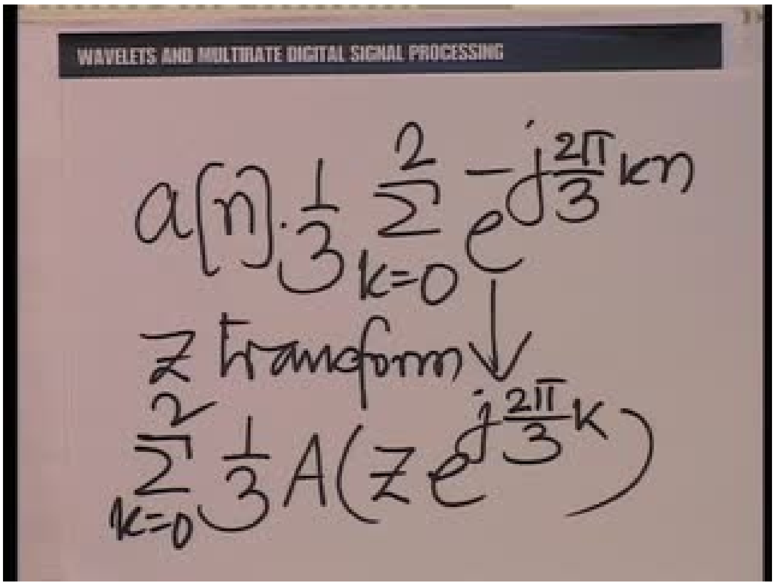

Then, a n times 1 third summation k going from 0 to 2 e raise the power minus j 2 pi by 3 times k n is going to have the Z transform, one third A Z e raise the power j 2 pi by 3 times k; that is easy to see some overall k.

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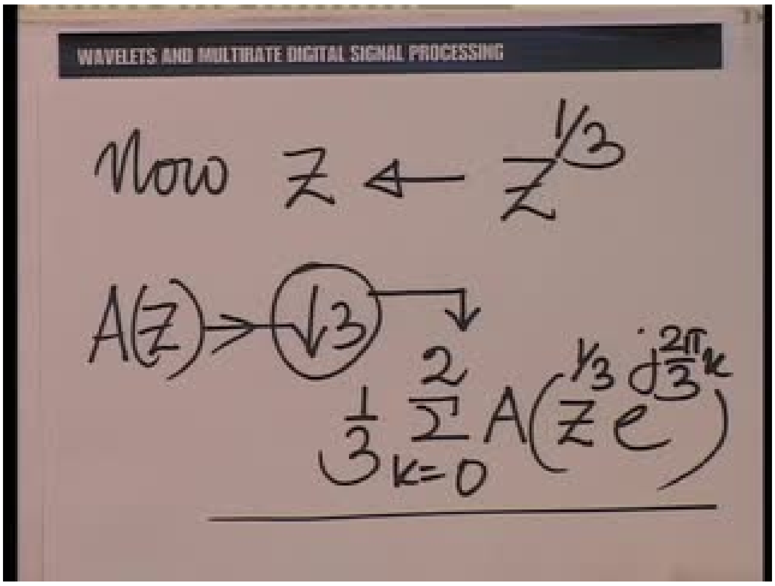

And of course, following that, what we are doing is to replace by Z by Z to the power 1 by 3; and therefore, when A in the Z domain is subjected to down sampling by 3, what we obtain is all and all, 1 third summation k going from 0 to 2 A z raise the 1 third times e raise the power j 2 pi by 3 times k.

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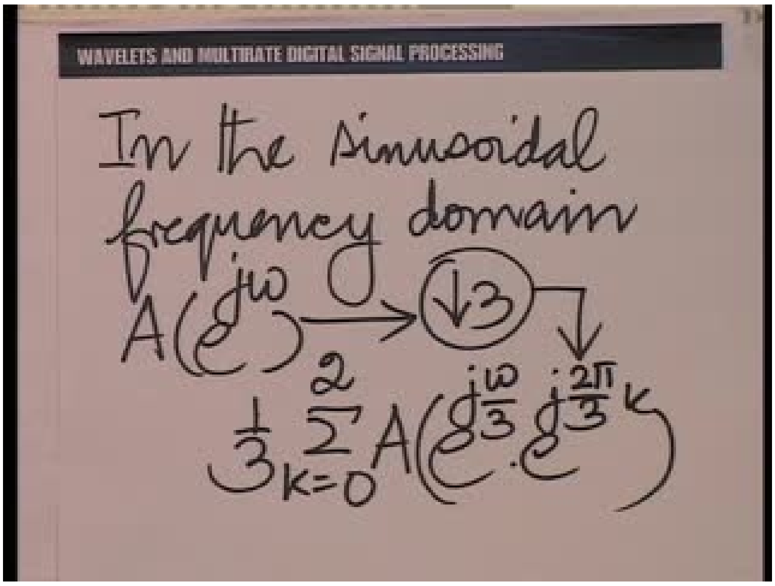

A very important result in it is own right and we need to interpret this in the frequency domain first, so in the frequency or in the sinusoidal frequency domain. So, in the sinusoidal frequency domain, A e raise the power j omega, when subjected to down sampling by 3 results in one third, summation k going from 0 to 2 a e raise the power j omega by 3 times e raise the power j 2 pi by 3 times k.

Now, we need to interpret this is the little more difficult to interpret; unlike the case of up sampling where the interpretation was very simple, we simply compress the whole thing by a factor of 3, but here, there is a summation involved, there is an expansion involved, there is a translation involved. So, we need to interpret this step by step.

So, let us look at this expression here; you see what we have here is, a e raise the power j omega by 3 e raise the power j 2 pi by 3 times k. For the moment, let us forget about omega by 3 here and let us focus our attention on this, j into 2 by 2 pi by 3 times k. In fact, if we do not write omega by 3 here, let us first write omega and see what happens.

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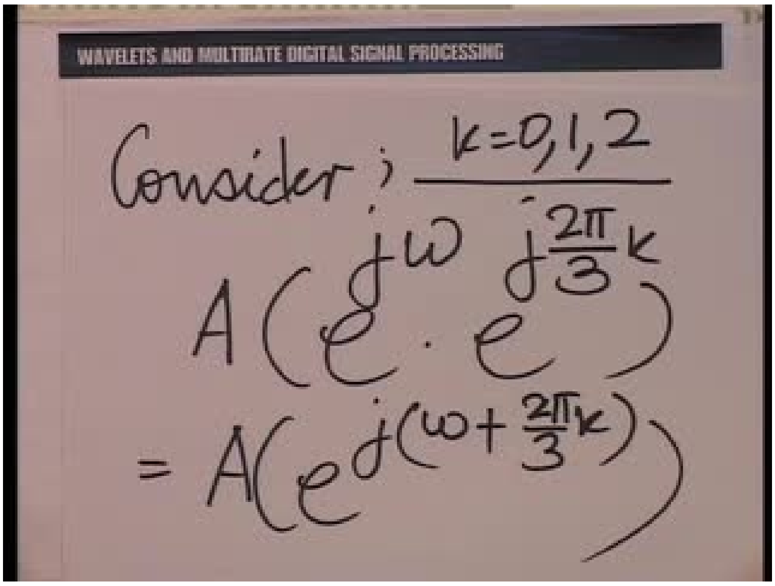

So, consider A e raise the power j omega times e raise times e raise the power j 2 pi by 3 times k is essentially a e raise the power j omega plus 2 pi by 3 times k, where k does not matters minus or plus, because anyway we are going to cover the three values 0, 1 and two modulo 3.

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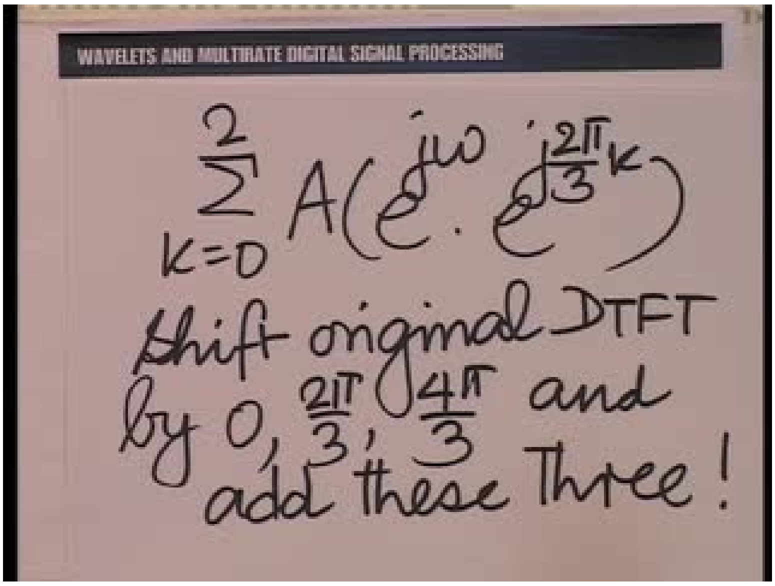

So, this essentially means, shift on the frequency access by 2 pi by 3 or multiples of 2 pi by 3. So, say, k is equal to 0, 1, 2; so, essentially, what it means is, take the original spectrum, take the original spectrum and shift it by 2 pi by 3 into 1, take the original spectrum and shift it by 2 pi by 3 into 2, and add all of these. So, summation k equal to 0 to 2 e raise the power j omega e raise the power j 2 pi by 3 times k, essentially means, shift the original spectrum - original DTFT - by 0 2 pi by 3 and 4 pi by 3, and add these three shift inverse of the spectrum.

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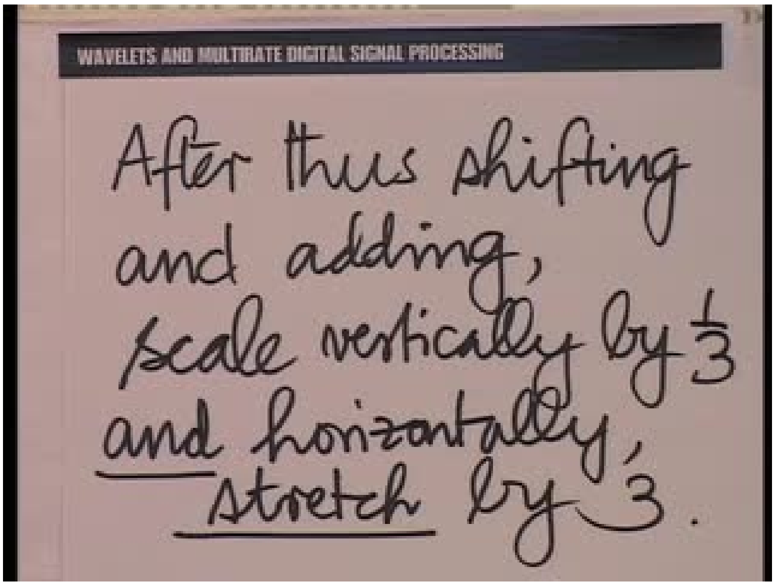

Now, multiplication by 1 by 3 here is a minor issue; it only means scale the spectrum by a factor of on the e by 3, but now the last step is to replace omega by omega by 3 and that is essentially an expansion step. So, what we are saying is, after thus shifting and adding, scale vertically by 3 rather by 1 third and horizontally stretch by a factor of 3, and that completes the frequency domain effect of down sampling by 3.

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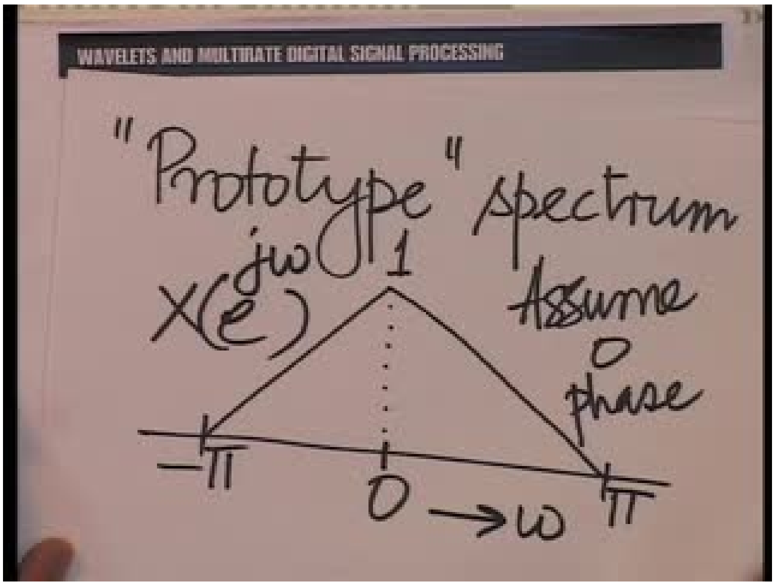

Now, to fix our ideas, let us what happens on the low pass branch, when we take the prototype spectrum. So, with the prototype spectrum, what I mean by a prototype spectrum is a spectrum which is very easy to understand, where we have distinct frequencies with distinct amplitudes. It is the simplest case of a spectrum, where we have distinct amplitudes for distinct frequencies and we have omitted the phase altogether. So, we have a straight line from going down from 0 frequency to pi and to minus pi on the other side, and we assume 0 phase to make matter simple; let us call this the underline sequence here, X of n. So, we have the spectrum capital X e raise the power j omega.

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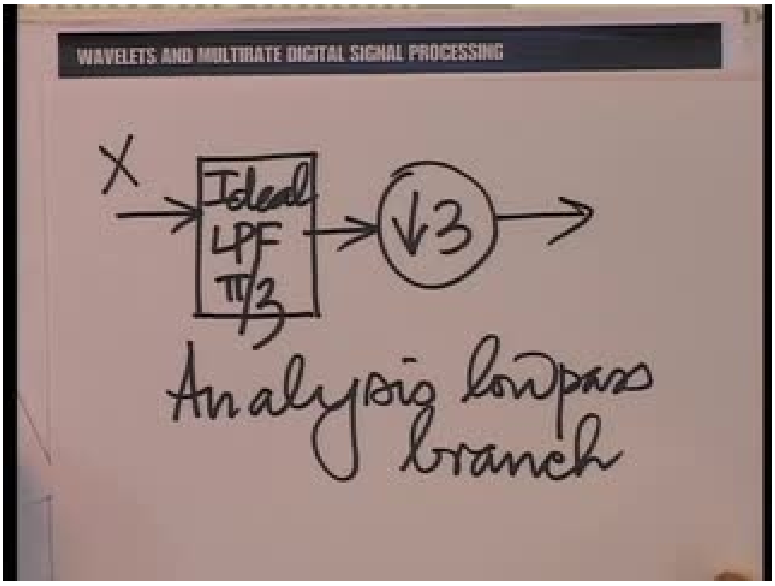

And we subject this spectrum to the low pass branch in the ideal three band filter bank. So, low pass branch is essentially low pass filter - ideal low pass filter - with the cut off of pi by 3, followed by a down sampler; and then, of course we have the analysis and synthesis side following the analysis side; this is the analysis side, but we will do the analysis of the synthesis side afterwards.

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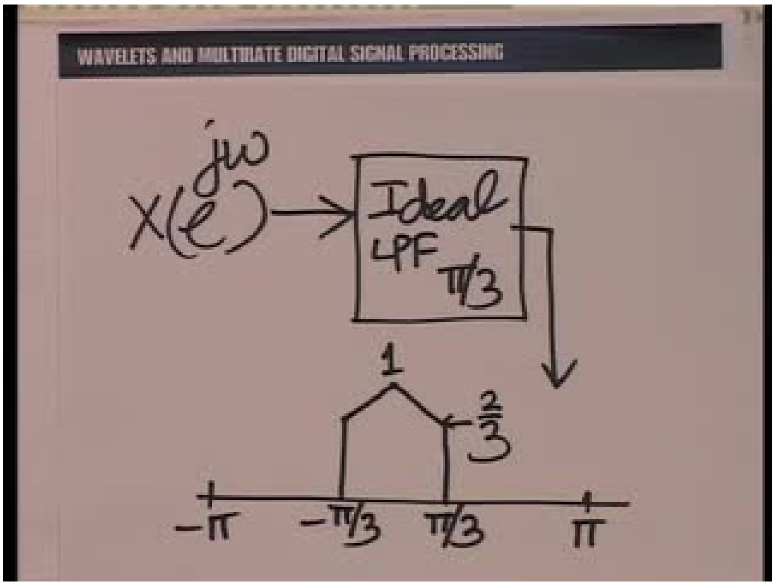

Now, we look at the synthesis part afterwards; let us see what happens, when we subject x to this part of the analysis filter bank. So, when X e raise the power j omega is subjected to the ideal low pass filter, we know what happens with the cut off of pi by 3; the out coming spectrum looks like this with a one here, and this is of height 2 by 3. Now, let us see how this would react, when subjected to the down sampler by a factor of 3. So, when down sampled by 3, what happens? Let us construct it piece by piece.

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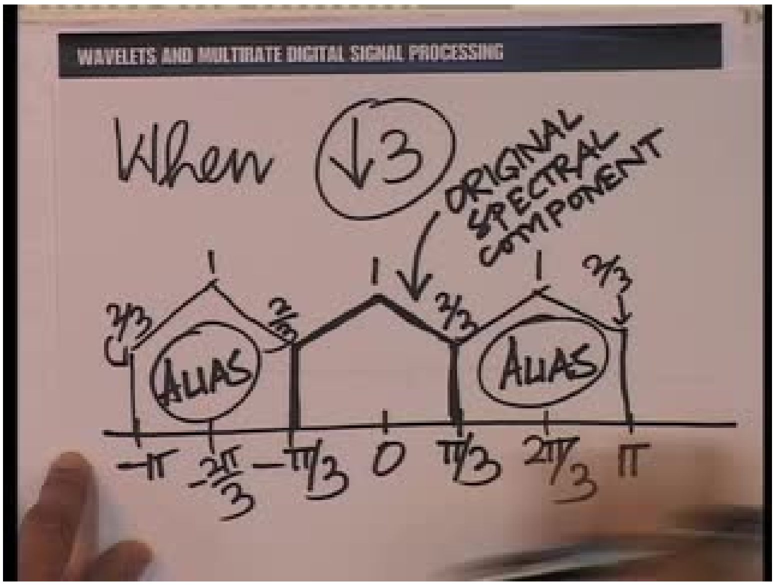

So, we have this original spectrum so to speak, which goes down to 2 by 3 here. Now, the expected it is translated to every multiple of 2 pi by 3; you must visualize that, this is repeated periodically at every multiple of 2 pi. So, what we have between minus pi and pi is also present between pi and 3 pi, is also present between minus 3 pi and minus pi. And for the moment, since we are now custom to dealing with this periodic discrete time Fourier transform, we shall straight away write down the result. So, of course, the shift by 2 pi by 3 is easy to visualize. In fact, what happens is, that this is brought, instead of 0 to 2 pi by 3 and there we go.

But what is also equally easy to understand, which is little more difficult to understand, is that, if when we shift by 4 pi by 3, what we are actually doing is to bring this back to lie between minus pi and minus pi by 3. So, shifting forward by 4 pi by 3 is equivalent to shifting back by 2 pi by 3, that is because all the shifts are modulo 2 pi. Remember, that is the periodicity of 2 pi in the discrete time Fourier transform.

So, all our shifts are modulo 2 pi, they are circular in that sense. So, shifting forward by 4 pi by 3 is equivalent to shifting backward by 2 pi minus 4 pi by 3, which is 2 pi by 3 again. So, what you obtain by shifting by 4 pi by 3 is the same spectrum brought back to lie between minus pi and minus pi by 3.

So, this is what happens, when we down sample by 3 on the low pass branch? And now, we could take a minute to identify the original spectrum, this is clearly the original spectrum so to speak - the original spectrum component, and this is an alias here and shows this. Now, this is not odd; after doing this, we do two more things; the first thing is to scale vertically by a factor of 1 by 3; And secondly, to scale horizontally by a factor of 3, which means to stretch by a factor of 3; let me take both the steps at once.

So, we get essentially this 1 becomes 1 third, 2 third becomes 2 9th, and 2 9th all the way there. Moreover, this pi by 3 is brought to pi, minus pi by 3 is brought to pi and all of these are then shift similarly.

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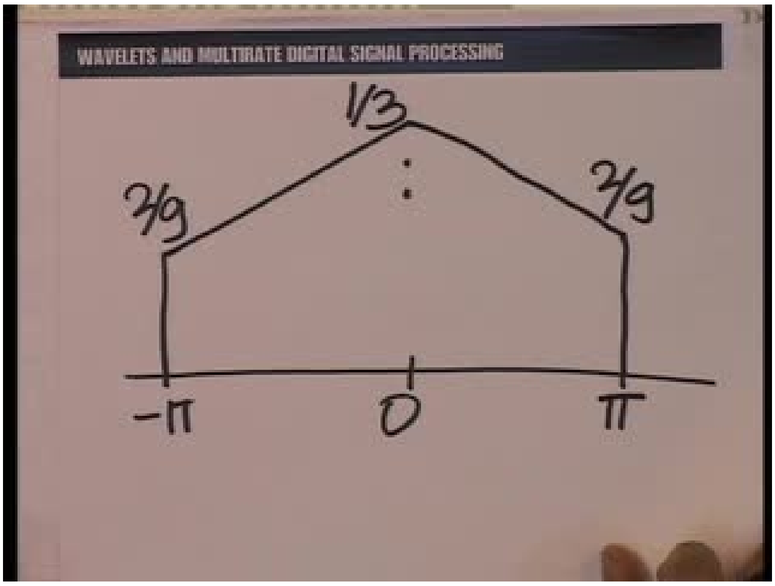

So, what we get all in all after down sampling by 3 is the following. This is 1 by 3 there, this is 2 by 9 and there is a stretching here. Now, it is easy to see, what happens when we subjected to up sampling by 3; so, let me up sample this by 3 again.

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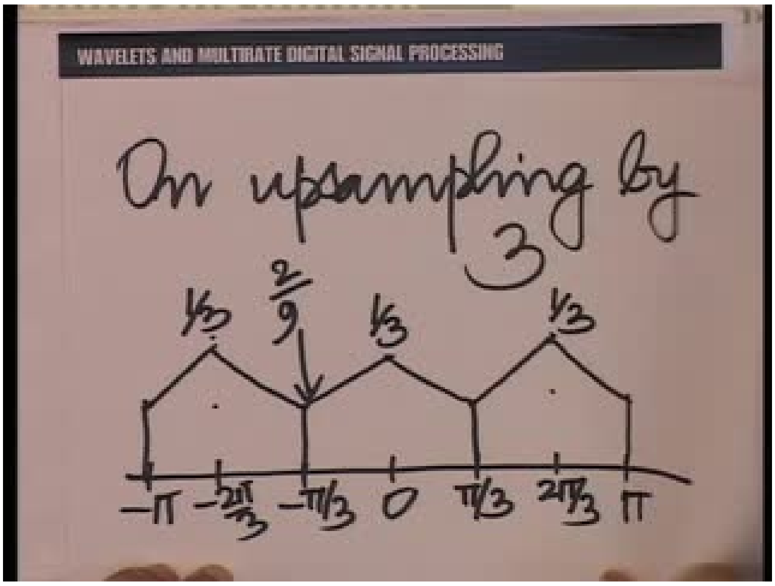

We go back here; so, you realize that an up sampling by 3, what is happening is a compression on the frequency access by a factor of 3. So, we see something very similar to what we had at the output of the down sampler, except that, there is a factor of 3 on the vertical axis; so, here we go.

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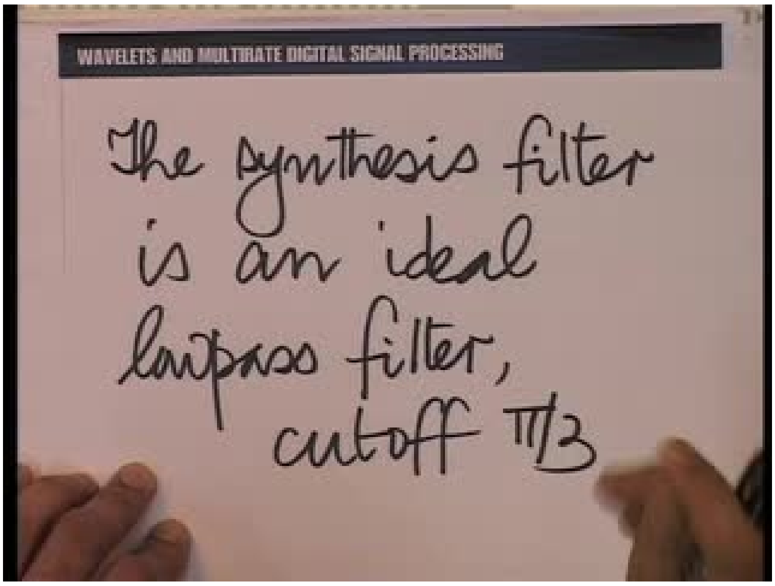

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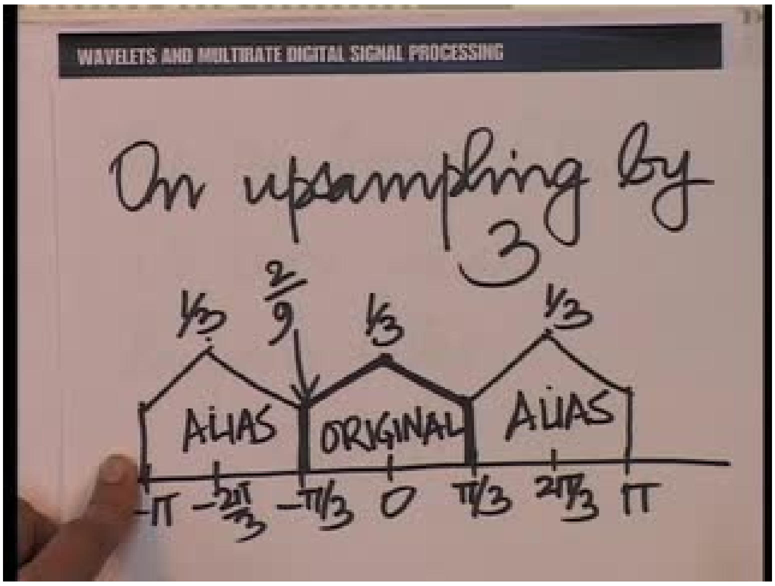

So, shape is all the same, except that, this is the 1 by 3 at the top and 2 by 9 there. Now, once we sit so clearly on this diagram, it is very easy to understand, what happens when we subjected the action of the synthesis filter. The synthesis filter is again an ideal low pass filter, cut off pi by 3, and once again we recognize if we well that, this was really the so called original spectrum and these were the carbon copies of the aliases.

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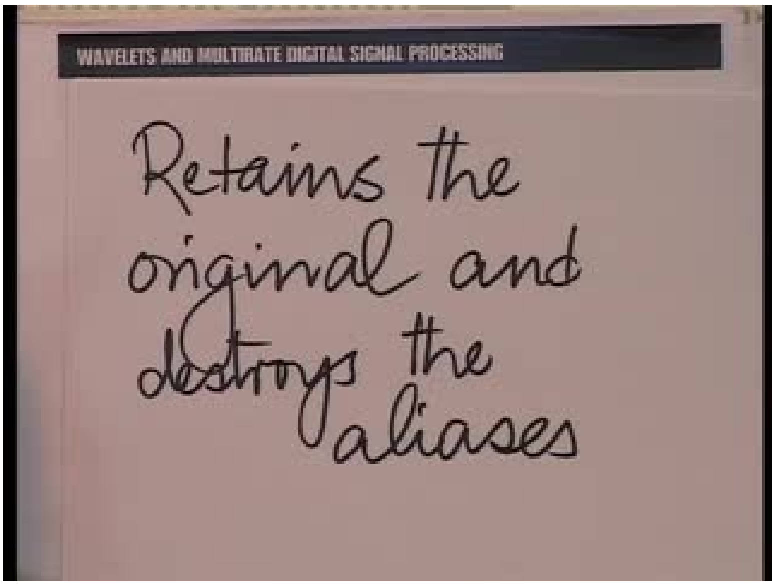

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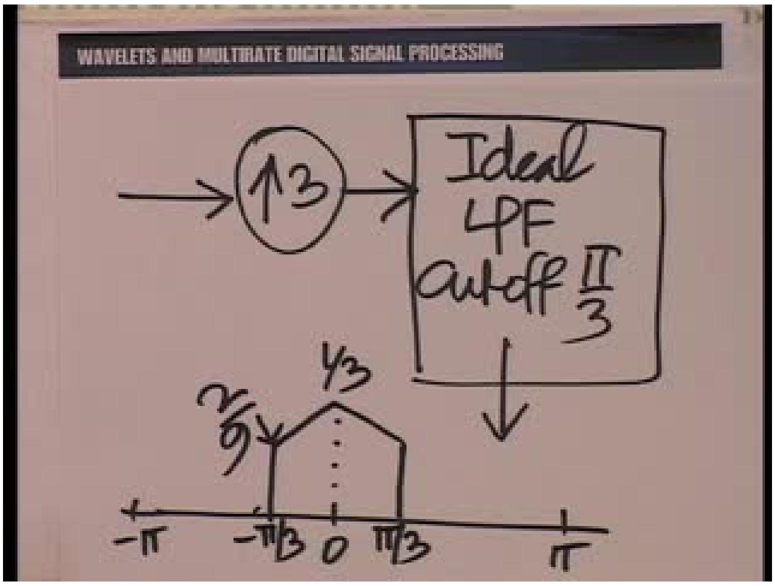

So, what the synthesis filter does is, essentially to retain the original and destroy the aliases; and therefore, what we have left after the synthesis filter. So, once we subject this to up sampling by 3, and then the synthesis filter, the ideal low pass filter with the cut off pi by 3, what we get here is essentially the original spectrum, but multiplied by a factor of 1 by 3, that is all, that is the only change.

1 by 3 there and 2 by 9 here; so, in fact, we did this analysis of the low pass branch carefully, because if we understand this analysis, then almost all the analysis for the 3 band filter bank to becomes crystal clear. Now, it will help for us do the analysis of the middle branch to fix our ideas further. So, what it we observe in general? On the low pass branch, the effect of the analysis part and the synthesis part is first to isolate the region between 0 and pi by 3, and then to reconstruct the same, after creating aliases between.

Now, we must again spend the couple of minutes in understanding why those aliases get created. The aliases get created because we wish to retain the total amount of data. You see what we are doing is, first to decomposing the frequency domain on each of the branches; subsequently, we do not want the overall amount of data to increase. So, whatever be the effective number of samples per unit time and the input at each of the analysis filter, is reduced to 1 third at the outputs of each of the down samplers.

So, total amount of information or data per unit time remains the same, 3 times 1 third. On the synthesis side, we are going from this multiple streams of data emerging from each of the down samplers back to the original data by reconstruction, alright. Now, let us look again at the middle branch that would give us a insight, because the middle branch is something very peculiar to the 3 band filter bank; you did not have an equivalent concept in the 2 band filter bank.

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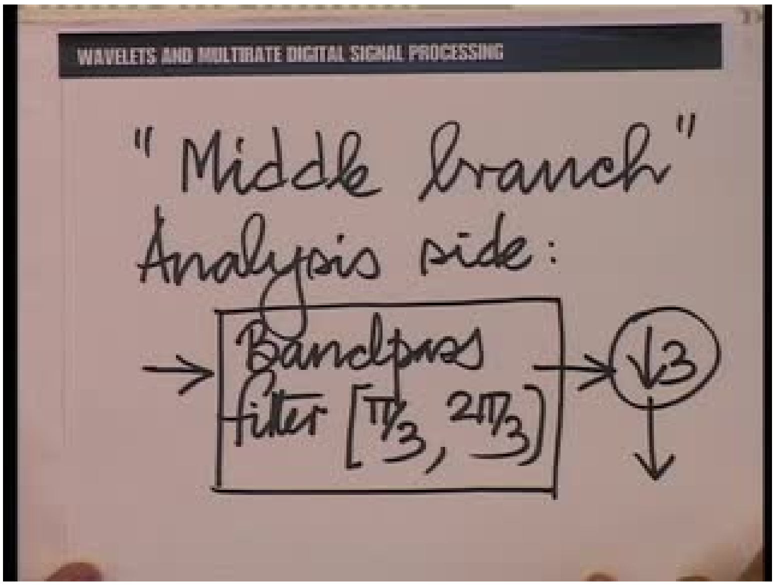

On the analysis side, the middle branch is essentially a band pass filter between pi by 3 and 2 pi by 3 - an ideal band pass filter of course - followed by down sampler of three. And let us take again the prototype spectrum, and see what happens, when we subjected to this analysis middle branch.

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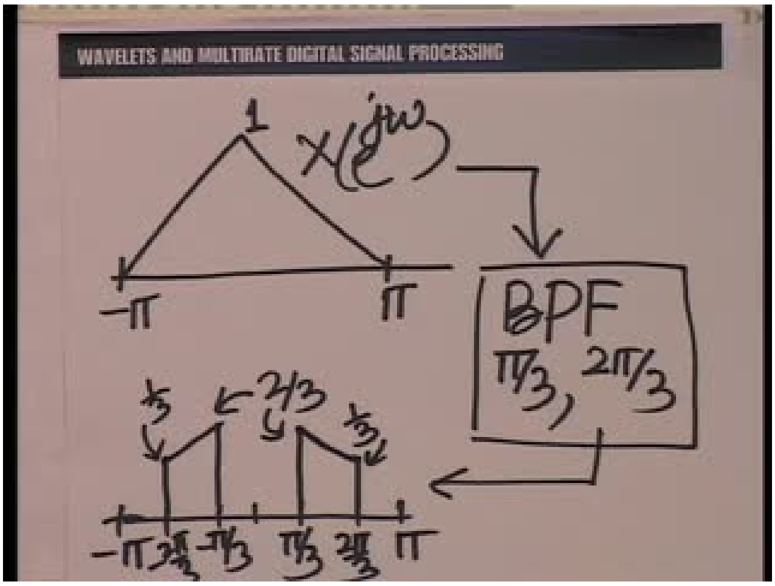

So, if we take the prototype spectrum X e raise the power j omega, subjected to the action of the band pass filter between pi by 3 and 2 pi by 3, we would get (( )), only between pi by 3 and 2 pi by 3; this would be 2 by 3 here and 1 by 3 there.

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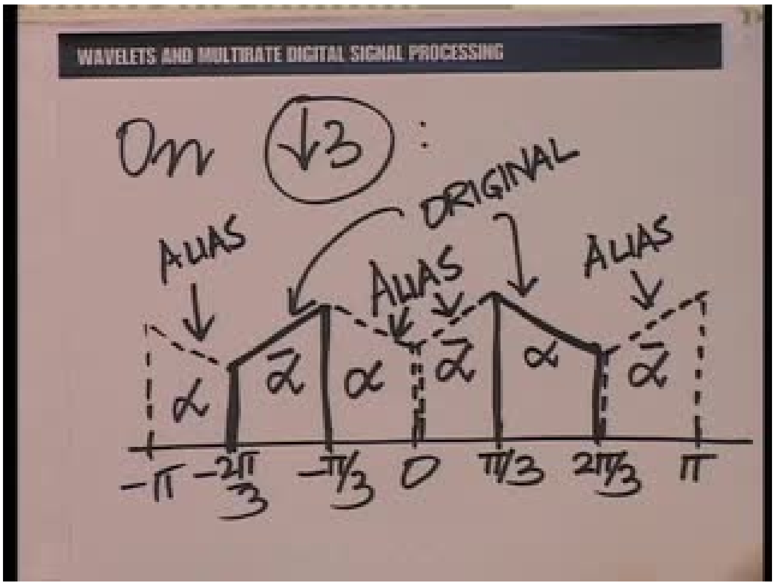

Now, as usual on down sampling by a factor of 3, we would be take this spectrum - this renal spectrum - after band pass filtering, shifting it by multiple of 2 pi by 3 and adding up the shift inverse; let me sketch what happens when we do so.

I need to expand the figure to explain properly. Remember, this is the original part of the spectrum so to speak; so, I will darken it. And I could call this spectral portion alpha and this spectral portion alpha conjugate. Now, we would shift this by 2 by 3 and by 4 pi by 3, 2 pi by 3 and by 4 pi by 3.

So, remember that all shifts are modulo 2 pi. So, we could think of shifting first backwards by 2 pi by 3 and then forward by 2 pi by 3, that is the same thing as shifting by 2 pi by 3 and 4 pi by 3 either forward or backward. When we shift this backward by 2 pi by 3, what we are going to get is this coming here.

So, something like this; so, we have an alpha appearing here. Now, when we shift this forward by 2 pi by 3, it goes outside the range pi and that would be taken care of anyway, when we consider the modulus here. Now, let us consider what happens? When we shift this forward by 2 pi by 3. So, when we shift this forward by 2 pi by 3, we bring this here.

Now, here it is convenient to do one forward shift and other forward shift. So, when we shift this one forward by 2 pi by 3, we bring this alpha back here; when we shifted again once more, we get the alpha bar here. When we shifted this backward by 2 pi by 3, we brought it here; and we shift once again by 2 pi by 3, we bring it here; so, we get an other alpha there. Now, you can see very easily that, once again we have the complex conjugate property we obeyed, so we have alpha, alpha bar here. If it is alpha bar, then it is alpha here; this is alpha here, it is alpha bar there; and if it is alpha bar there, it is alpha here; not only that, now we can also see what happens on the synthesis side.

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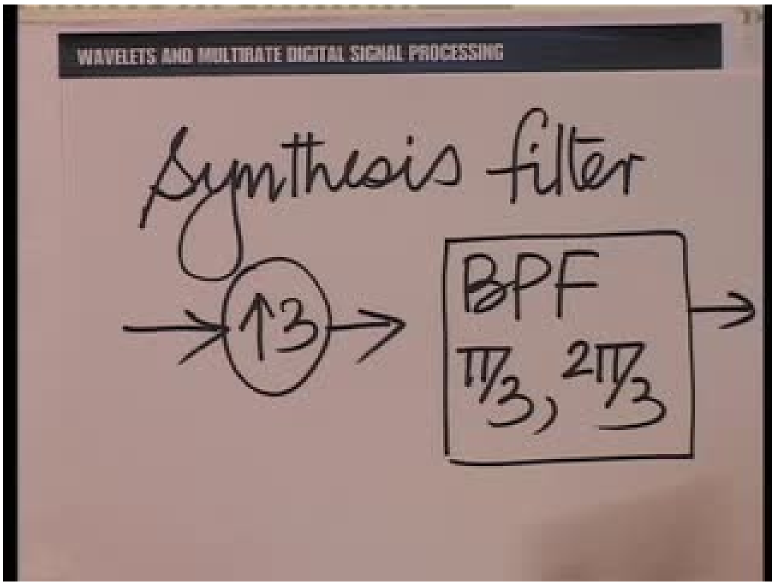

On the synthesis side, these aliases are removed; so, this is an alias, these are aliases here, and this is also an alias here, and alias there. So, without repeating all the discussion, I can straight away summarize once again. The synthesis filter which is essentially a band pass filter, once again between pi by 3 and 2 pi by 3, following an up sampler would retain the original and through away the aliases; you see, remember, on down sampling, what is going to happen is, this is going to be scaled by a factor of 1 by 3 on the vertical, this is going to get expanded by a factor of 3 on the horizontal, on up sampling by 3 it is going to conflicted once again. So, after down sampling by 3 and up sampling by 3, you come back here except with a scaling of factor of 1 by 3. I have not shown all those details again here, I would left it for you to complete those details; let me understand what happens.

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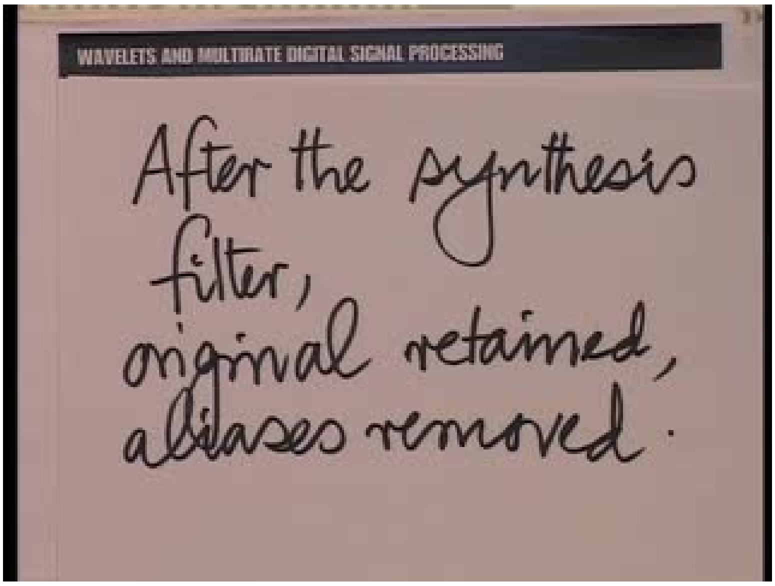

So, after this, after the synthesis filter, the original is again retained and the aliases removed. Now, I leave it for you as a student to complete the analysis for the third branch, in 3 band filter bank; it is very similar, and if one goes to it carefully, it would help fix one's ideas completely. What we have done here, is to analyse the ideal 3 band filter bank; and actually now that you have done it, you can see that going to the m band case, for m greater than 3 is extremely simple, at least conceptual.

But rest is sure going from ideal to practical is a very big job in the 3 band filter bank and even also in higher order filter banks. In fact, it is a big job for two reasons: one is designing the filters in the discrete time filter bank, and the second is to interpret the generalization of the multi resolution analysis. What do you mean by a multi-resolution analysis, when we talk about the 3 band case?

For example, in the 2 band case, essentially the multi resolutional analysis implied that, you would have a peel of one sub space every time. Now, as you can see, you have a peel of two sub spaces; each time, there are two detail sub spaces. So, if you go back to the case of 2 band filter bank or the diadic multi resolutional analysis, each time we took the low pass version or the low pass branch and iterated on it; and the interpretation was, that we took the course information and decompose it further into course and specific information to that scale.

Now, would be a saying is going down closer, each time implies peeling of two parts of specific information at that scale; so called middle frequency information as so called high frequency information. And this is the case for m equal to 3; for larger m, there are m minus 1 sub spaces been peeled off each time. So, we could look at these in greater detail in subsequent lectures.

Now, what is it else that we do in subsequent lectures. We would like also to look at more applications; we would like to look at the generalizations, not only to m band but also to multi dimensions. And we might what to look at generalizations to the cases, where the up sampling and the down sampling is rational rather than integer. So, with this, then we shall look forward to some more expositions on wavelets and multirate signal processing in subsequent lectures. I conclude this fortieth lecture, thank you.