Advanced Digital Signal Processing – Wavelets and Multirate
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Lecture No. #04
Wavelets and Multirate Digital Signal Processing

A warm welcome to the fourth lecture on the subject of wavelets and multirate digital signal processing in which we intend to build further, the connection between signals or functions in L 2 R and vectors, and therefore, we wish to build further, the idea of thinking of functions as belonging to linear spaces and characterizing them in a manner, slightly different from what we were doing in the previous lecture.

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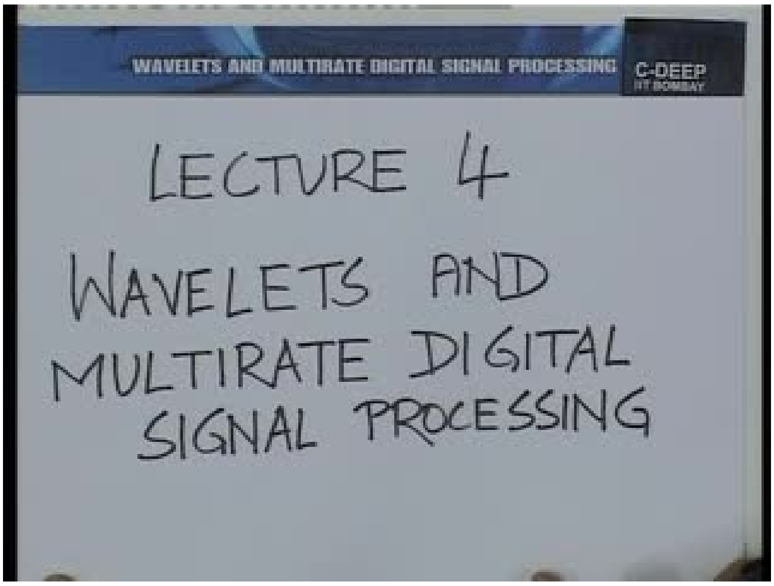

So, just to put our discussion in perspective, this is the 4th lecture on the subject of wavelets and multirate digital signal processing and what we intend to discuss in this lecture is the following, let me put down the points, one by one.

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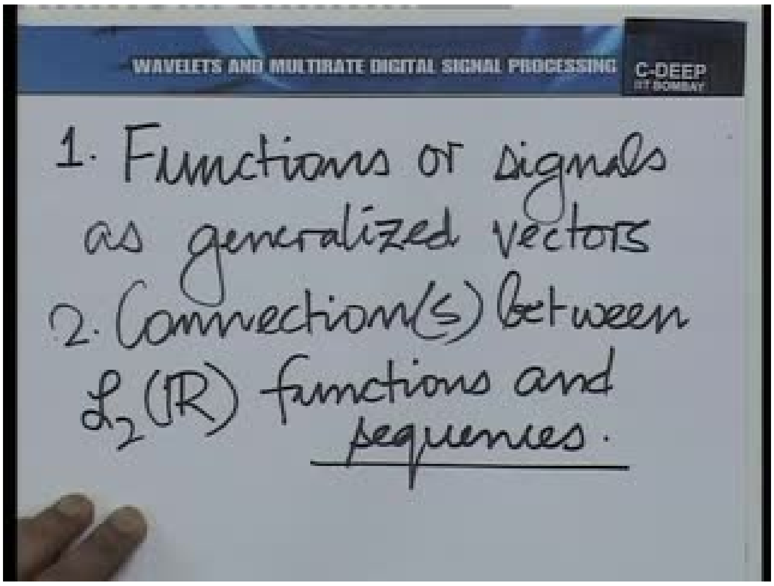

The 1st thing that I wish to talk about today is to think of functions as generalized vectors.

This idea is going to be useful to us in many different contexts in this course. So, we need to understand this connection between functions or signals and vectors in depth; we shall spend some time on it today.

Secondly, the connection between L 2 R functions, connection or connections between L 2 R functions and sequences, we wish to understand this in greater depth.

So, what we are going to show in the later part of this lecture is that one can intimately relate processing of a function to processing of an equivalent sequence and whatever we are doing to try and gain information from or modify a function, can be done by equivalently processing or modifying, that sequence corresponding to the function.

Let us then, embark on the 1st of these 2 objectives now. You see, let us begin by asking what characterizes a vector after all? Let us take a minute and reflect.

What characterizes a two-dimensional vector, for example? A two-dimensional vector is essentially characterized by 2 coordinates, which are independent, we call them perpendicular coordinates. Actually, the idea of perpendicularity there is also intimately related to the idea of independence.

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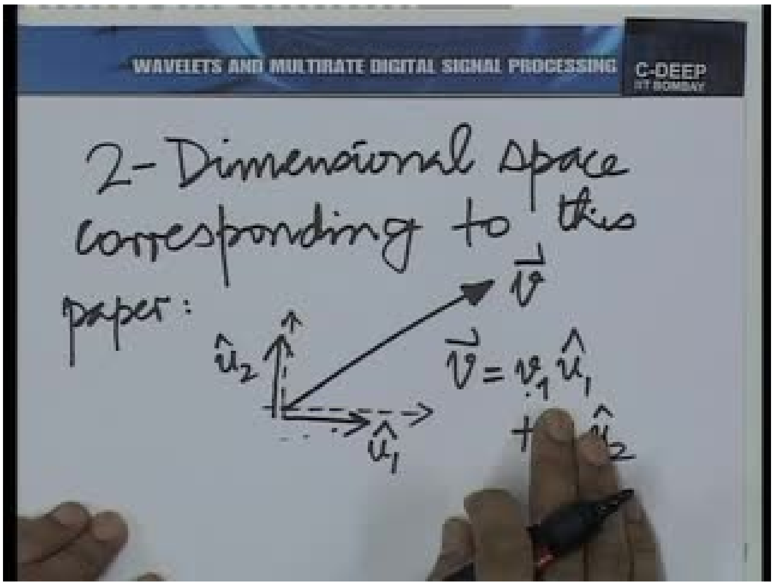

So, for example, let me treat the plane of the paper as a 2-dimensional space; the 2-dimensional space corresponding to this paper. Well, let us take any vector on this 2-dimensional space, so this vector be v, I am marking it as v.

There are many different ways to characterize this vector. In fact, notionally, an infinite number of ways and one of those ways is to choose the following 2, so called, perpendicular axis. So, we choose one axis like this and another axis like this and choose a unit vector along each of them. So, I have, say, unit vector, let me call it u 1 cap along this axis and another unit vector u 2 cap along this axis, and then I could write v 1 or I could write this, sorry, just the vector v uniquely as, say, v 1 times u 1 cap plus v 2 times u 2 cap, whereby v 1 and v 2 characterizes vector v uniquely in this 2-dimensional space, with respect to the coordinates system generated by u 1 and u 2, and there is an infinity of such coordinate systems.

In fact, one infinity of such coordinate systems can be generated simply by rotating this coordinate system of u 1 u 2. It is very easy to see, that if I take this structure u 1 u 2 and rotate it by any angle in this 2-dimensional plain, it would give me a new coordinate system. So, there is infinity of orthogonal coordinate systems in 2-dimensional space and in fact, there is also a relation between all these infinite orthogonal coordinate systems, simple enough. And orthogonal coordinate systems are not the only kinds of coordinate system for a 2-dimensional vector. So, for example, the same two-dimensional space can be described by the following different coordinate systems.

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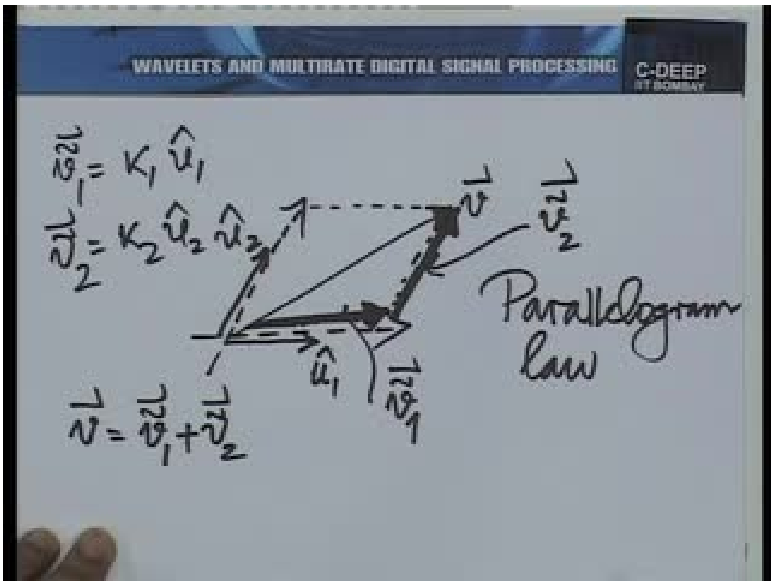

So, I will draw the same vector v and it is perfectly alright to choose a coordinate system something like this. I could choose one coordinate like this, another coordinate like this and of course, I could again have the unit vectors in these 2 directions, u 1 cap, so to speak, u 2 cap and I could express v in terms of u 1 cap and u 2 cap. Indeed, I could complete a parallelogram here, so using the parallelogram law, I could draw a line parallel from the tip of this vector to this u 2, another one parallel to u 1 from the tip of the vector and it is very easy to see, that this dot dash vector here plus this dot dash vector here gives me v. Let me highlight, that dot dash vector.

This vector here plus this vector here gives me v. Let me call this v 1 tilde and is a vector, and let me call this v 2 tilde that is again, a vector. Of course, we have, v is v 1 tilde plus v 2 tilde and it is very easy to see, that v 1 tilde as a vector is some multiple of u 1 and similarly, v 2 tilde as a vector is some multiple of u 2.

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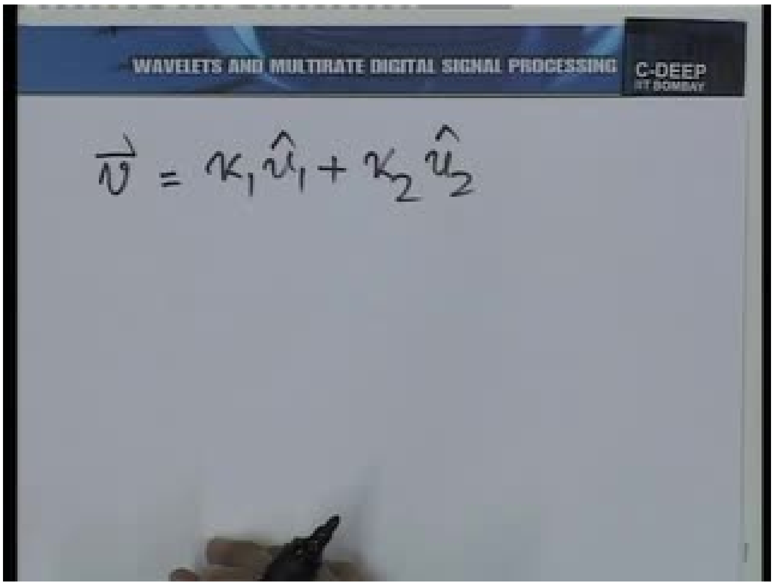

Thereupon, I have, v is some multiple of u 1 plus some other multiple of u 2, k 1 u 1 plus k 2 u 2. The only catch is determining k 1 and k 2 is a little more difficult than determining the constants in the previous representation. In fact, let me go back to that previous representation.

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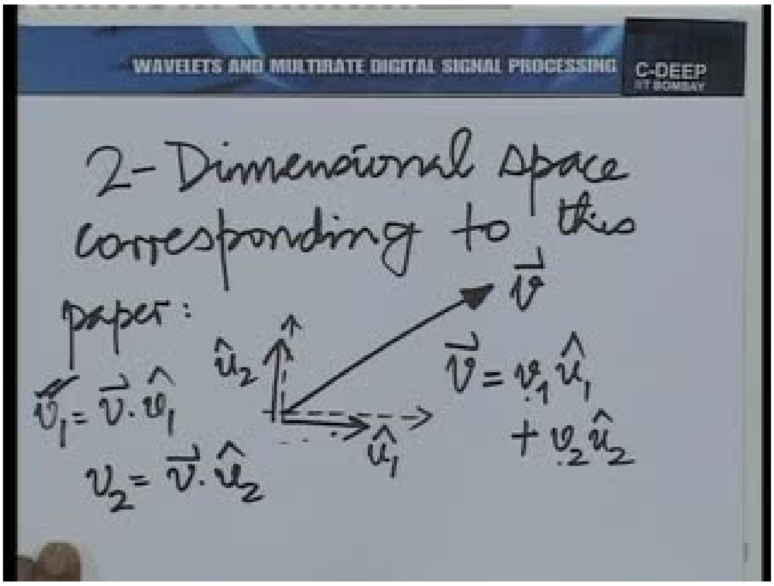

I had this representation previously, where v is v 1 u 1 cap plus v 2 u 2 cap and remember, v 1 and v 2 here, of course, are constants and very easy to obtain because I can simply obtain them by taking a dot product of v with u 1 cap and v with u 2 cap. So, in fact, in the sense of dot products, v 1 is indeed v dot u 1 cap and v 2, I mean, v 1 is a coordinate not as a vector, v 2 is a coordinate, is the dot product of v with u 2 cap, simple enough.

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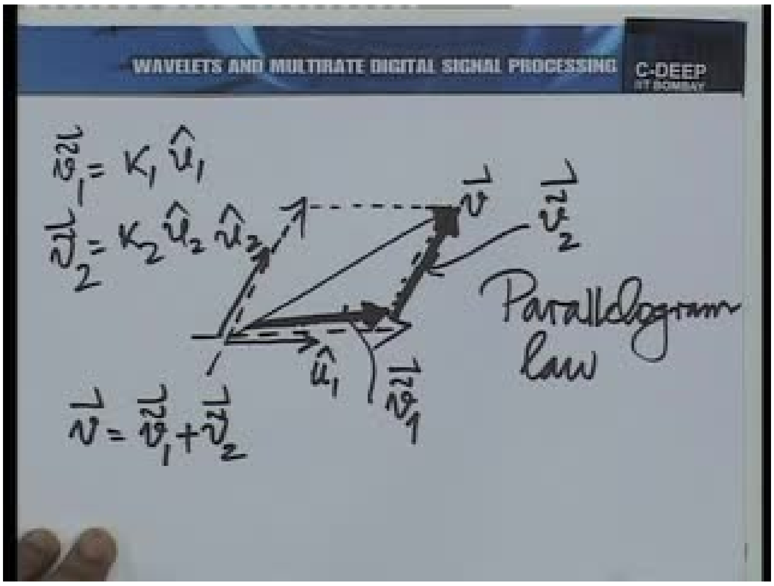

Such a simple relationship does not exist in this context. While we are not hard put to describe the process by which we obtain k 1 and k 2, it simply says, construct the parallelogram; expressing this analytically is a bit of work.

So, it is definitely very clear from this example, that an orthogonal or a perpendicular coordinate system has its advantages. It is always nice to have a perpendicular coordinate system in two-dimensional space to represent any two-dimensional vector. The same idea can, of course, be extended to three-dimensions too and then, one could also conceive of more than three-dimensions, four-dimensions, N-dimensions and then, in principle, an infinite number of dimensions too. Now, there again, when we talk about infinite dimensional situations, we have countably infinite and uncountably infinite finer points, but for the moment, infinite is difficult enough.

So, infinite dimensional vectors, in fact, lead us to the idea of functions. Now, it is a little difficult to understand infinite dimensional vectors all at once, so to progress towards infinite dimensional vectors, it is easier first to start from finite dimensional vectors of larger and larger dimension, and all that we need to do is to understand, that what characterizes the dimension of a vector is really the number of independent coordinates, that it has.

For example, a three-dimensional vector has 3 independent coordinates, a four-dimensional vector would have 4 and N-dimensional vector N, and an accountably infinite dimensional vector would have accountably infinite number of dimensions or countably infinite number of coordinates.

By countable we mean, we can put the coordinates or the dimensions in one to one correspondence with the set of integers. So, we can talk about the 0th coordinate; we can talk about the 1th coordinate; we can talk about the minus 1th coordinate; the minus 2th coordinate, and so on, so forth.

What are we talking about here then, if we talk about an infinite dimensional vector? We are, in fact, talking about sequences; we build up the idea from there.

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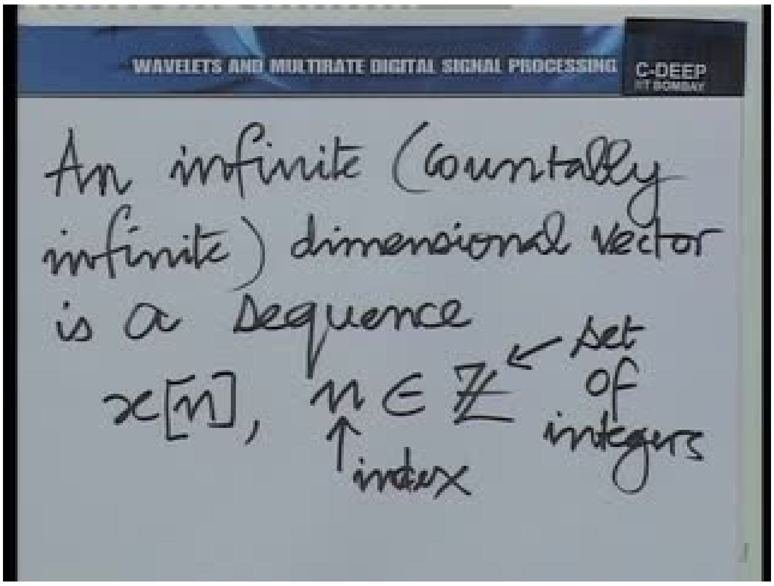

So, here we are, let us make a note of this. An infinite dimensional vector or rather an infinite - countably infinite - dimensional vector is, essentially, a sequence. So, for example, we have a sequence x of n, where n belongs to set of integers over all the integers; recall, that this script Z is the representation of the set of integers and this is called the index variable.

So, now, we have a different interpretation for sequences. A sequence is like a vector and each n is a different dimension of that vector; I think that is important enough for us to write down explicitly.

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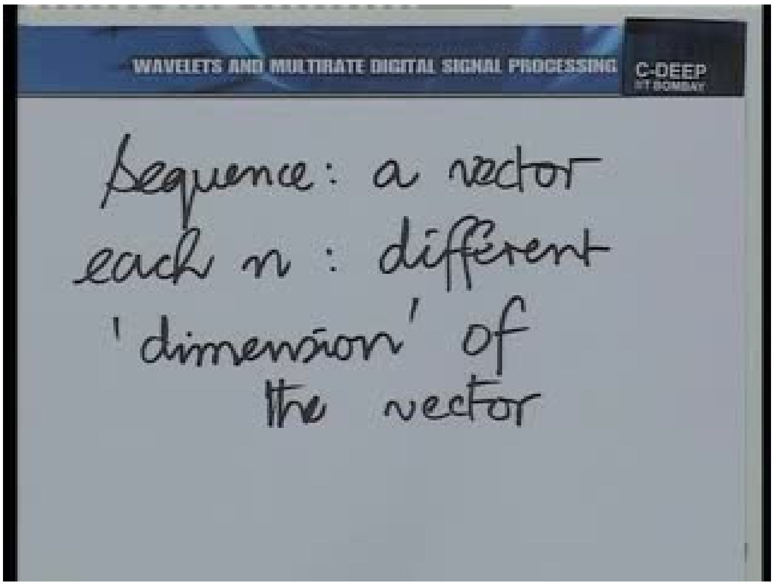

So, a sequence is a vector. Each n is a different dimension of the vector and once we have this analogy, then extending other ideas of vectors to this context is not difficult at all. For example, adding 2 vectors, simple, add the sequences point by point; multiplying a vector by a constant, very simple, multiply each point of that sequence by that constant.

What we would like to do now is to extend some of the other ideas of vectors that we have. Some of the geometrical ideas to this, this context of infinite dimensional vectors and one of the very useful ideas that we have in the context of vectors, is the idea of a dot product. How do we take the dot product of 2 vectors in two-dimensional space? So, let us recall.

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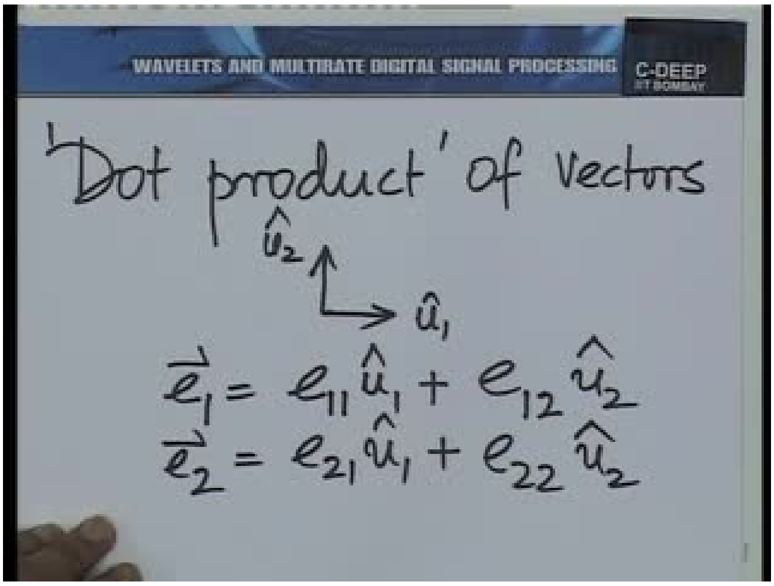

So, suppose, for example, we choose a pair of orthogonal coordinates. So, we have u 1 cap and u 2 cap, as we did some time ago, orthogonal to one another, perpendicular to one another. And we have 2 vectors; let us call them e 1, which has the coordinates e 11 and e 12. So, e 1 is e 11, u cap, u 1 cap plus e 12 u 2 cap, and similarly, e 2 has a vector has the coordinates e 21 u 1 cap plus e 22 u 2 cap.

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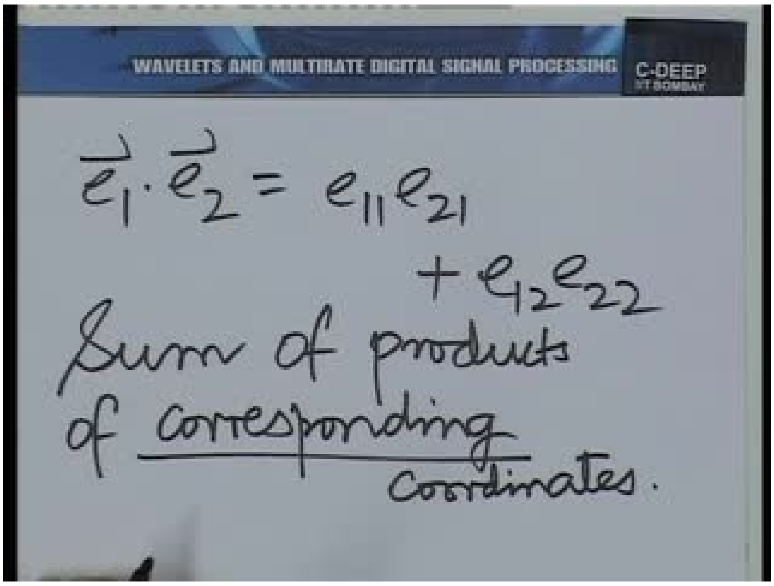

Then, the dot product of e 1 and e 2, e 1 dot e 2 as we write it, is essentially, e 11 e 21 plus e 12 e 22. So, it is the sum of products of corresponding coordinates; two-dimensions, easy enough to understand; three-dimensions, easy to extend; in fact, N-dimensions, equally easy to extend.

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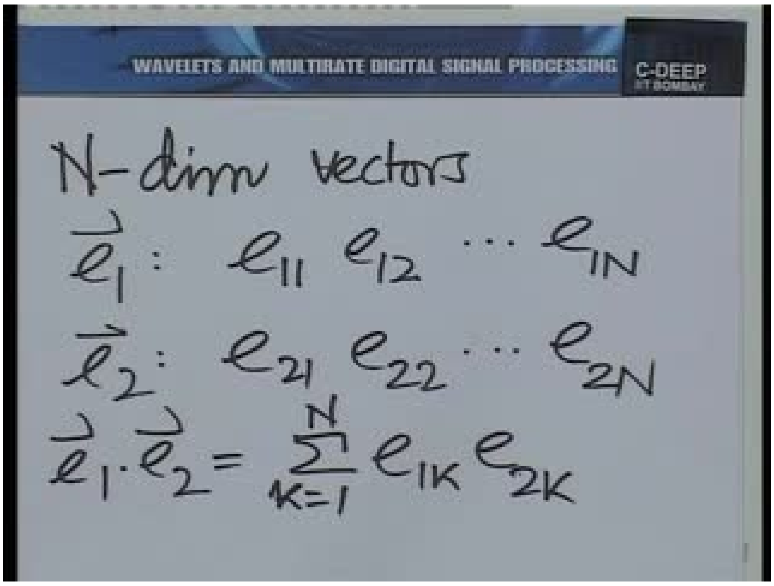

Suppose, we had 2 N-dimensional vectors characterized by coordinates, say e 11 to e 1N. So, you have 2 N-dimensional vectors, e 1 characterized by coordinates e 11 e 12 up to e 1N and similarly, e 2 characterized by the coordinates e 21 e 22 up to e 2N.

Then, of course, e 1 dot e 2 is easy to express if we generalize this. It is essentially, summation K from 1 to N; e 1K times e 2K. So, dot product generalized to N-dimensions; of course, we assume these are orthogonal coordinates.

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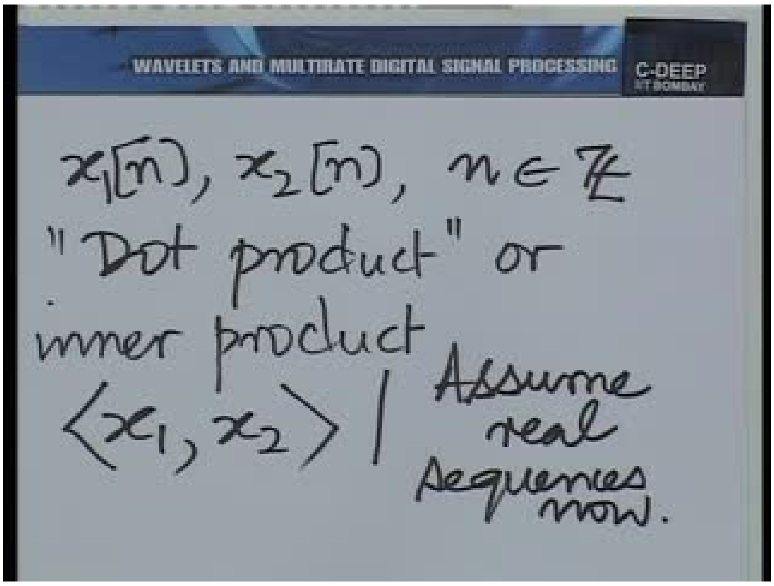

Now, we can even take this to infinite dimensions. So, we can think of the dot product of 2 sequences, let us say, x 1 and x 2. So, we have here, for example, 2 sequences x 1 n and x 2 n, defined over the set of integers n, over all the integers. They are, so called, dot product or inner product, as the formal name is. So, we see, instead of dot product, now you would like to use a term inner product to generalize and we denote the inner product this way.

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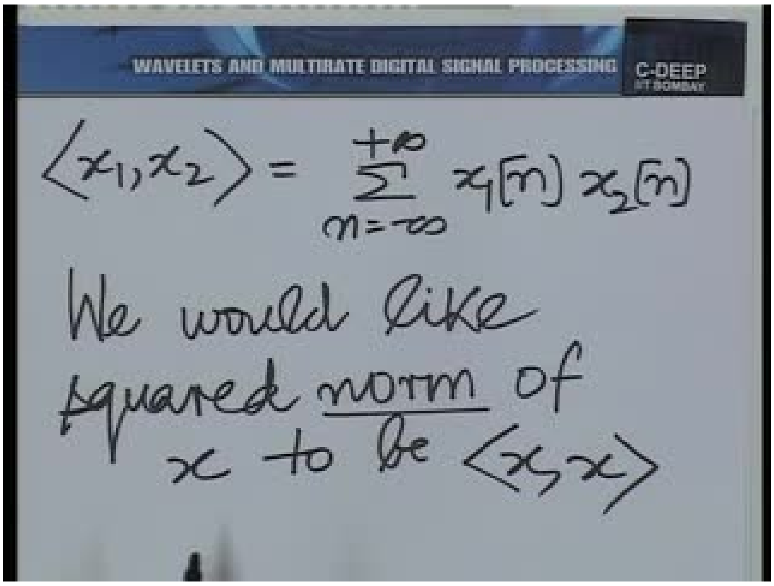

For the moment, let us assume, these are real sequences for the moment. In that case, if we generalize, it is easy to see, that the inner product of x 1 and x 2 is simply summation on n, n running from all the way from minus to plus infinity, x 1 n times x 2 n. And of course, it is clear, that the dot product or the inner product, as we are going to call it, in this generalized situation is commutative, that means, if I interchange the rows of x 1 and x 2, the result does not change.

However, we would like this inner product or dot product notion to give us some of the powers and some of the conveniences that the dot product offers in the context of vectors. One so called convenience, so one such, so called, interpretation or meaning, that we derived from the dot product is the notion of magnitude. So, in fact, one could think of the notion of magnitude as induced from a dot product if one desires, or in other words, one could calculate the magnitude of a vector by using the notion of a dot product as one path towards the calculation of magnitude.

Incidentally, the word magnitude of vectors is used for small dimensional vectors, like 2 and 3 dimensional, but when we go to these generalized situations of N-dimensional vectors or countably infinite dimensional vectors, we replace the word magnitude by the word norm.

So, we say, that we would like the squared norm of x to be the dot product of x with x, as is the case with vectors. So, if you recall, A dot A, where A is a vector in two or three-dimensions, for that matter, is the magnitude squared of A. The same should hold good here. When we take the dot product of a sequence with itself, it should give us the squared norm of that sequence, where norm is a more general word for the magnitude.

In fact, in L 2 R, the norm is representative of the energy, but at this moment we are not talking about L 2 R because we have not yet come to the situation where we are dealing with functions of continuous variables. So, we will postpone that interpretation for a minute, not very far away from now, and once again come back to sequences.

Even for sequences, when we take the dot product of a real sequence with itself, we indeed get something that we likened to energy of the sequence. So, it is not uncommon to refer to the dot product of a real sequence, or for that matter, sequence with itself as the energy in that sequence. Anyway, I kept emphasizing real for a good reason. When we talk about the magnitude of a vector, or for that matter, there is more generalized word norm, what is it that we expect of a magnitude? We want the magnitude or the norm to be a non-negative number and in fact, strictly positive if that vector is non-zero.

So, there are the following things that we demand of this concept of norm or magnitude, let us write them down, it is a useful and a powerful idea to have around us. So, what do we want of a norm?

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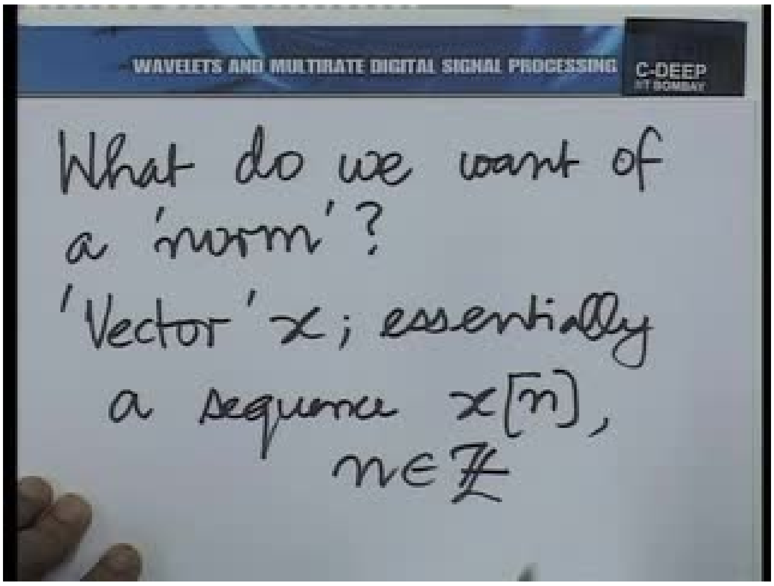

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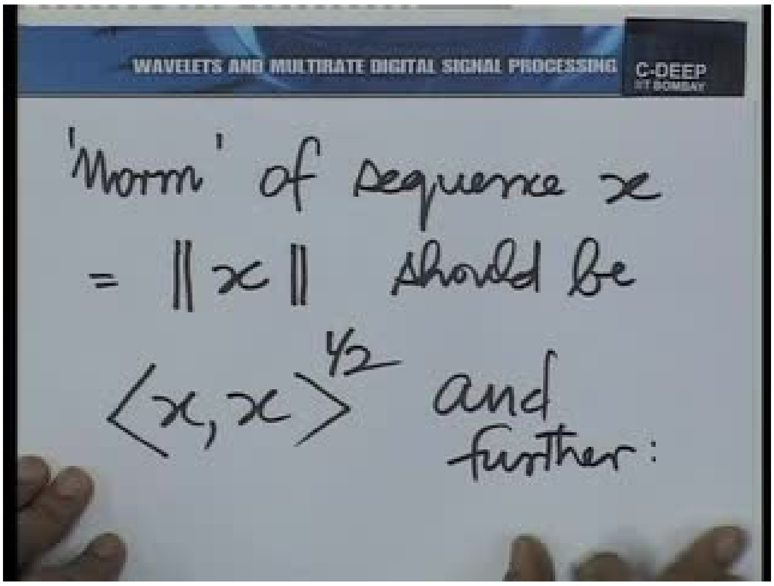

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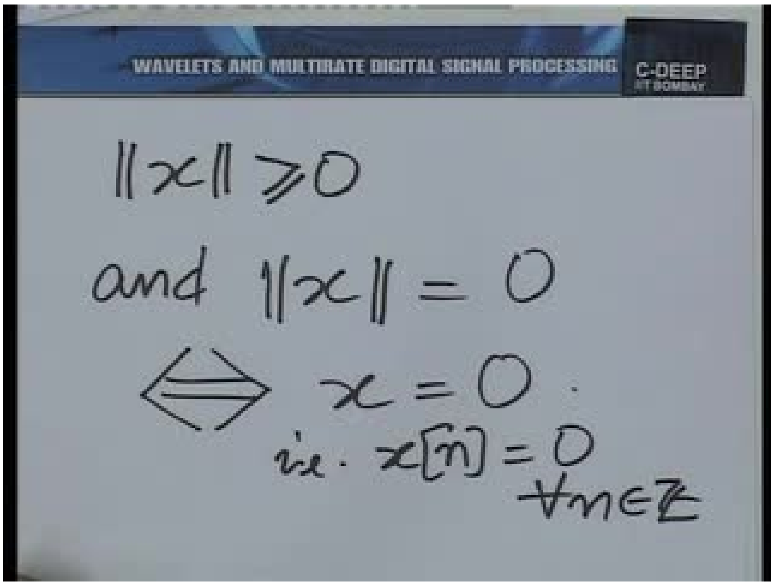

So, if I have a vector x, essentially a sequence x n, n over the set of integers, then it is norm, which we shall denote in the following way. We denote it like this, should be essentially, the dot product of x with x square root, and further, we would want norm of x to be non-negative. And if at all the norm of x is 0, that implies, and is implied by the sequence itself being 0 everywhere; that is, x of n is equal to 0 for all n belonging to set of integer. This is important, so we do not want that norm to be 0 unless the sequence itself is the 0 sequence.

A non-zero sequence, even if it is non-zero, at one point must have a non-zero norm and a 0 sequence must have a 0 norm.

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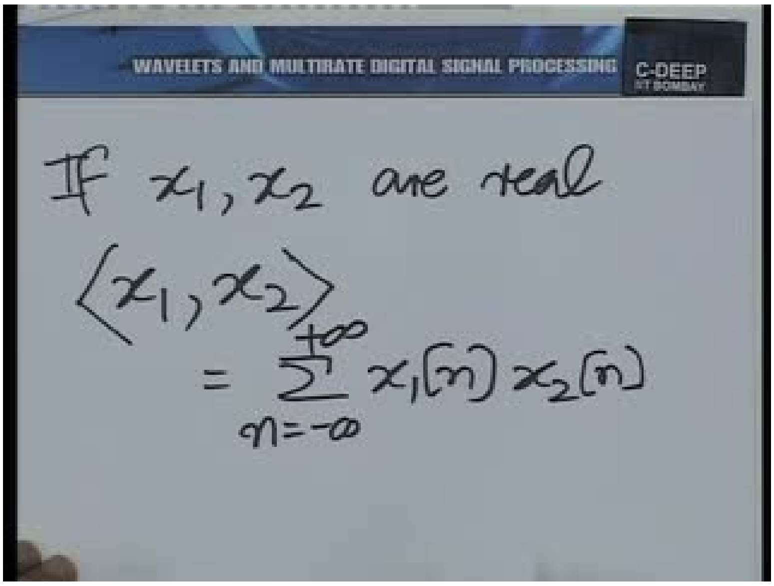

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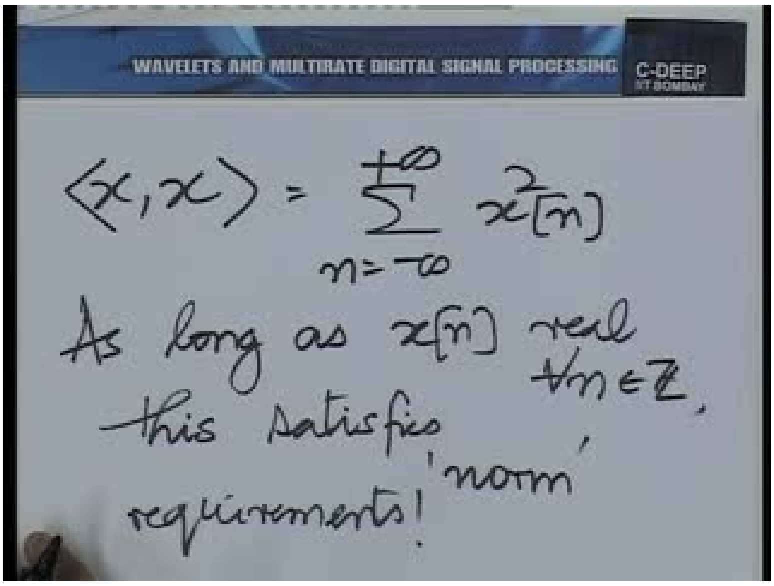

Does are dot product satisfy this? Well, for real sequences, it does. If x n is real, rather, if x 1 x 2 are real and we take the following definitions, the dot product of x 1 and x 2 is, essentially, summation on n going from minus to plus infinity x 1 n x 2 n, then the dot product of x with x is essentially summation n running from minus to plus infinity x squared n, and as long as x n is real for all n belonging to Z, this satisfies the requirements of norm.

It is non-negative and it is 0 if and only if the sequence is identically 0, but what if this is complex. So, we have to allow complex sequences too. One of the coordinates could be complex and in fact, the situation could be such, that x squared n could be plus 1 for one of the coordinates and minus 1 for some other coordinate in that case, because when you square a complex number, nothing guarantees the output is going to be non-negative. In fact, nothing even guarantees the output is going to be real, where is the question of non-negative? So, this definition is not going to work when x 1 and x 2 are complex sequences in general and we need to tweak the definition a little.

Well, it is not that difficult after all. What we want is that for every coordinate you must get a non-negative quantity when you take point by point products. So, all that we need to do for that purpose is to complex conjugate the 2nd argument in that summation.

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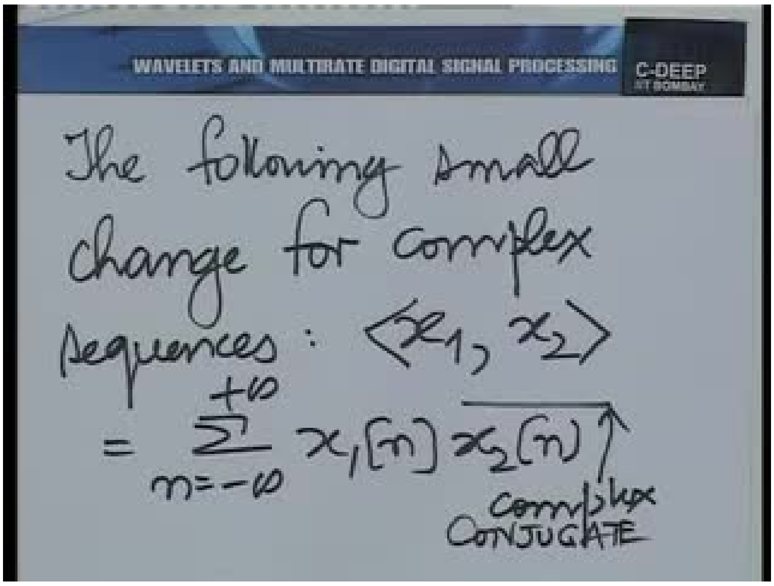

So, the small change for complex sequences, we will do our job. Dot product of x 1 with x 2 is summation overall n x 1 n x 2 bar n, where bar denotes the complex conjugate.

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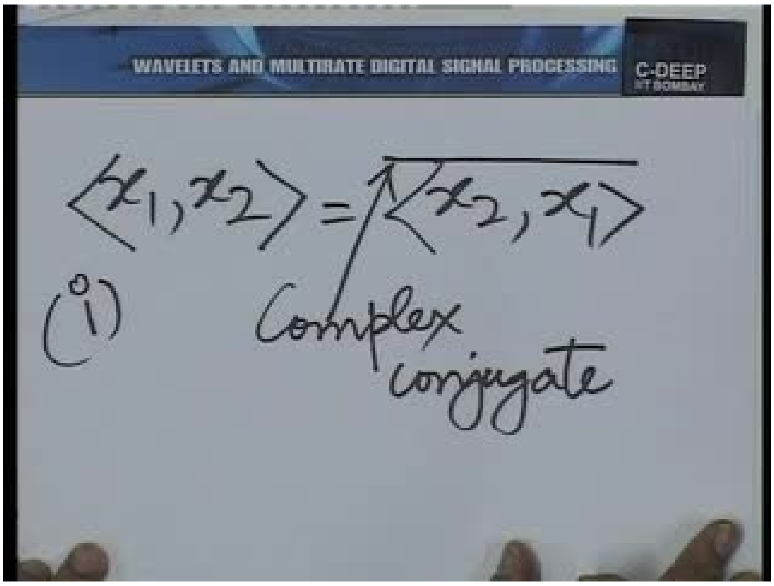

Now, one point to note here when we make this little change is that, that commutativity property is lost. So, if I take the inner product x 1 with x 2 and then, if I take the inner product x 2 with x 1, there is a complex conjugate relationship and this is the more general requirement of a dot product. In fact, this is the simplest way in which one can define a dot product between sequences; there are many other ways, again infinite number of ways, but at this moment we shall not go into the other ways, they will only confuse us. This is what is called the standard inner product, but one can have many other nonstandard inner products, which obey the following conditions.

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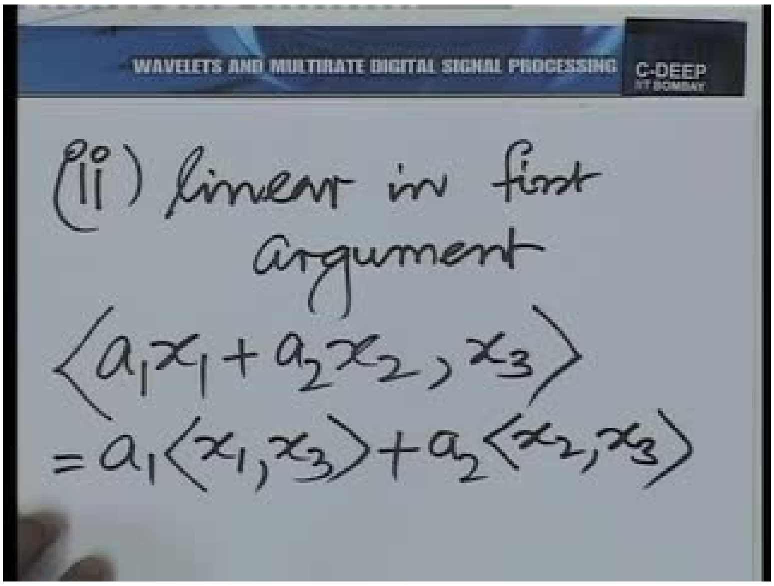

The 1st condition is this that we write down here; the inner product of x 1 with x 2 is the complex conjugate of the inner product of x 2 with x 1. Secondly, the inner product is linear in the 1st argument. In other words, if I take a 1 x 1 plus a 2 x 2, where in general a 1 and a 2 could be complex and take the inner product with x 3, it is essentially a 1 times the inner product of x 1 with x 3 plus a 2 times the inner product of x 2 with x 3; this is the 2nd requirement of an inner product, linearity in the 1st argument.

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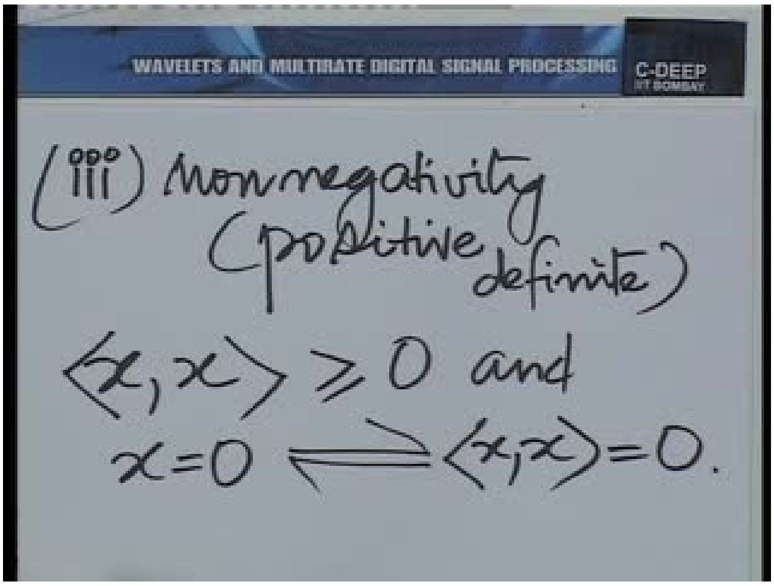

The 3rd requirement of the inner product is what we have been building towards all this while, namely what is called the positivity or non-negativity. In fact, positivity is more appropriate, positive definiteness, namely the inner product of x with x is always greater than equal to 0, and x equal to 0 implies and is implied by the inner product of x with x being 0.

In fact, any operation between 2 sequences, x 1 and x 2, which obeys these 3 conditions, is called an inner product and the standard inner product that we have just described is one such, which we shall use very frequently. So, in the discussion, henceforth, when we say inner product of sequences, we mean the standard inner product unless otherwise specified.

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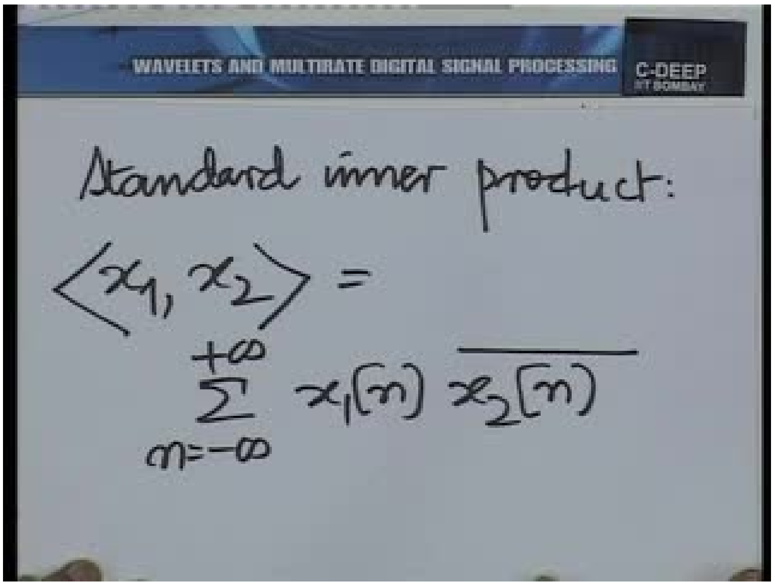

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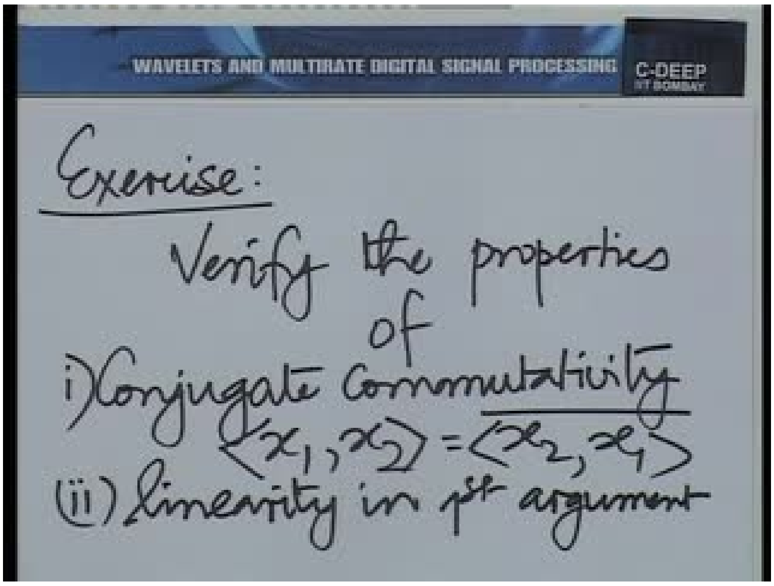

So, let us just verify this for completeness; let us verify this for the standard inner product. The inner product of 2 sequences, x 1 x 2 is essentially, the sum n going from minus to plus infinity x 1 n x 2 bar n definition. The 1st property, as we said, is complex conjugate easy to verify. So, in fact, I leave it to you as an exercise; verify the properties of what is called conjugate commutativity, the first property and linearity, linearity in the first argument. I leave it as an exercise, easy enough to do.

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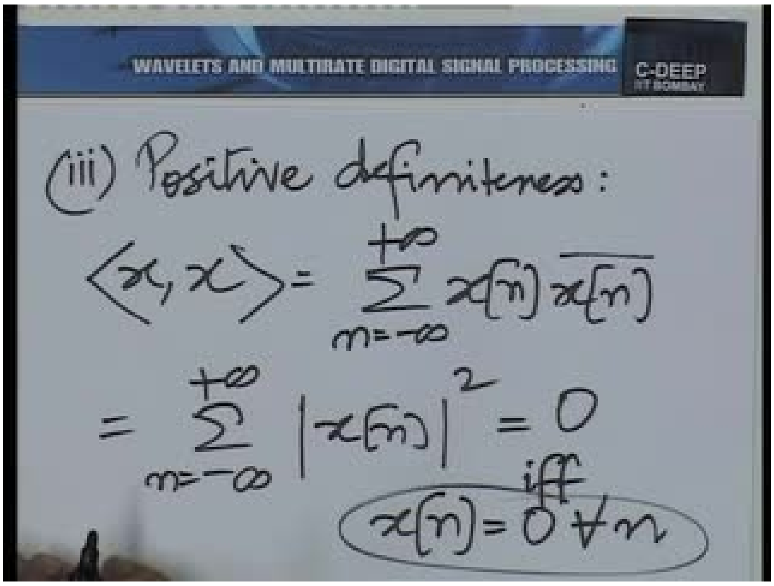

But we shall, because it is so important, verify the 3rd property, the positive definiteness. Indeed, if we take the dot product of x with x, it is summation n going from minus to plus infinity x n x n bar. We should, summation n going from minus to plus infinity mod x n squared and it is very easy to see, that this is equal to 0 if and only if x n equal to 0 for all n.

Even if one of the coordinates is non-zero, that particular mod x n squared is going to be non-zero and it is going to contribute a positive term and of course, it is very easy to see, that each term for every end, I mean, is strictly positive if x n is non-zero, so far so good. So, now, we have build up the idea of inner product or dot product between 2 sequences, which is going to be useful to us.

So, we move from two-dimensional to three-dimensional to N-dimensional, n is finite and then, to countably infinite dimensional.

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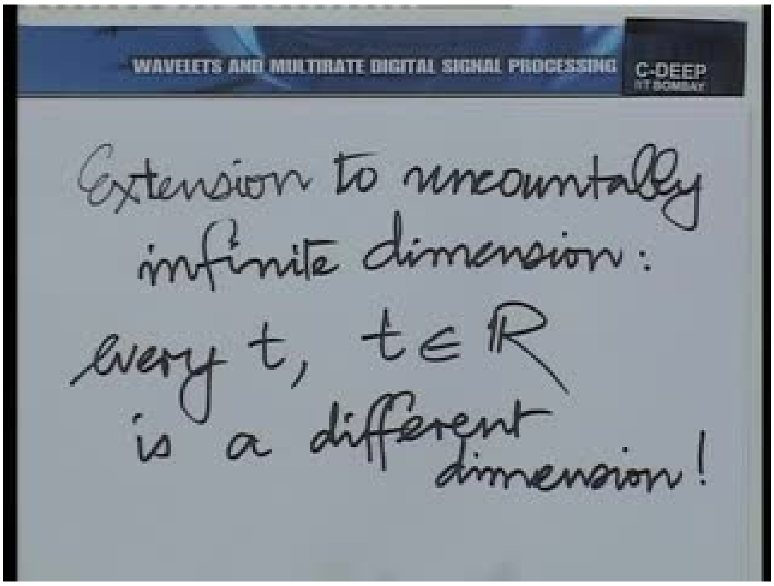

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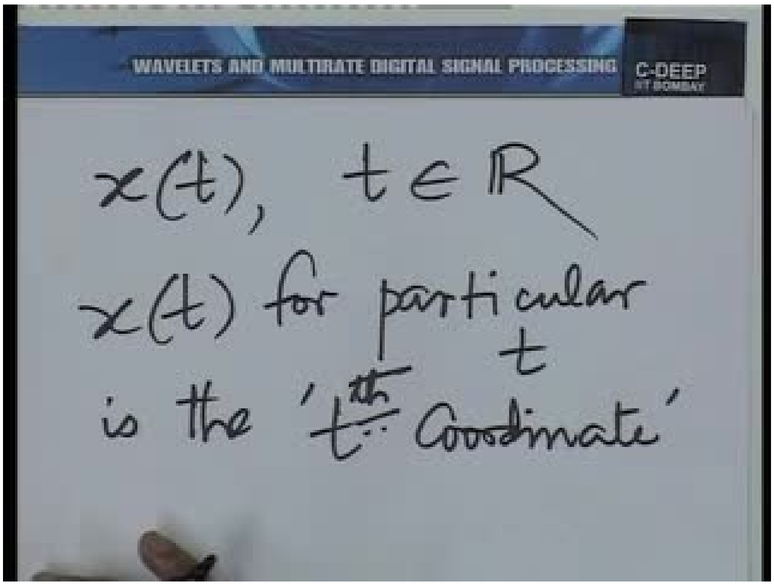

Now, let us move to uncountably infinite dimensional. So, suppose I take a function of the continuous variable t, how can I extend these notions? So, extension to uncountably infinite dimensions, well this is going to be very difficult in general, but very easy in particular. If we simply accept, that every t, for real t is a different dimension, simple. So, if you have a function x of t, t over the real numbers; x of t for a particular t is the tth coordinate, so to speak, and there is an uncountably infinite number of such coordinates indexed by the real numbers.

So, in principle, in a given function you have complete liberty to put down the value of x t, at every different point t, the only catch is we have agreed, that we would like to make the functions square integrable. So, that, that puts some restriction on x t, but not a very serious one, even so.

Now, you know, dealing with infinite dimensional spaces, if we wish to do it very rigorously and very, very carefully and you know, to satisfy the fastidious mathematician is a difficult job and we do not really intend to do that, all the way, in this course. If some of us do wish to take that puritanical perspective, one of course, would benefit from it in some ways and one could look up a book of, on function analysis, but what we wish to do is, rather to give intuitive understanding of some of the concepts at different places.

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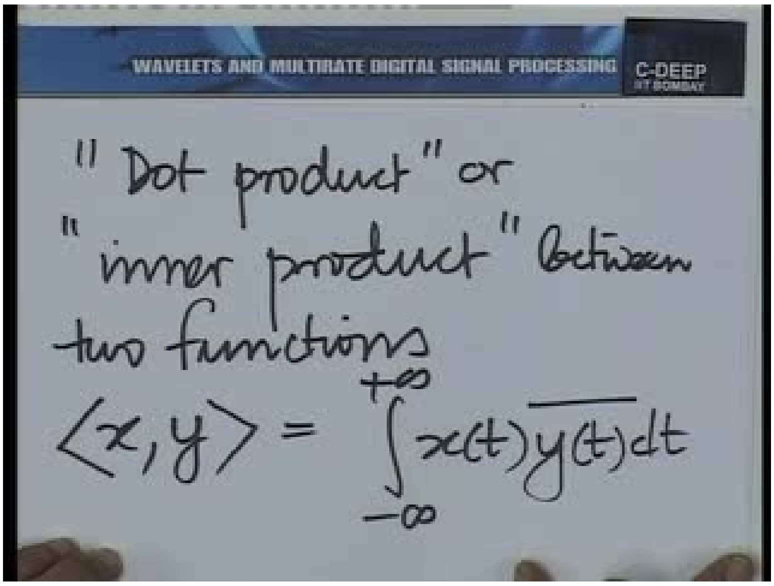

The intuitive understanding will not be different from a more rigorous understanding for those specific situations, but it may not quite be complete. Even so we would not suffer too much in our study of wavelets, in our applications of wavelets if we take this intuitive path to some extent, not all the time, I mean, to some extent in the context of dealing with infinite dimensional spaces. So, with that little prelude, let us come back to this uncountably infinite dimensional space of functions on the real line, in which case we can generalize. So, we can generalize the notion of a dot product or inner product between 2 functions.

Essentially, if I take 2 functions, x and y, both on the variable t, that dot product is not going to be a summation any more, but integral. So, x t y bar t dt, taking that idea further of multiplying corresponding coordinates and instead of summing, you now integrate. So, the integral replaces the operation of summation here.

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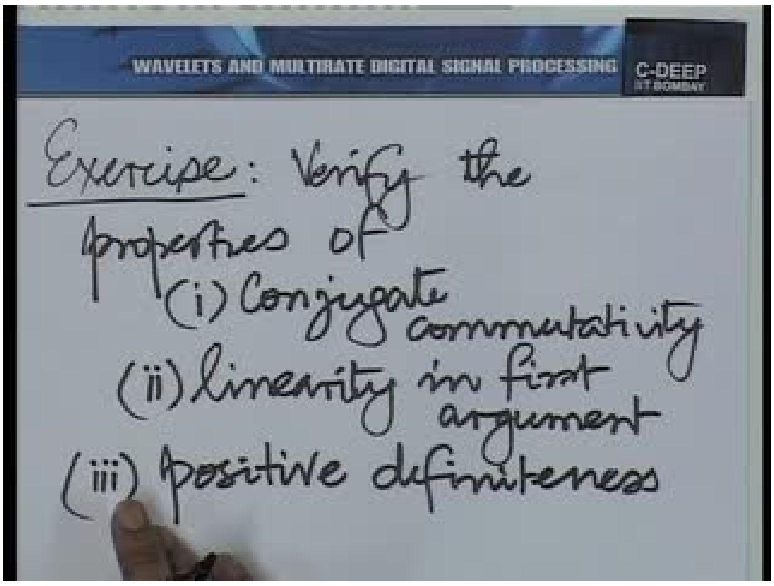

Now, of course, it is easy to verify and I leave that as an exercise to you the properties of linearity and the commutativity and so on. So, I leave it to you as an exercise here; verify the properties of conjugate commutativity. In other words, if I interchange the order of the arguments, there is a complex conjugation involved, 2nd of linearity in the 1st argument.

So, if I take a linear combination of 2 vectors or 2 functions in the 1st argument, then the corresponding inner products are also similarly, linearly combined and 3rd, positive definiteness. So, I leave this to you as an exercise, but what I wish to emphasize at this point is the famt Parseval's theorem of which we are aware in the context of the Fourier transform. So, let me recapitulate, that very important theorem in the context of the Fourier transform.

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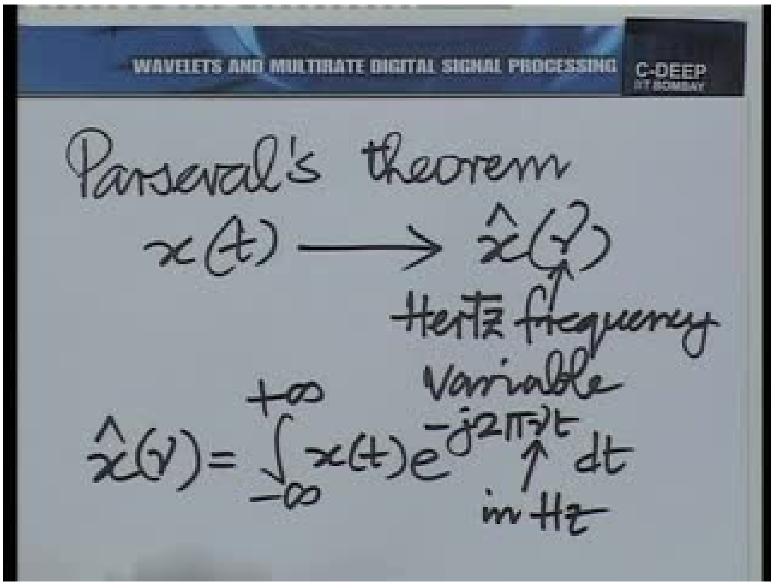

And let us also give an interpretation to it. You see, the Parseval's theorem, as we know it for continuous functions says, that if x t has the Fourier transform, now I am going to use the frequency, hertz frequency variable, so this is the hertz frequency variable, nu. In other words, what I mean by that is that the Fourier transform of x t is, essentially, integral x t e raise to the power j 2 pi nu t dt integrated overall time t. So, this is the hertz frequency variable in hertz.

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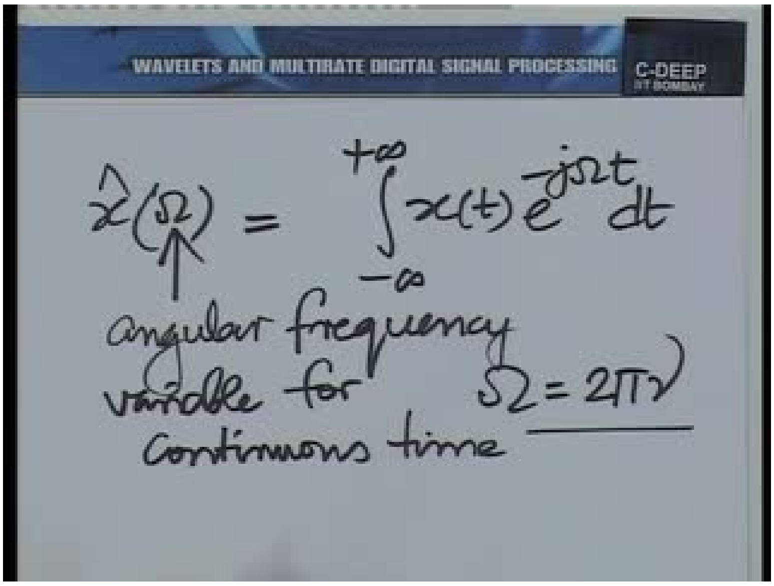

Recall, that you can also have an angular frequency variable. So, for example, you could write x cap of omega now and use this capital omega. When we are talking about continuous time we are going to follow some, notions of, different notation for continuous time and discrete time.

So, we use this as the angular frequency variable for continuous time in which case x cap omega is x of t e raise to the power minus j omega t dt. And there are simple relation between capital omega and nu, omega is 2 pi nu angular frequency in hertz frequency. Well, simple things, but we should put down all our cards in the beginning, so we do not get confused later.

Now, again this is a little bit of abuse of notation because I am using x cap of capital omega here and I am using x cap of nu there, and depending on the context, I must interpret either hertz frequency in the argument or angular frequency in radians per second in the argument.

Normally, from the context, it shall be clear and if there is some confusion likely, we will make it clear by expressive statement, but remember that from the context we should be clear, whether we are dealing with hertz frequency or angular frequency radiance per second. Anyway, with these details, let us come back to the Parseval's theorem.

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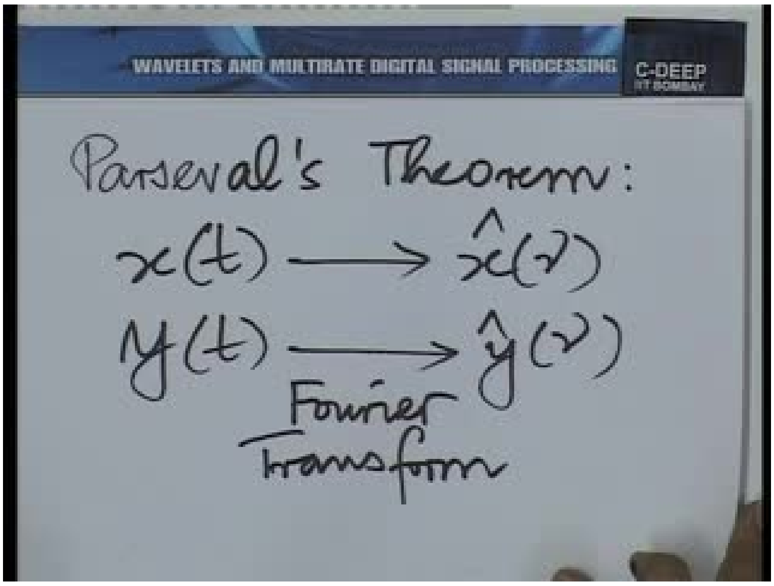

What does a Parseval's theorem say? The Parseval's theorem says the following - if you have the Fourier transforms of x and y, so if x t has the Fourier transform, let us use the hertz frequency variable x cap nu and y t has the Fourier transform y cap nu, this arrow denotes the Fourier transform, then there is an equivalence of the Fourier transform inner product and the time inner product. That is what the Parseval theorem says in our language, now.

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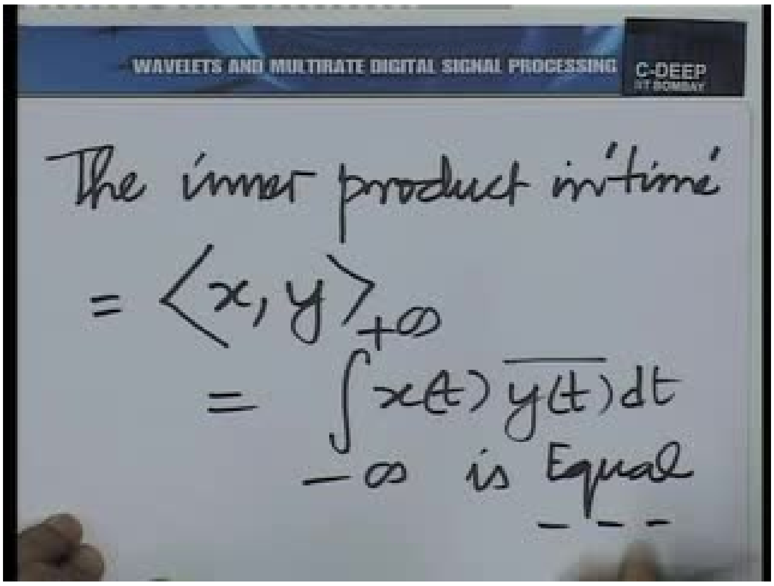

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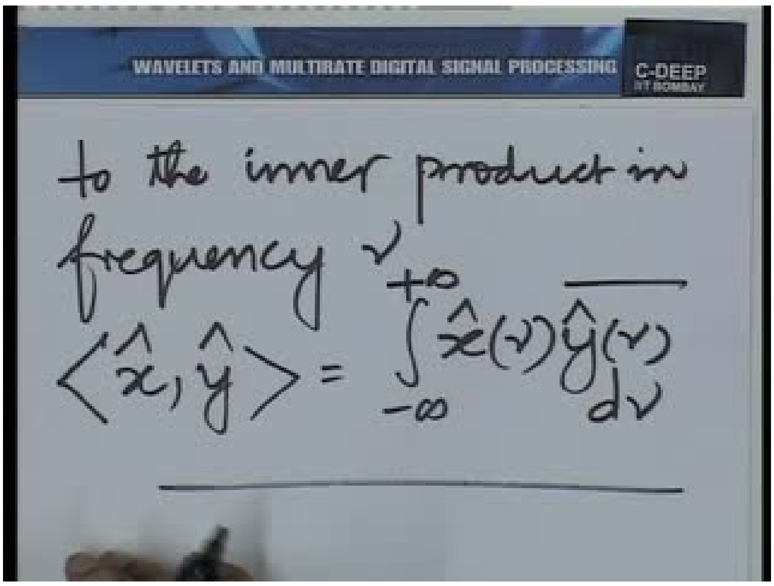

So, the inner product in time, so to speak, is equal to the inner product in frequency. In other words, if you take x cap and y cap and construct their inner product in the same way, treating the frequency as the independent variable or the argument…

Now, this is very beautiful and a very powerful interpretation of the Parseval's theorem. When we talk about the inner product perspective, we were, we have a very different way of looking at Parseval's theorem and in fact, if we really think of it a little more deeply, Parseval's theorem become so much more intuitive when we talk in terms of inner products. And let me take a minute now to show you why it is so intuitive.

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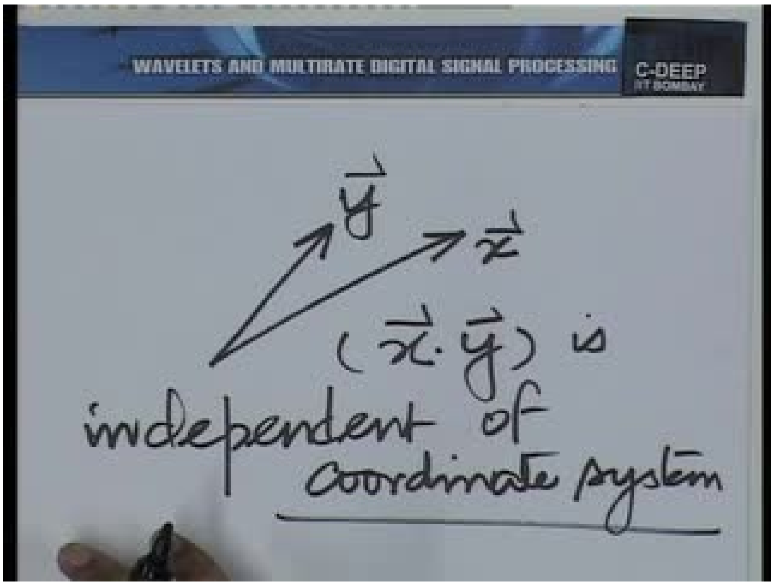

Indeed, what Parseval's theorem says, in the language of inner product is this and let us do the same in 2 dimensions and then, it will be absolutely, amply clear.

So, I have 2 vectors, let us call them x and y. Now, what Parseval's theorem says is x dot y is independent of the coordinate system, simple enough. What coordinate system we choose to represent x and y does not affect the inner product, that is what Parseval's theorem says, in a way. And to strengthen…,

See, it may not be obvious to you, why Parseval's theorem relates to this statement, it is obvious for two-dimensional vectors that the inner product is or the dot product is independent of the coordinate system. What is not obvious is, why is this related to the

Parseval's theorem? Well, towards that we need to go back to what x cap nu really is in a way and that will become clear if we write down the inverse Fourier transform.

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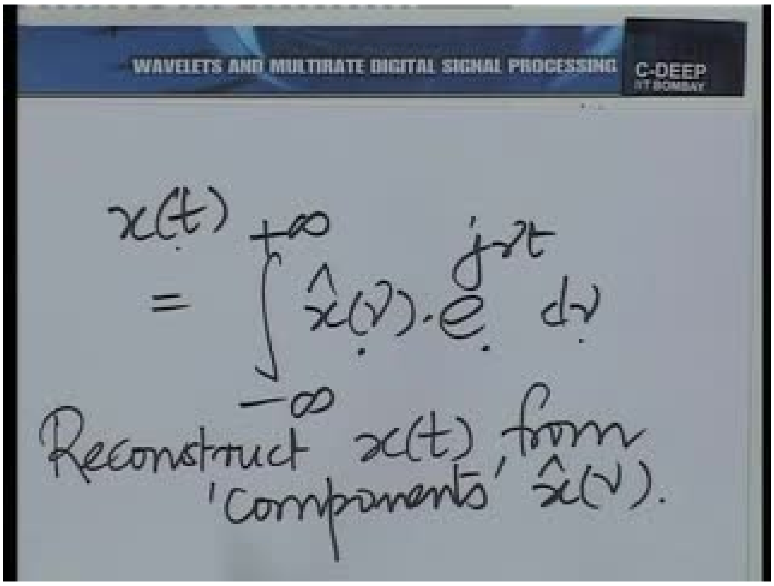

So, we can write down x t in terms of the inverse Fourier transform as x cap nu e raise to the power j nu t d nu, nu is the hertz frequency variable again. So, in a way, what we are saying is, we are reconstructing x t from its components. Each of the x cap nu for different values of nu is a component here and this is the way we have reconstructed x t from its components, and in reconstruction we have used these vectors. Each of these e raise to the power j nu t is like a vector, is a function of the real axis.

The only catch is e raise to the power j nu t is not in L 2 R function, so we have to deviate little bit there, from our discussion. But if we choose to ignore that fact, we have essentially taken these coordinates, multiplied them by the corresponding, so called, functions along each of the coordinates nu and added them to get the function x t.

So, each of the x cap nu is like a different expression of the same vector x in a different coordinate system. So, what we are saying in Parseval's theorem is that the dot product is independent of the coordinate system. Whether we choose to use the standard coordinate system of time to represent the function or the slightly less obvious coordinate system of frequency to represent the same function, the dot product remains the same.

So, these and some other such interpretations are what are offered when we represent functions in terms of vectors or when we think of functions as generalizations of the ideas of vectors.

And now, for the last remark in this lecture, which we shall build on even greater in depth in the next, namely, what is a connection between functions and sequences, continuous functions and sequences.

Just to initiate the discussion here, without completing it, completing it or rather taking it further, we shall do it in the next lecture, but just to initiate the discussion let us go back to the idea of piecewise constant approximation.

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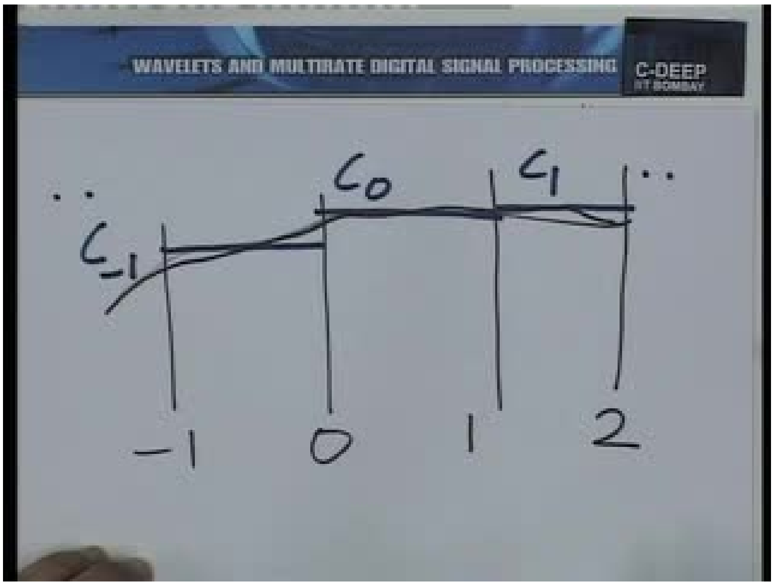

So, suppose we have this piecewise constant approximation of this function on intervals of length 1. So, I take the standard unit intervals and I make a piecewise constant representation of a function. So, I have this, so let the values be, let us say, C minus 1 here, C 0 there, C 1 there and so on, so forth.

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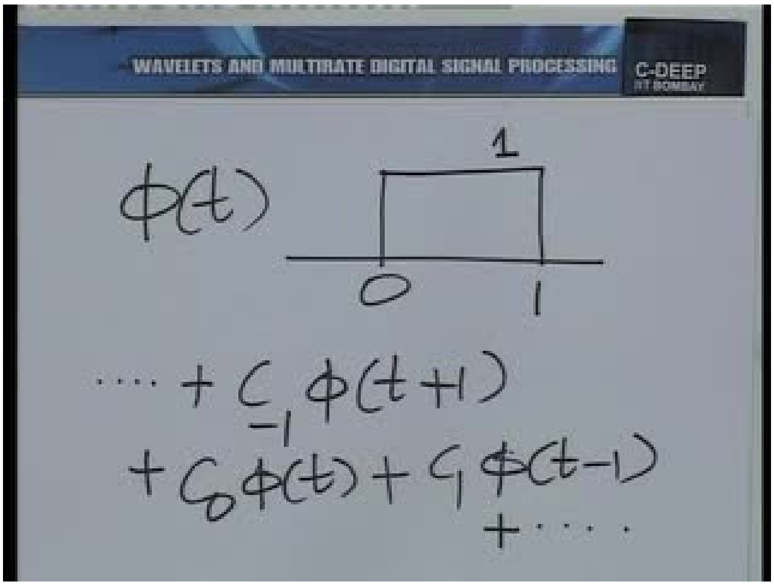

Now, it is very easy to see, that if I take the basic function phi t, describe this way, 1 between 0 and 1 and 0 elsewhere, then this piecewise constant representation can be written as C minus 1 among other terms phi t plus 1 plus C 0 phi t plus C 1 phi t minus 1 and what have you afterwards.

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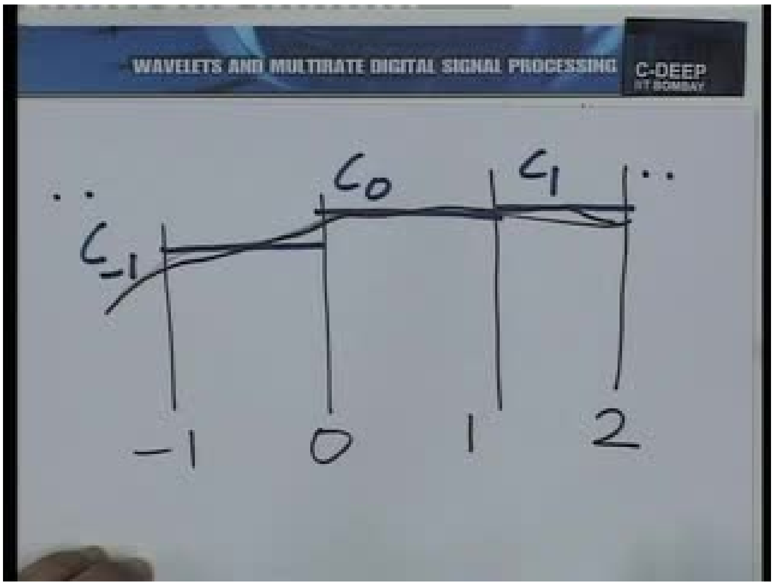

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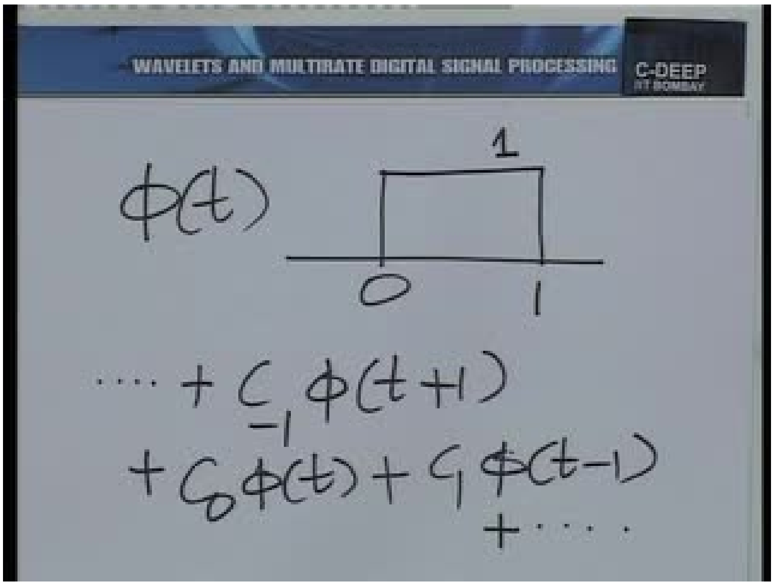

So, to conclude just this introduction of this correspondence, we can note, that equivalent to this piecewise constant representation, that I have here, this function in v 0, that we talked about last time, equivalent to that function is the set of values: C minus 1, C 0, C 1, and so on.

So, the sequence C n, n overall the integers is equivalent to that piecewise constant function in v 0. Any of them can be constructed from the other. From that piecewise constant function we can construct the sequence, from the sequence we can construct the piecewise constant function, given phi t.

Now, this equivalence is what we shall take further and delve into deeper in the next lecture and in the next lecture, we should also build further these ideas of vector functions and sequences.

Thank you.