Advanced Digital Signal Processing aWavelets and Multirate

Prof. V. M. Gadre

Department of Electrical Engineering

Indian Institute of Technology, Bombay

Lecture - 39

A warm welcome to the 39th lecture on the subject of Advanced Digital Signal Processing, specifically Multirate Processing in Wavelets; in the previous couple of lectures, we have looked at some very interesting x positions by students, who looked at applications of wavelets and multirate digital signal processing. One of the successes and one of the joys in conveying a subject is to see how well it is reflected in the understanding of the student.

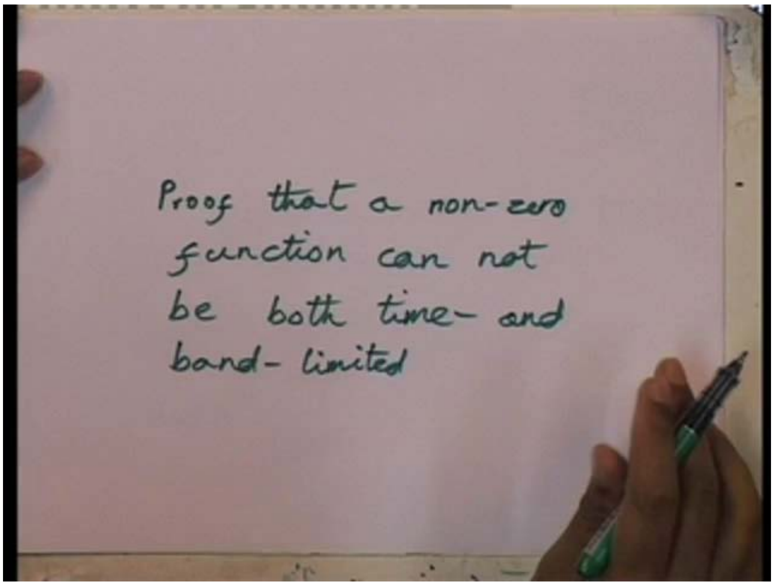
And more so, if the student could actually challenge the teacher by giving an alternate proof or by rectifying an inaccurate statement, and by providing a more accurate statement with a more accurate x position. So, among the audience of the course has been other than the students who presented some excellent application presentations, in the previous couple of lectures. One student who pointed out an inaccuracy in the way in which we, dealt with the very fundamental principle but, you know perhaps not as accurately as it should have been.

So, he pointed out that although it was correct, that a function cannot be compactly supported both in time and frequency, he also pointed out that the reasons for this had not been accurately explained; and I was very happy to see that, the worst is alternate proof which he brought out, which I shall now ask him to present, in this lecture the 39. So, I will present to you (()), one of the students, one of the audience of the course, who then looked carefully at this issue of trying to be compactly supported in both domains time and frequency, and came out with the much better explanation of the inability to have compactness in both the domains.

I have asked (()) to present himself both the theme, which is going to talk about, the concepts that underlie the proof and the proof itself, so here we have (()) to follow.

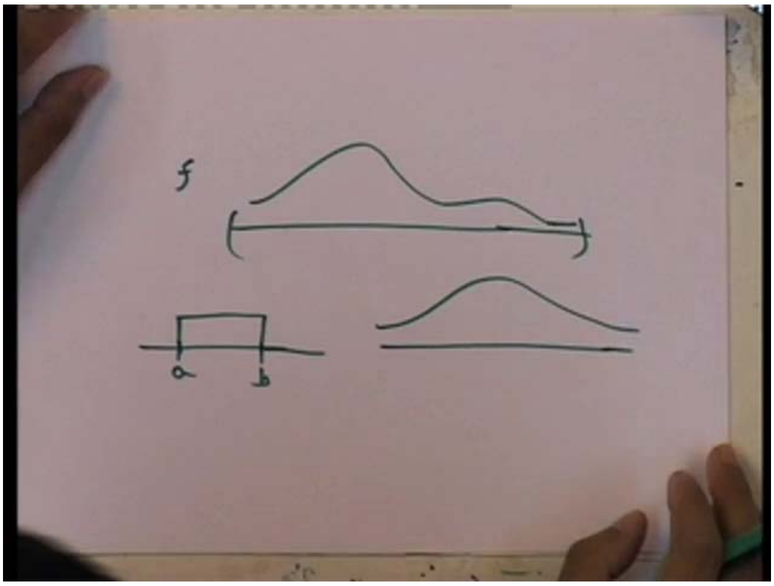
Hello everyone, today we will be looking at a particular proof of a theorem the stunt, the uncertainty principle in Fourier transform.

(Refer Slide Time: 03:32)



So, here is what we are going to look at proof that a non-zero function, cannot be both time and band limited at the same time, basically this we mean, that a function cannot have a compacts support both in is time domain and (()). So, these already a well known theorem some proof's of this exist, we will have a one of the existing proof's and also the that I have function it up, so here is what you mean by time limitedness and band limitedness or compactness in general.

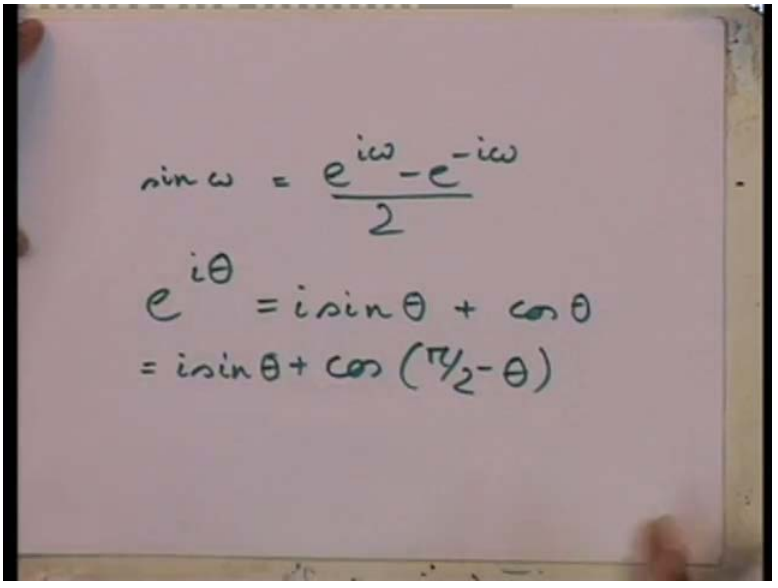
(Refer Slide Time: 04:38)



So, suppose your function f which we draw like this, with the support of the function we mean the interval, outside which in the function is 0 for example, if I take a rectangular pulse like this, in the support of the function is this, the states a to b in the support is just the interval a to b. On the other hand, if I take the Gaussian function, the support is a whole layer length. So, function is a to b limit or set to have a compact support, when this support is finite in length for example, for the rectangular pulse.

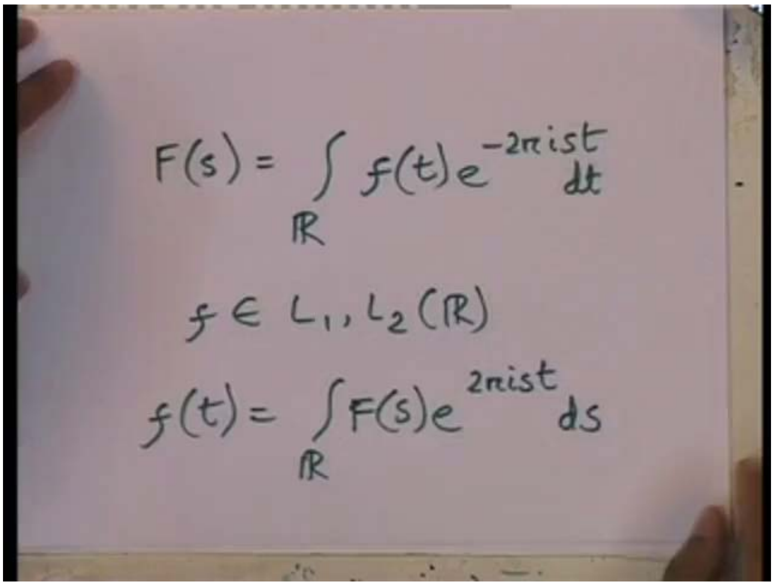
So, let us first looking to the general background on which we will you working, so we will be considering Fourier transform, which is one we have looking in to a signal from its frequency domain, by frequency we means sin waves, sin waves define frequency. So, over here, we will be considering translates of sin waves, and that linear combinations, so if you do it in the complex domain, we can bring out exponentials.

(Refer Slide Time: 05:58)



Because, we will split up sin functions like this, sin omega can be written as e to the power I omega minus e to the power minus i omega by 2; so if I want to write the complex exponential e to the power i theta, I can write it us i sin theta plus cos theta but, again cos theta is a translate of sin theta which is pi by 2 minus theta. So, basically like all other transforms the general idea of Fourier transform is to project a function on to the space of e to the power i omega t or e to the power minus i omega omega t.

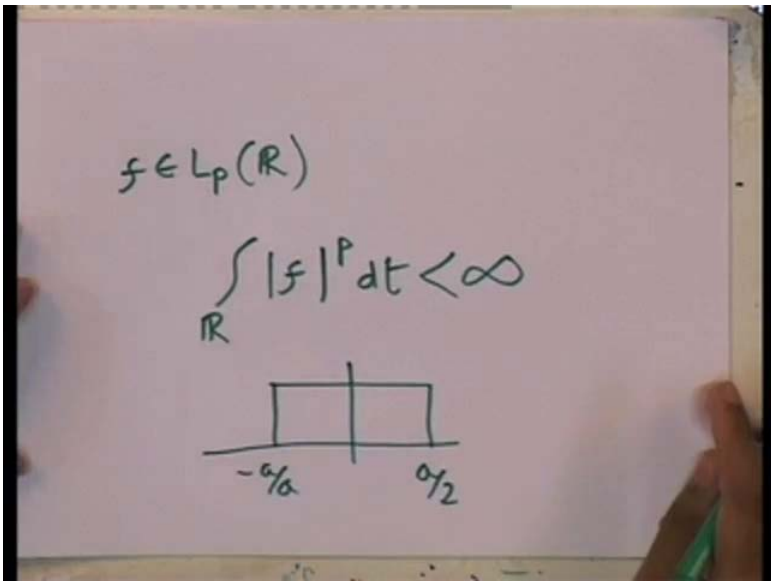
(Refer Slide Time: 06:58)



Mathematical formula is given like this, for function f in the Fourier transform if it exist is given by F of s equal to f t into e to the power minus 2 pi i s t integrated over d t for the whole of the real life, the function f is taken over the whole of the real line, the domain of the function. For the inversion of this formula, we need some conditions and efforts.

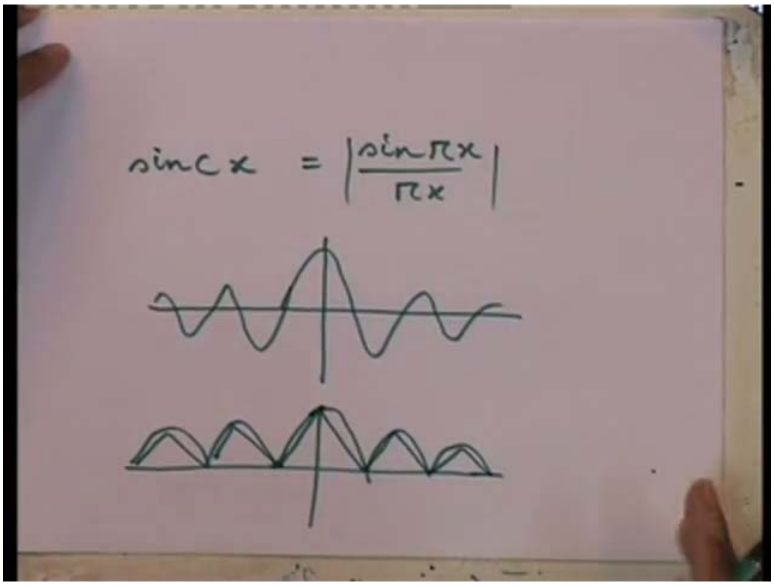
So, if the inversion holds for generally when a function f belongs to L 1 or L 2 R, then the inversion formula holds. And just given by a similar formula, there is a such change in the sin, in the dependent variable at the exponential part; so over come across L 1 and L 2 or in general we also consider this spaces L p for functions.

(Refer Slide Time: 08:20)



So, let us define that spaces first, by a space L p, if is either a function belongs to L p R, what you mean by that is the p x power of the modulus of the function, when indicated it over the wholly real line, say this is t, this is finite. For example, the function which we took rectangular function set goes from minus a by 2 to a by 2 this belongs to L p basically, all L p's for t real numbers.

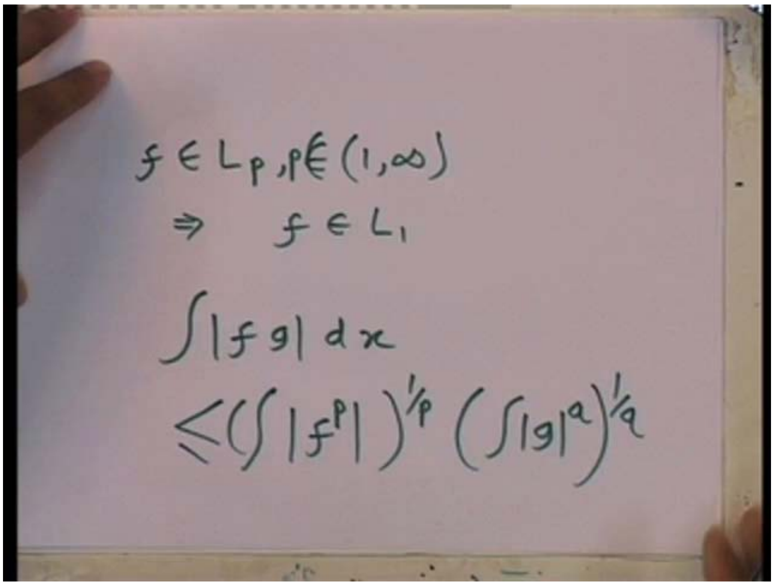
(Refer Slide Time: 09:03)



On the other hand, the function sinc x, by sinc x we need sin pi x by pi x, so if you draw and goes at x equal to 0 then unit goes to 1, so it goes something like this, but this function does not belong to L 1. Because, if you take the modulus of this, the areas become positive here, the areas cancel out a bit, and the final integral becomes pi by 2 on the other hand, here this part become positive, and if you fit in to a triangles in to this we can see that the any of the triangles diverge; so, the area of modulus index is also going to diverge.

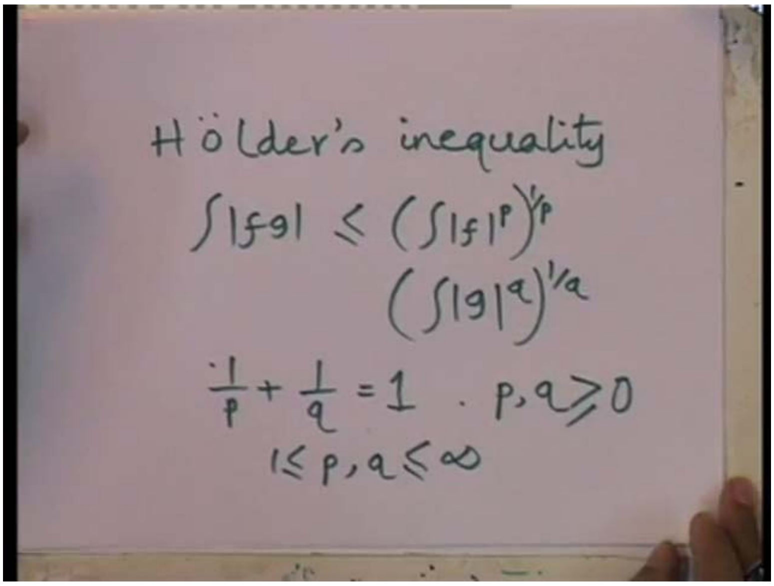
So, in this new proof will be extending, will be a first considering functions which belong to classes L 1 and hence, will be extending the class of functions to general L p, p from 1 to infinity. So, this very general inequality, which states that when a function belongs to L p, p between 0 to p between 1 to infinity, it also belongs to L 1.

(Refer Slide Time: 10:30)



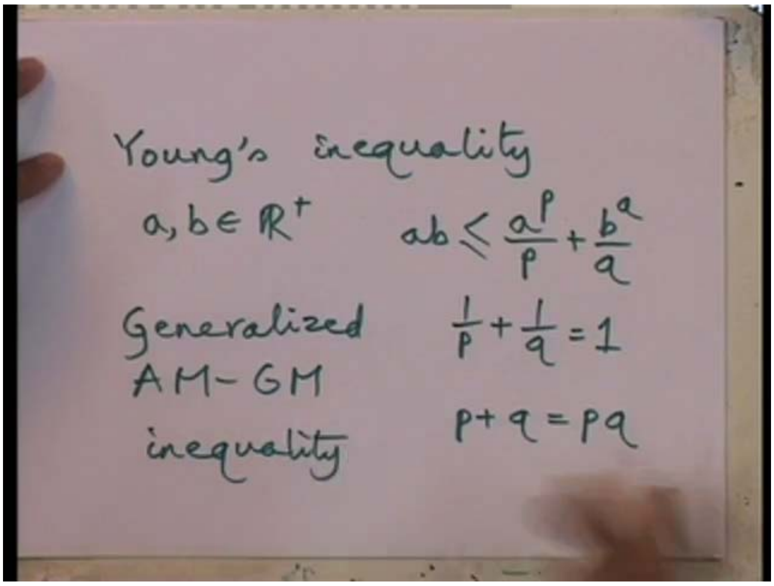
So, mathematically f belonging to L p, p belonging to 1 to infinity imply that f belonging to L 1, so for the time limited function, this can very easily be proved with holders inequality; holders in equally states that for two functions f and g, this integer is less than equal to, so you have the condition for holders inequality to hold.

(Refer Slide Time: 11:26)



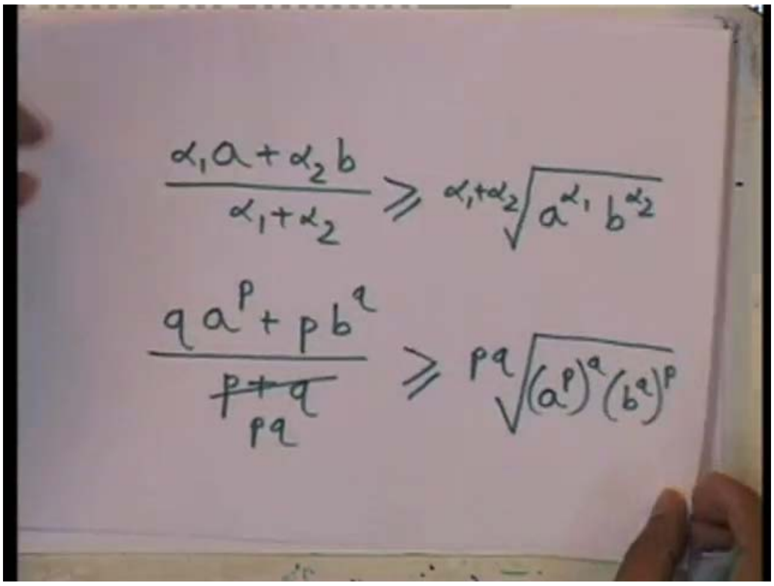
The main point is 1 by p plus 1 by q has to be equal to 1 and of course, p and q both positive, from this we can get the range of p and q, see if we take p to be less than 1, then q becomes negative. So, p can only go from p and q both can only go from 1 to infinity, equal to infinity is also allowed, if I allow 1 on one side, so here is the proof of this theorem.

(Refer Slide Time: 12:40)



This is another inequality called young's inequality, which states the following fact given any two real numbers, positive real numbers a and b, a b is less than a to the power p by p plus b to the power q by q, this thing is very easy to prove. If we use the generalized AM-Gm inequality of course, the same conditions hold over here, 1 by p plus 1 by q equal to 1 or otherwise p by q is equal to p q.

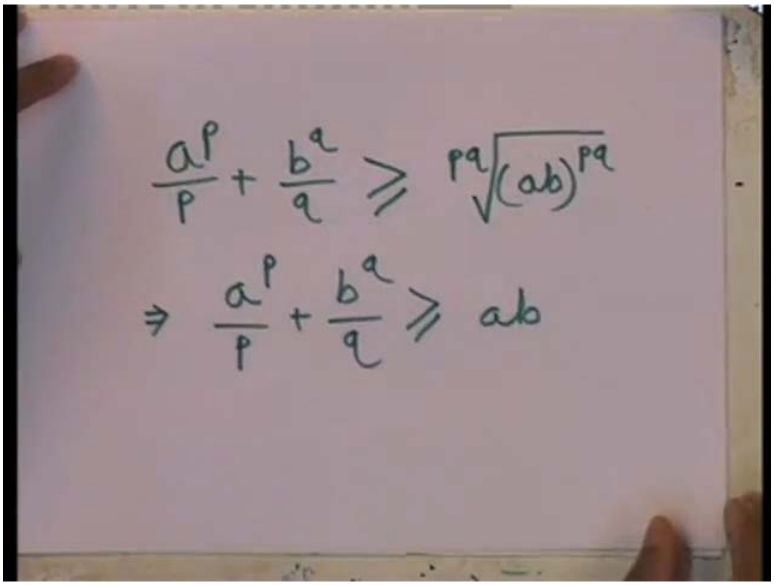
(Refer Slide Time: 13:46)



So, for the proof of this we will be using the generalized AM-GM inequality, here is the generalized inequality given two numbers a b, and alpha 1 alpha 2 are the weights of a and b, we have this result; a and b are positively real numbers. So, y here let us try to fit in the numbers which we have been given, let us take alpha 1 to be q and a to be a to the power p alpha 2 to be p and P to P to be b to the power q, below if we write p 1 p plus q, we can as well write p q instead of this.

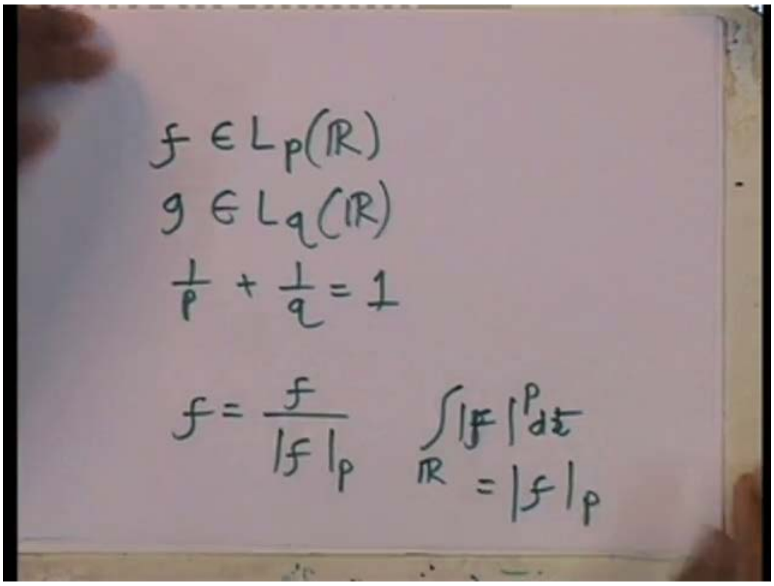
Since, we already know that 1 by p plus 1 by q is equal to 1 similarly, on the other side we get p q and the p q eth root of a to the power p whole to the power q and b to the power q whole to the power p.

(Refer Slide Time: 15:18)



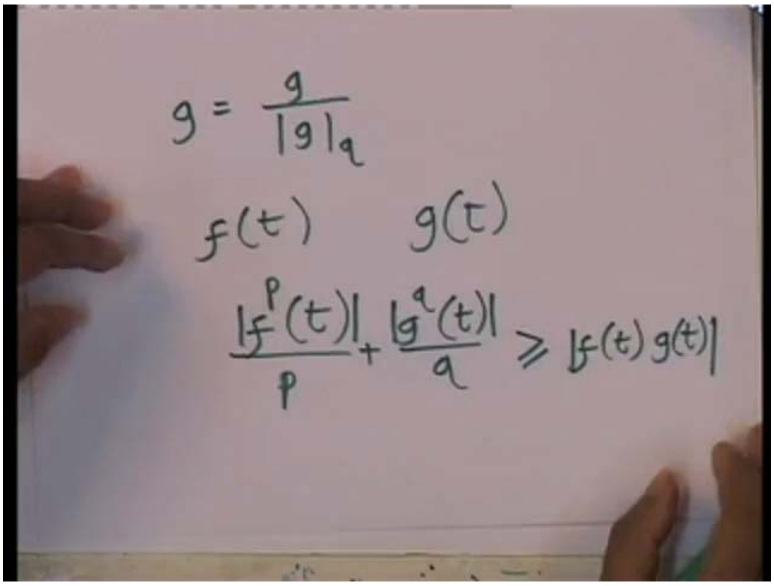
Taking real real values of p and q and as their not 0, we can cancel this and what we get is a to the power p by p plus b to the power q by q on the left hand side; on the right hand side we get the p q th root of a b to the power p q, this is greater than equal to so finally, you have got the young's inequality.

(Refer Slide Time: 15:59)



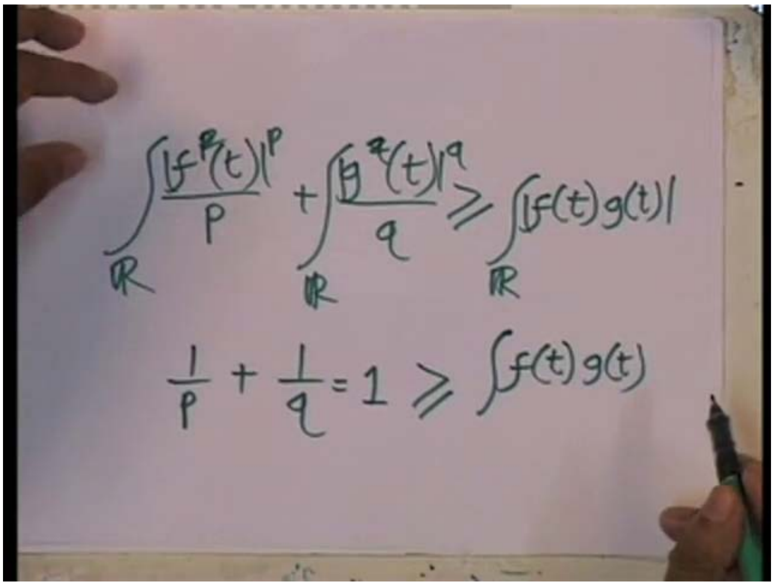
Now, say there are two functions f and g such that, f belongs to L p and g belongs to L q with the same conditions 1 by p plus 1 by q equal to 1. So, from young's inequality, let us define redefine f s, f is equal to f by the p eth norm of f, the p eth norm is this over the whole of d t.

(Refer Slide Time: 16:51)



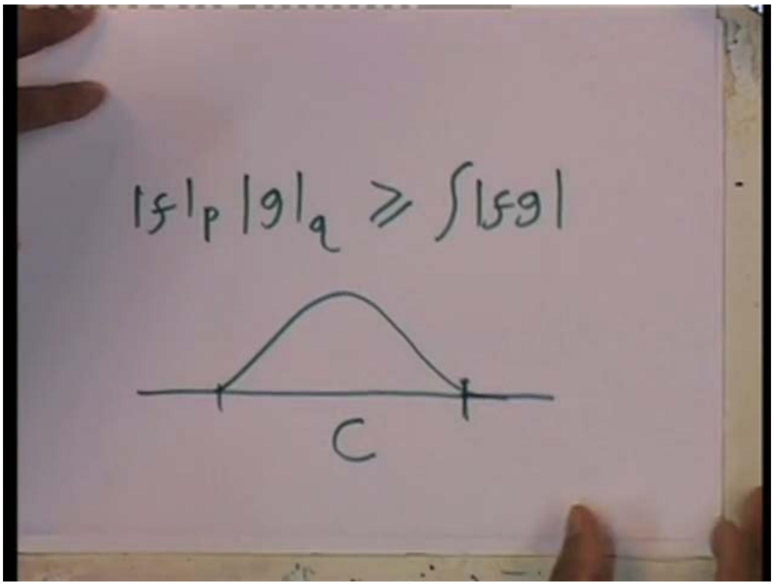
g is also redefine the same way g by q eth norm of g, now let us put in this f and g point wise as numbers f (t) and g (t) in the young's inequality. So, f (t) to the power p by p g (t) to the power q by q is greater than equal to f (t) g (t), f (t) and g (t) can be comprisal functions also, so we take numbers by defining the module.

(Refer Slide Time: 17:51)



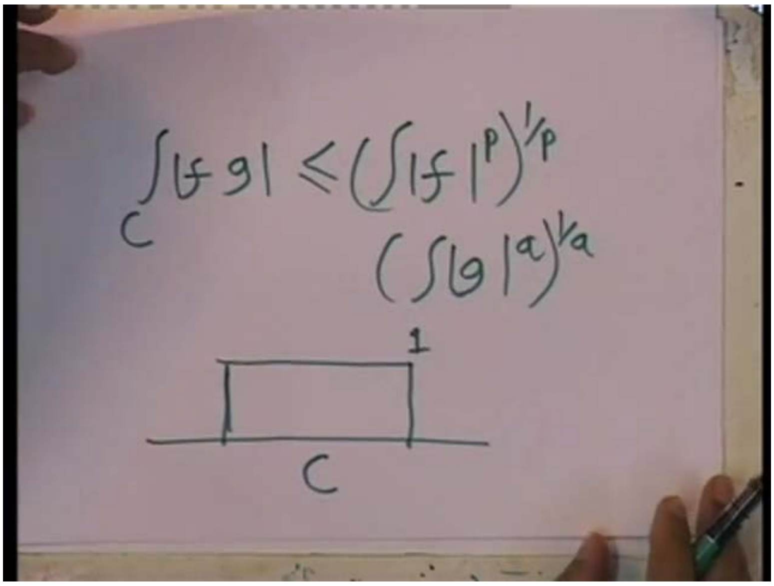
So, if we integrate this over the whole of the real line, you basically get out holder's inequality, because on one hand, we are going to get 1, because is the weighted p eth norm of f (t) already. We get 1 by p on one side plus 1 by q on the left hand side which equals 1, and on the right hand side, we have got mod f (t) in to mod g (t), the integral where f (t) and g (t) are weighted.

(Refer Slide Time: 18:59)



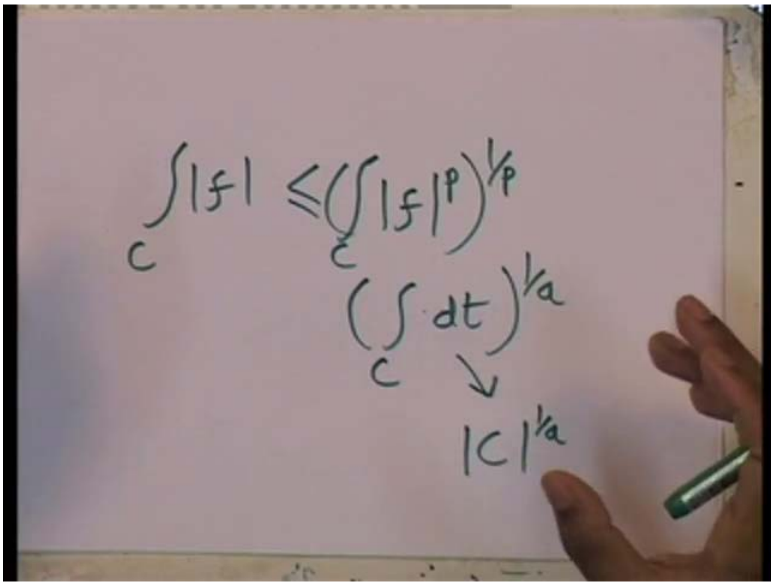
Now, if you bring out the weights and multiply with 1, we get that the p eth norm of f into the q eth norm of g is greater than equal to the integral of the modulus, so this basically proves holder's inequality, how are we going to use this in our proof. So, say function f is time limited, can we show the defeat belongs to L p class it belongs to L 1 class also, yes we can by choosing a suitable g, so say the function that we are interested in is non-zero, on this particular interval C only, compactly supported the C and 0 everywhere.

(Refer Slide Time: 19:47)



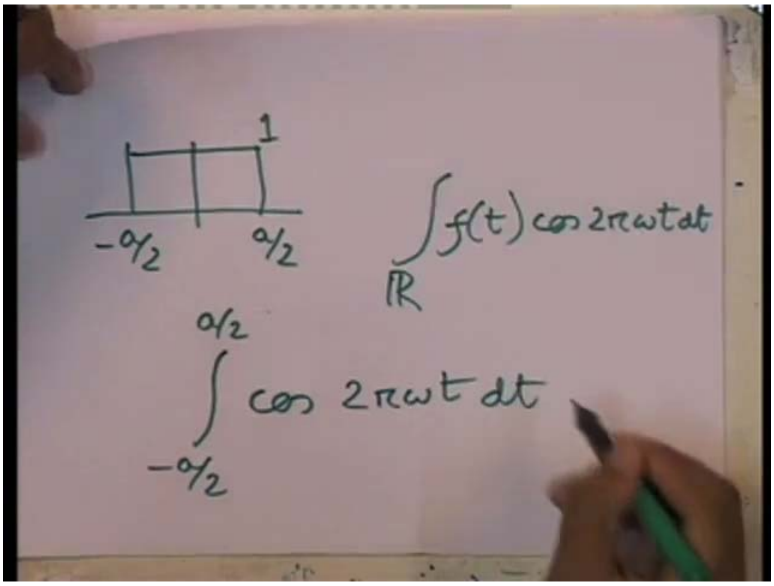
So, we can write down only on C, because everywhere else the function is 0, the holder’s inequality; now we can choose g in such a manner that g is 1, only on the interval C and 0 everywhere else.

(Refer Slide Time: 20:24)



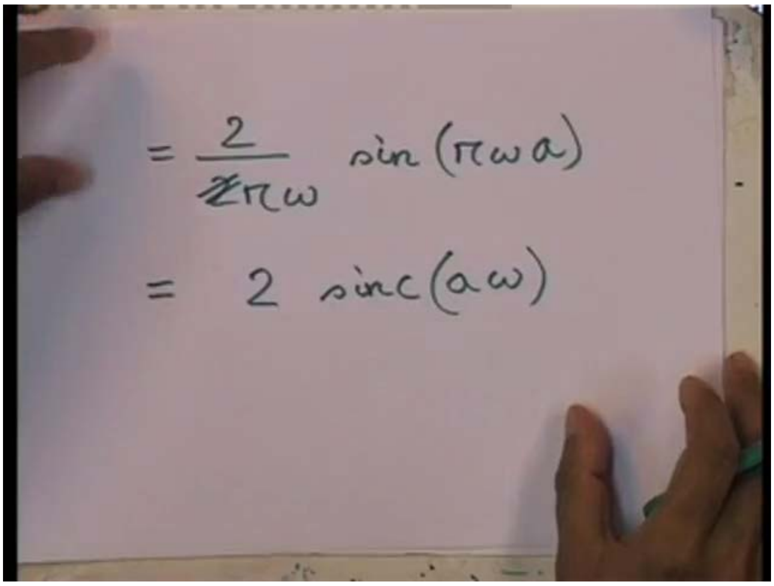
In that case on the LHS only mod is remains on the RHS thus the p eth norm to the power 1 by p of f and g has been replace by 1 on the compact support C, so 1 into d t to the power 1 by q, which will basically give me out the measure of C to the power 1 by q. So, on the right hand side it is a norm of f p, which we already know is finite and the measure of C is also finite as a f has been assume to be time limited, on the LHS at this is less, this has necessary f should necessarily belong to L 1. Next will move on to some tools that will be using for this whole thing, the tools are particular Fourier transforms, and in linear algebra the (()) on matrix.

(Refer Slide Time: 21:34)



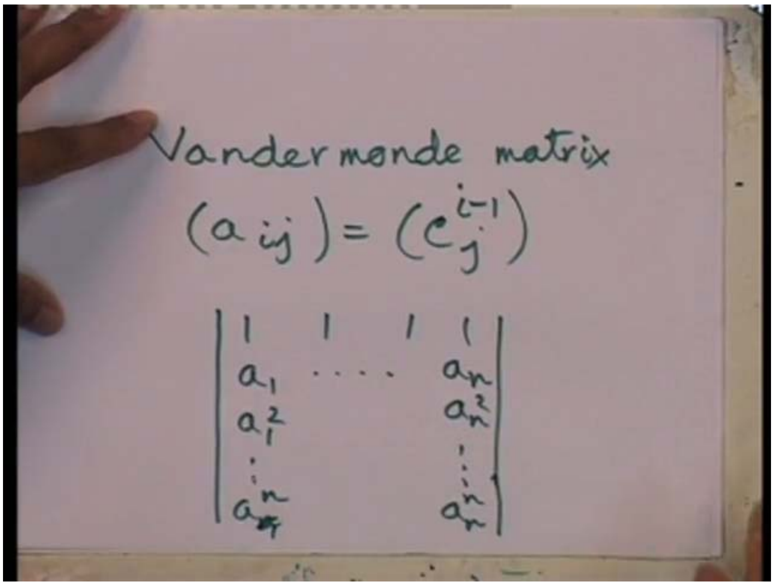
So, you will be considering the Fourier transform of the rectangular pulse, it is defined as you have a value 1 between a by 2 to minus a by 2 and 0 everywhere else, so this is the real symmetric function; in Fourier analysis, we know that it real symmetric function gives the real symmetric functions. Otherwise, what we can do is in the in the Fourier integral for a symmetric function f t, we can just take the cos 2 pi omega t, because the sin omega t will give as a 0. So, let us see what comes out over here, now the interval changes from minus a by 2 to a by 2, because all the rest is 0.

(Refer Slide Time: 22:35)



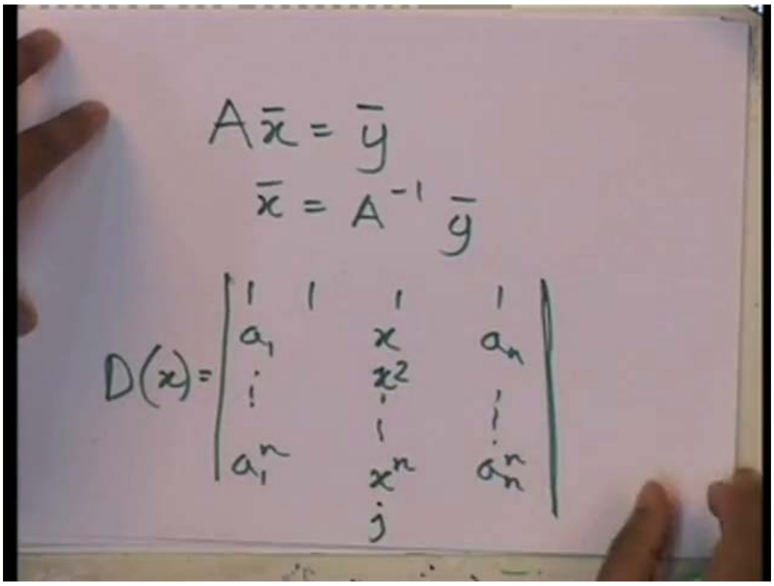
And the way we define the sinc functions, this is the sinc a of omega and this structure of the sinc function, the factor a is only inside the sin part is very crucial for a proof.

(Refer Slide Time: 23:14)



Next that Vandermonde matrix, this matrix is defined as C j's at the different values for different columns, and their corresponding powers in different rows i minus 1; so when it is written down matrix becomes like this, a 1 and square up to say a n, a 1 to the power n on the other side, if it is a n then it goes as a n square and stops at a n to the power n.

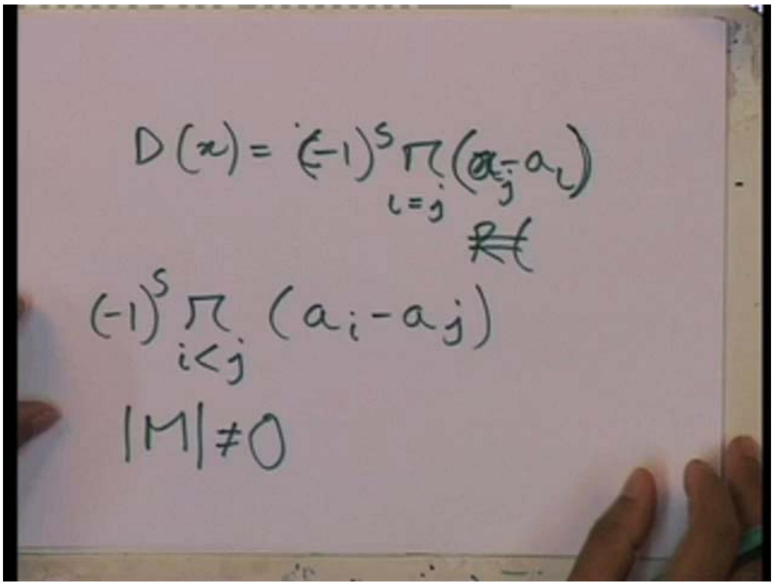
(Refer Slide Time: 24:14)



So, whenever we have a linear equation A x equal to y vectors, we always looking at invertability of a, so that when we know y we can solved x like this, now Vandermonde on matrix very nice property for which, it can be shown to be invertible, if we just know the entries. So, it is how it works, so a 1 to the a 1 to the power n and in between, let us replace any particular column by x, so it goes as x x square till x to the power n. Now, let us take the determinant of the matrix, so we can take the determinant here already, let us call it D (x), this will be a polynomial in x, with the maximum power x to the power n.

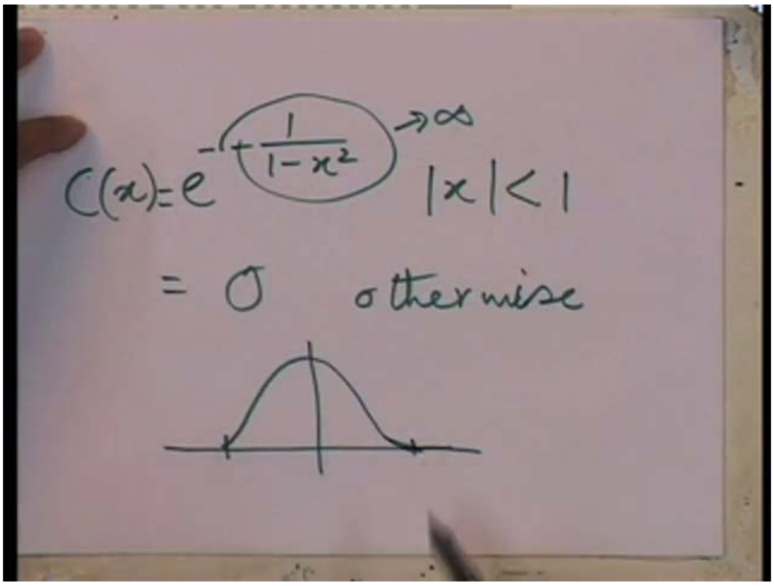
If we put x equal to a 1 if we put x equal to a 1 this two columns are going to be the same, and the determinant will come out to be 0. Similarly, for all other a n's except put this x at the j eth column, except a j for all other a's this determinant is coming out to be 0. So, the polynomial D (x) will be of the form some constant which is (minus 1) to the power some power into pi (x minus a l) l goes from 1 to m but, not equal to 0 and the rest part, this rest part can be found out, now let us replace x by a j, so it is (a j minus a l). Similarly, in this matrix we can replace other columns by x also, and with the same operation; so for each and every column, there will be a product like this, but in the determinant the highest power is a 1, that we can have is a 1 to the power n with that limitation, what finally comes out they combining all this product is a i minus a j, say i less than j, so that we do not repeat and i is not equal to j ever, in that case it is just 0.

(Refer Slide Time: 25:57)



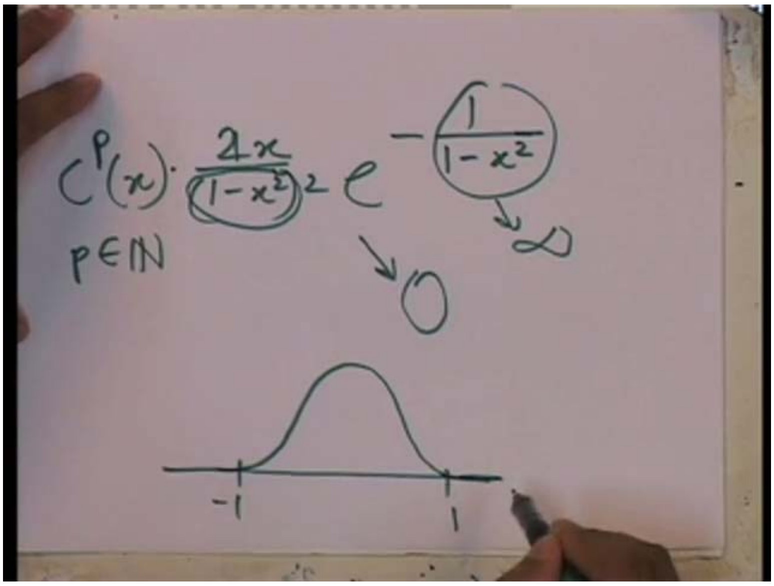
Now, matrix can is invertible whenever the determinant model is not equal to 0, so over here whenever my entries in the first row a 1 up to a n, and mutually distinct we get that the determinant is not equal to 0, so it is always invertible, in that case. Next will move on to a theorem, that a function cannot be compactly supported both in it is time and in it is Fourier domain; so the function only non-zero functions will be considering, because in our proof we will be showing that the only function which is allow to have this property is the identically 0 function.

(Refer Slide Time: 28:18)



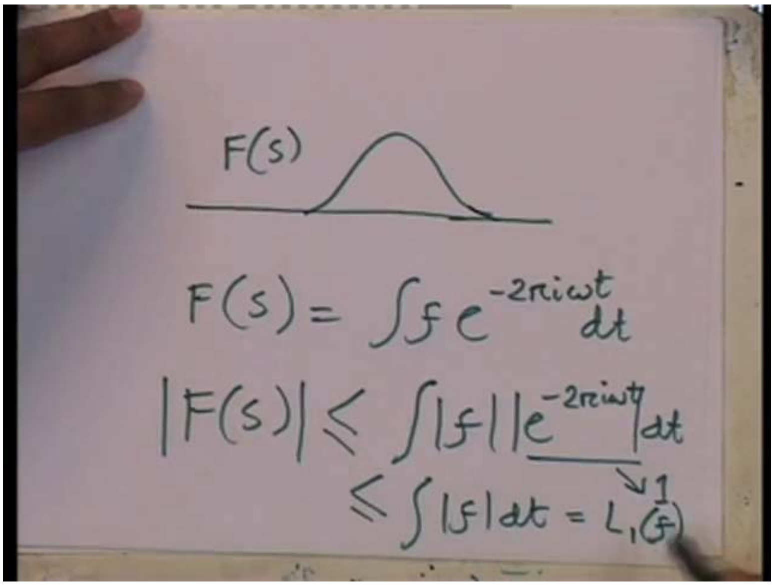
So, let us first consider some very smooth functions which have compact compact support for example, in the function e to the power in between minus 1 to 1 and 0 otherwise; say it can be shown the function has the structure like this e to the power x falls off, when x goes near 1, this part goes near infinity, and because of the minus sign c x tends to at 0.

(Refer Slide Time: 29:11)



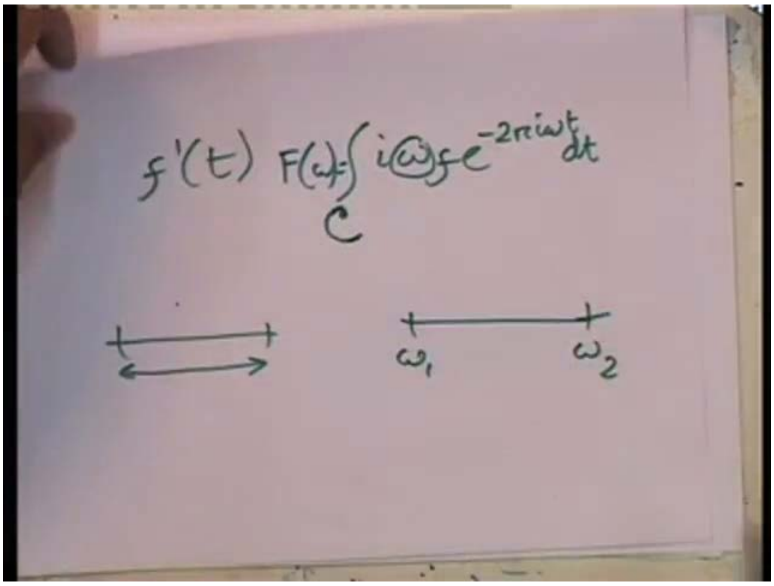
Not only that, if we take the derivative C s C s, what we get is the minus, but cancels out now as x goes near 1 this term goes infinity as before, and thus function part also on the other hand, the low part of this thing goes towards the infinity but, exponential if also faster than any polynomial that can be made. So, overall this thing turns at 0, if we take the element, and this wholes all derivatives not only C prime x it also all C p x p belonging to natural number, from this it can be shown that the function is very smooth said minus 1 and 1, this function and all its derivatives end towards 0. So, can we have a very well be a function like this also, for which the Fourier transforms is completely supported; so the various proves, which use various certain things for analyticity holonomicity and all those things.

(Refer Slide Time: 30:49)



So, how we first state one proof, and give a briefly overview of x when a function is band limited, and say the function f belongs to L 1, then as F (s) can be written like this, within have a bound on the modulus of F (s), now this is 1, if this is the L 1 of (f).

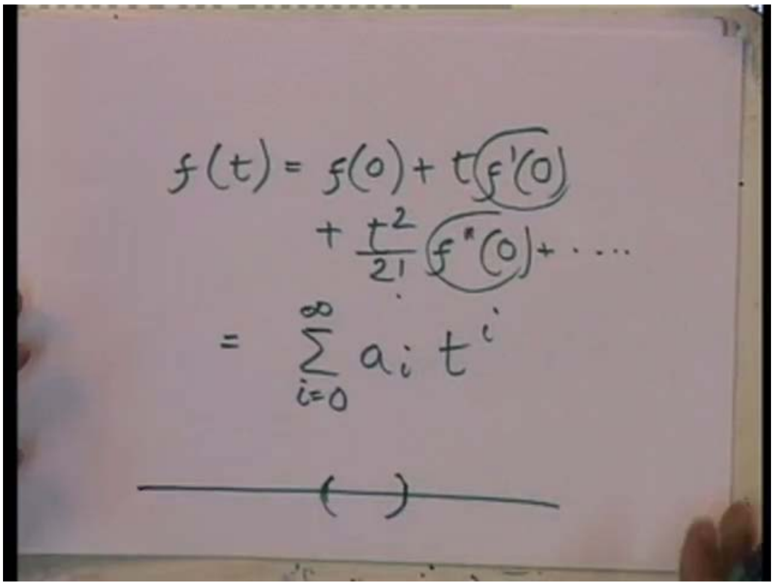
(Refer Slide Time: 31:52)



The Fourier transform of the derivative of f f dash (t) if it is exist it is going to be i omega f into e to the power minus 2 pi i omega t d t; now there as we f to be time time limited. So, instead of the whole line, the only interval we can handle now, we will be handle now is the compactly supported interval C. Now, over here also because, omega is in C only t sorry, t is in C only when we take the modulus again we will get a bound, so f dash t.

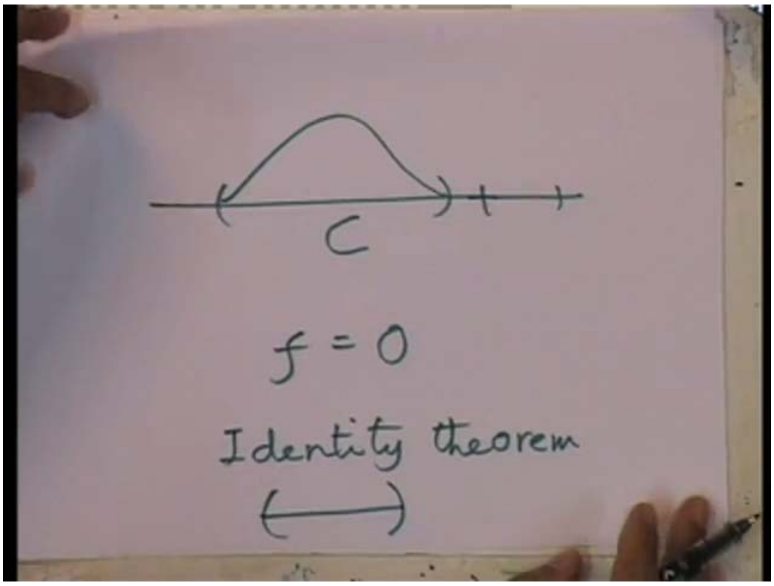
We can show that all these derivatives as f dash t whole that all points, when f is both time limited and band limited. So, over here, f is band limited, so for only some interval omega 1 to omega 2 we would have been non-zero f omega, and for the rest that would have been 0.

(Refer Slide Time: 33:18)



So, that words we can write the function f (t) by the expansion as f (0) plus t into f t (0) and so on; now t to the assumptions f all these number exists, so this is basically a power series expansion. And this wholes for all t, in that case by the identity theorem, if this power series is 0 and any open interval it is going to be 0, everywhere on the real line.

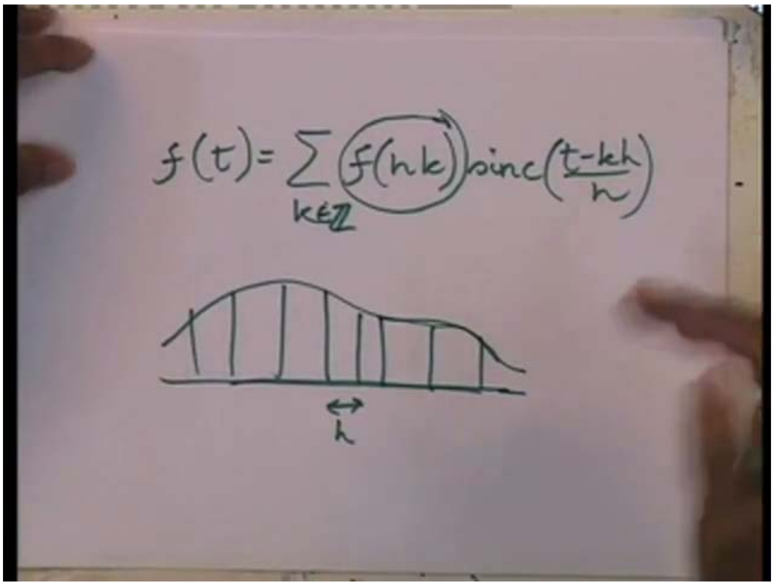
(Refer Slide Time: 34:17)



So, compactly supported f of course, the power series is 0 outside the compact support C, in that case only the 0 function, f equal 0 is allowed, because a function 0 by the identity theorem; this theorem states that a function f, which is holonomic which is the power series everywhere is defined completely, by any it is addition any opponent. Another whole what a version of this proof is new proof is that all this previous proves, that there is in holonomicity or celebrate a theorem this (()) complex analysis, and request mastering complex analysis, you have to know quite simulate of complex analysis to note this basics.

On the other hand, we have many other ways of looking into signals the unlimited signals of course, we have come across a (()) the construction formula or the channel we take cardinal series whatever it is called, the reconstruction formula, is a way to looking to this theorem with their..

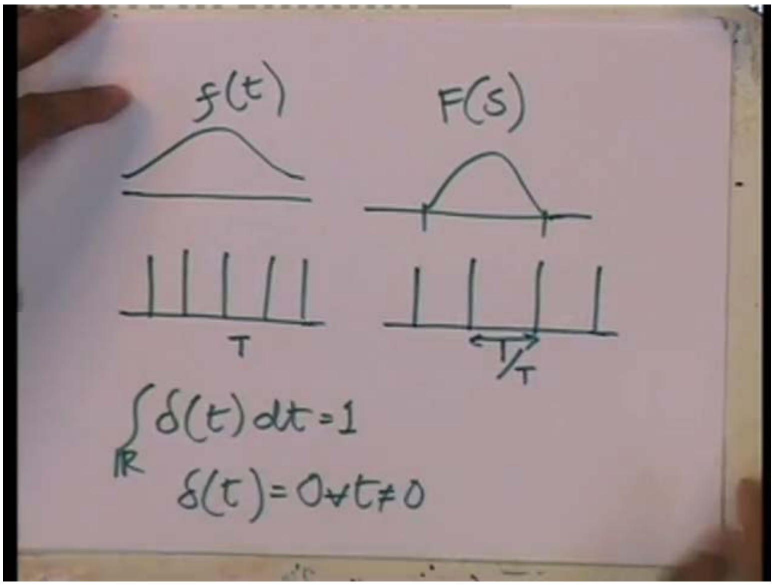
(Refer Slide Time: 35:43)



So, in channel revaluation signal processing he give this formula, which is called formula (()), what is the revaluation above 10, so this formula wholes when f is band limited, this f (h k) is in the samples, so say this the function f like this, can we have an operation for measuring the function f (t) repeat the value of f (t) at each an every point, instead of that if there is an limitation on f over here, the limitation is in the Fourier domain its compactly supported.

So, can we remove the redundancy if there is some, so redundancy in the Fourier domain because, it is the unlimited we can take only the samples that we are going to intervals, so of sampling time it goes our all case and from that we can get back our f (t), we will show graphically how this works.

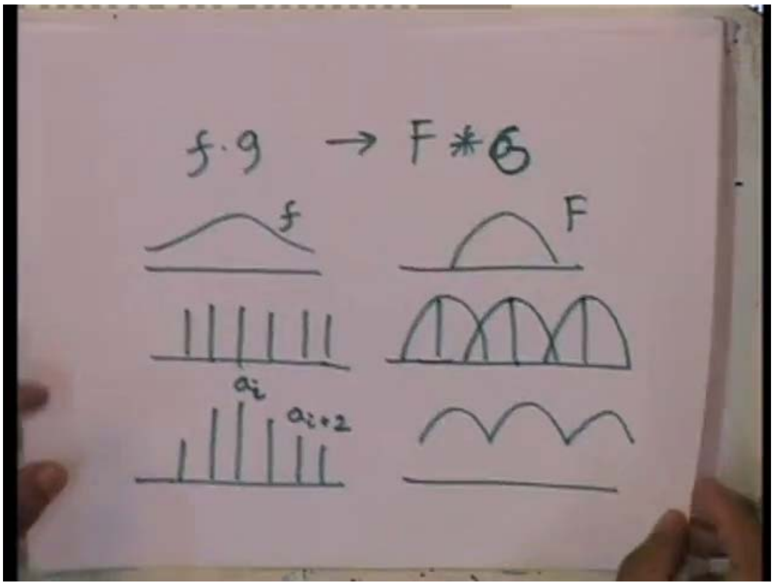
(Refer Slide Time: 36:59)



So, say a function f (t) is compactly supported in its Fourier domain, we will using distribution theory for this function, for this proof but, in the actual proof, I will used just real analyst no distribution theory, but for the sake of gravity will be showing here with distribution theory which equally holds. So, if we taken dirac delta train functions, by delta function we mean say function delta function is defined like this, the integral over the whole real line is 1 but, the function is 0.

But, all points except t equal to 0, so this is not actually a function in the normal sense, it is a distribution, for dirac delta train, on the other hand side we also get another dirac delta train, series of pulses like this, so this is t the difference between this is 1 by t.

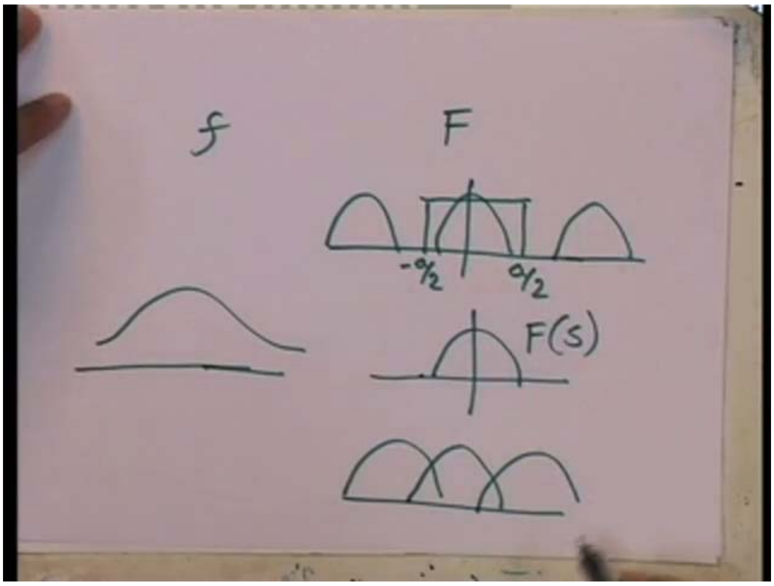
(Refer Slide Time: 38:37)



Now, we know that multiplication in the Fourier domain implies convolution in our real set the time domain, and visa versa. So, here you multiplying the time domain with delta functions, and this gets the sampling it just picks up the values at this particular points, and real it is 0, so we get so we delta delta functions a a i a i plus 2 second method.

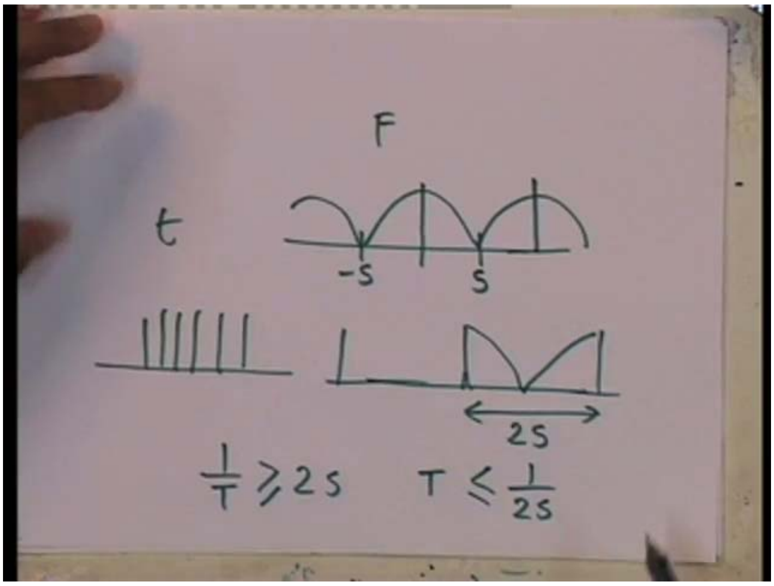
On the other side capital F is getting convert the delta function, this another property of delta function that its convolution with any function gives the same function there. So, over here this whole capital F will be repeated along the delta train at each and every delta, it will be repeated, and the whole thing will be summed up, final outcome is something like this.

(Refer Slide Time: 40:04)



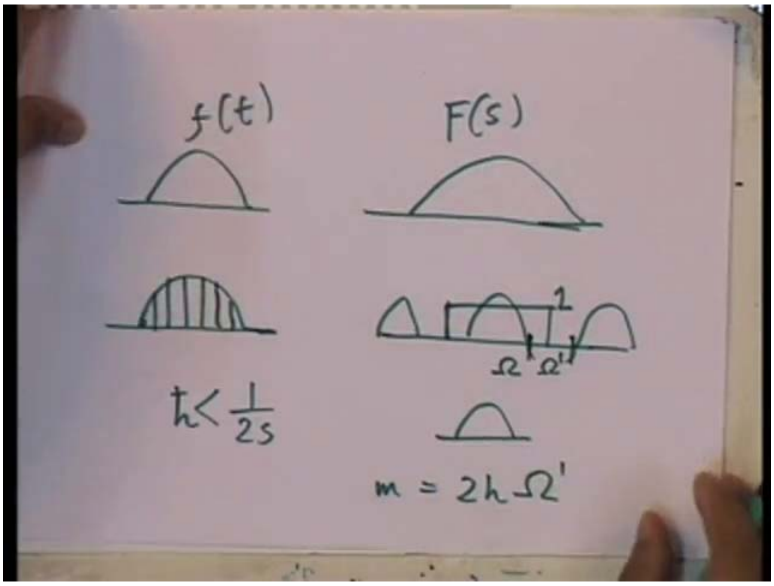
Now, can we reconstruct the signal from this, see we had the repetition repeated like this, if we did ideal low pass filtering with a rectangular pulse over here, on the other hand side, on this side we would have get actual F (s) back. Now, Fourier transform is a one to one operation, so on the other side the corresponding operation should revise back f only, so over here we can already see, what is going to happen the threshold conditions, say the repetitions the F (s) over over lapping like this, we cannot do the ideal low pass filtering.

(Refer Slide Time: 40:58)



So, the condition for ideal reconstruction is that the capital F loads have to be well separated, and this is the threshold condition weather exactly touch, say this is band limited by S minus S to S, in that case our repetition rate has to be good enough, so that they become well separated. If we make the delta trains closing the time domain, in the frequency domain there going further part; so threshold condition this is 2 S and 1 by T, so 1 by T should be great than 2 S, great than equal to 2 S, so T should be get less than 1 by 2 S, and when this equality that is a (()) threshold condition.

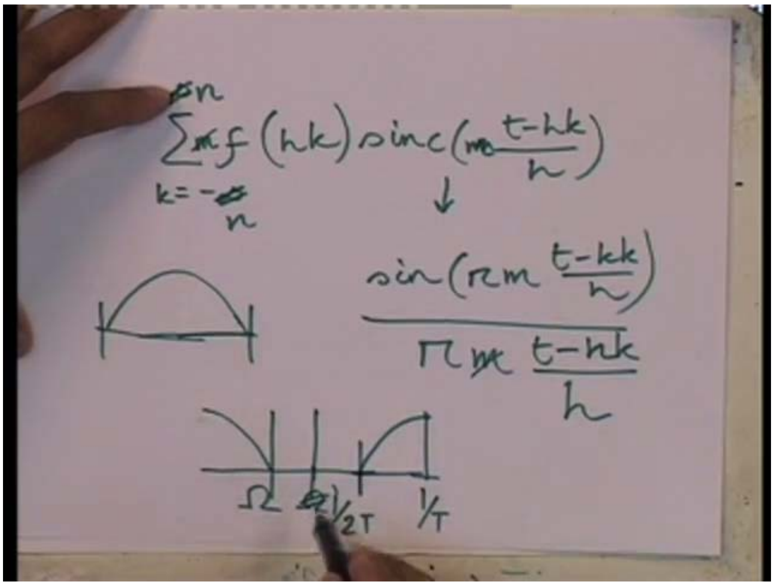
(Refer Slide Time: 42:03)



Now, let us consider function which are time limited and band limited, so after multiplying the delta train, we get loads like this, and sampling over here for f (t), which is compactly supported in the time domain, we get only a finite number of samples which are non-zero, every ever else at 0. On the Fourier domain, if we take t to the sufficiently small or h which will over here, on the other side the well separated which 0 in between the whole intervals which is 0's.

Now, we are freedom and reconstruction, we can multiply this whole repetition with a rectangular function, which goes from this point to this point, because it is just 1 its going to give as F (s), till this rectangular function remains with the this range. So, we have some other freedom, let us call the rectangular function it works to sigma prime and sigma is the band limit, so will define a parameter m which is 2 h sigma, 2 h sigma prime.

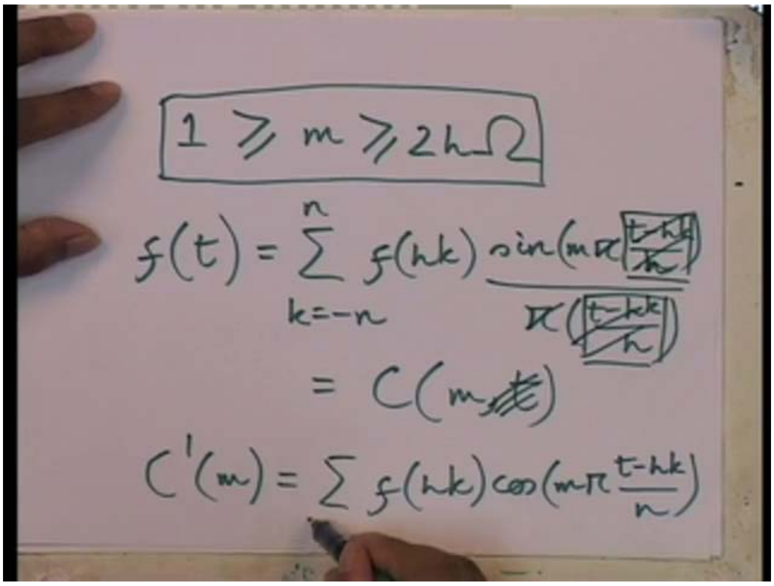
(Refer Slide Time: 43:53)



On the other hand, in the time domain the corresponding operations, we get series to be the cardinal series at the time limit series to be f (h k) sinc of here the m comes in, and this factor of m in the from. Now, f is time limited, so k will not go from minus infinity to infinity, it will stop somewhere say at n, so now this is a finite series.

Now, what I told before, the property of sinc function in the sinc becomes sine (pi m t minus h k by h) due to this structure this sinc function this to m’s a going to cancel out, which is very necessary proof. Now, what was the condition on m, we can find out exactly, but here is sigma, here was a repetition, so sigma prime had to be little, this was say 1 by t, so if we take sigma prime lies between 1 by 2 t to sigma then this exactly holds.

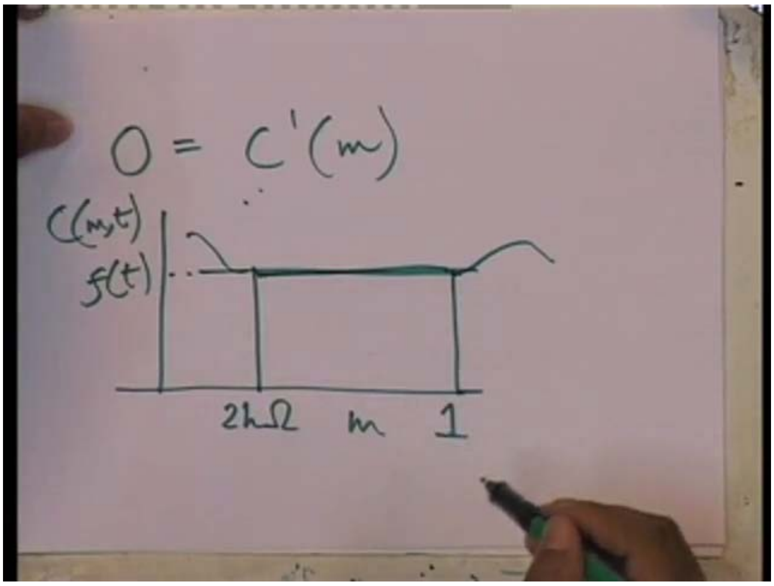
(Refer Slide Time: 45:39)



We can derived that within, whenever m lies between 1 and 2 h sigma the reconstruction formula holds, in the reconstruction formula holds the series cardinal series converges to f (t), now k goes from some finite number n minus n to n. Now, when this condition is satisfied the m for all real m like this before this wholes exactly, in that case we can take this cardinal series to be a function of m and t, and if we f (t), f (t) becomes an number t minus h k by h becomes becomes numbers, so where freedom of m, which we can manipulate.

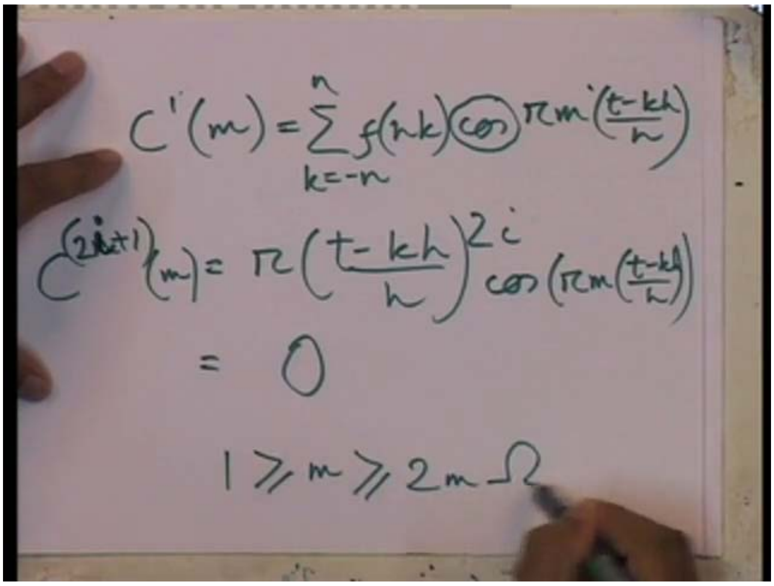
Let us take the first derivative of C (m, t), but t is fix, so we want try t any more, so C prime (m), t minus h k by h is construct get cancel, pi also get cancel, and we are left with f (h k) cos (m pi t minus h k by h k) go from minus n to n.

(Refer Slide Time: 47:52)



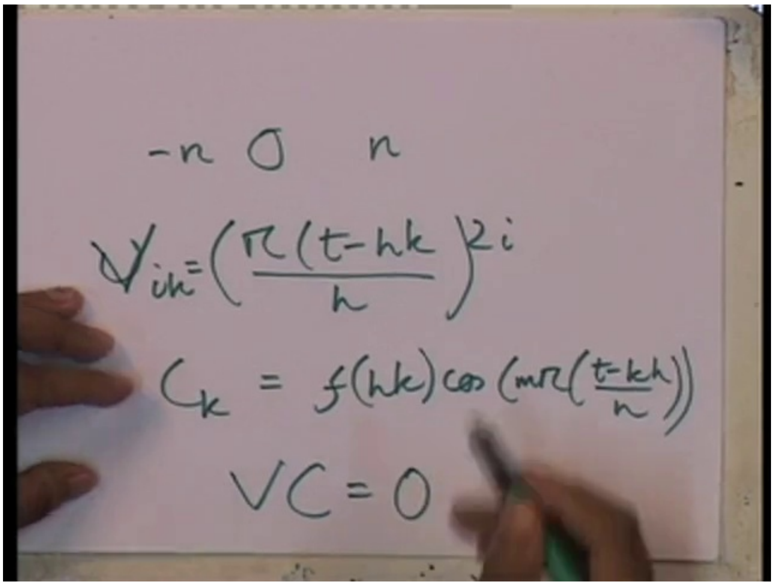
Now, f (t) was already fixed, so when we take the derivative with respect one it is the 0, so C prime (m) 0, and here this finite series, if we write it graphically, so we have plot of C (m, t) with respect to m. So, when m lies between 2 h sigma and 1, this is going to give us f (t), we do not know it is going on its going on outside but, over here we sure that there is a constant function, so we can take as many derivatives as we want of C (m, t) with respect m.

(Refer Slide Time: 48:47)



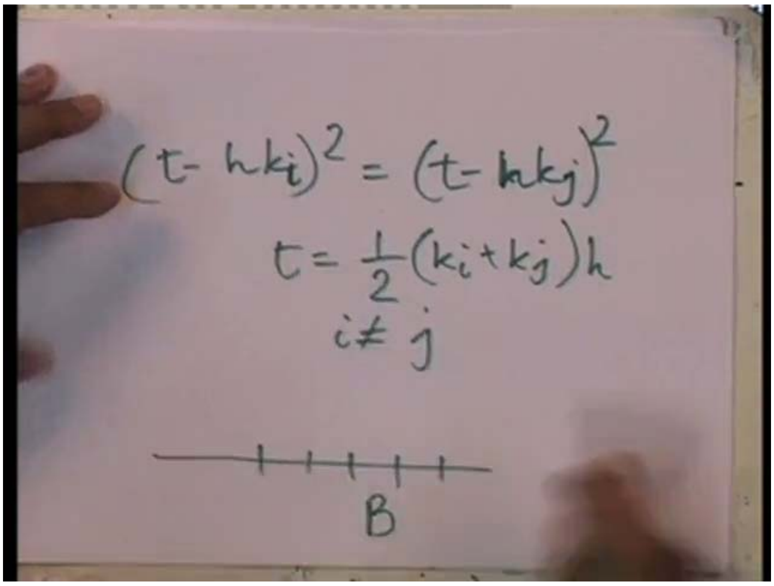
So, let us take second order derivatives of C prime (m), second order derivatives with respect to m for this cos cos it will just denote (pi minus t k h by h) to the power 2 i, so say this is i eth second order derivative 2 i plus 1. And all of them are going to be identically 0, because C m is a constant function, when m is like this, we can write this whole thing down as a matrix equation.

(Refer Slide Time: 50:15)



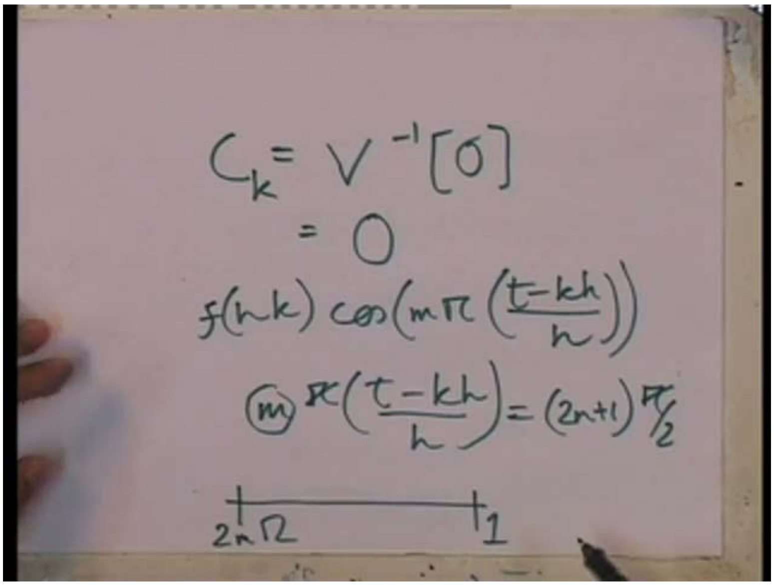
So, say how many samples did you have its minus n 0 and n, so 2 n plus 1, we take 2 n plus 1 equations like this, doing the derivatives; so the random on structure, random on to come later on, so we get a structure like this, and C k. So, what what we are going to get the V C equal to 0 writing down all the 2 n plus 1 equation.

(Refer Slide Time: 51:18)



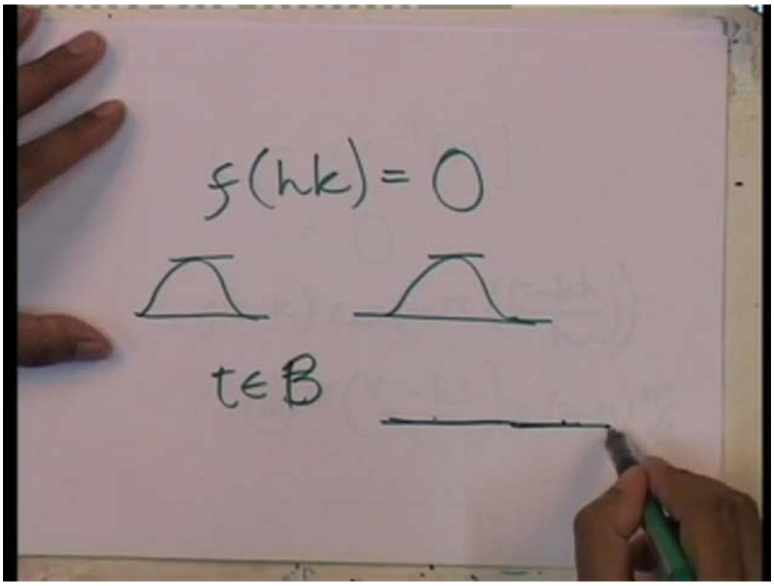
Now, this V has a random on structure, it is going to be invertible as long as (t minus h k) square k 1 k i square is equal to (t minus h k j to k j) square this whole and this thing only holds for a finite number values of t, i not equal to j; now we can work on the whole of the real line, other than this values of t, let us call this set of t as B.

(Refer Slide Time: 51:59)



For all other t, we get that C is equal to V inverse into the 0 vector, that so it is absolutely 0, so C is 0, C k is 0, for all k in that case either f (h k) 0 or the cos part is 0 or both of them. Now, the cos part can be 0 only when m pi minus this thing, this whole thing equals odd multiple of pi by 2; so only first in finite countable number values of m this can hold, but in our proof we are the whole range of m, so this is not going to holds for all m, if we take any other.

(Refer Slide Time: 53:03)



This will give that for all the other end f (h k) is equal to 0, now it can be show that for a time limited in a band limited function, the function as very smooth, so all the derivatives exist the bounded on both the sets and all this thing. So, if this things happen our cardinal series C m, a cardinal series is going to converges exactly not only point point wise but, exactly it is going to converges f (t), so if we f (h k) is equal to 0 or this thing equal to 0 (Refer Slide Time: 54:03).

So, if it is converges to f (t) on the other side, and let left with an f (t) equal to 0 nothing else but, we are missing the set B, t belonging to B, but we already know that f is continuous, so if finite number of points that is non-zero and further 0 but, cannot hold it has to be 0, on the whole, so it is we have finally, got there only the function f equal to 0 is allowed to have this property.

We had here a very interesting exposition of the proof of the inability to have simultaneous compact support time and frequency (()), I must appreciate the effect by the young man to look again at the concept carefully to identify a number of basic inequality, then equalities underline the proof and to present it in such a low set manner. Not with standing the fact that there has been small, glassing over points during the proof which we can control, the idea is come out beautifully and he has been able to put cross the ideas in a very accurate, in a very beautiful manner. So, even though at places we might a found smooth in accuracy in writing, which you know one thing can do I thought I mentioned that, because when the audience listen to it, it might be found that there are some places were symbol and so on or little bit of over loading of the symbol and so on.

But, that that happens I think we can control that but, otherwise I think the proof will be what are beautifully and I think we should appreciate this young man for the beautiful proof and the beautiful way which is explained the idea, and it satisfying the two ways it is satisfying, because it tell us technically how what why it is to perceive the subject of at least trying to be reasonably compact this or we know going to towards or reasonable support in one domain, where other is compactly supported. We know that there are wavelets that are compactly supported, now we also note from the discussion from slogan, that in the time domain in the other domain they can be, the frequency domain they can be compactly supported.

But, the challenge is how close you can go to constraining them in the other domain, so challenge becomes ever green once again, that is a technical reason, the personalize reason is that, here is a young man towards look at the concepts in the course carefully. And put in this own thought his own creativity to augment the ideas of the course, I encourage all students who at some time of the other listen to this lecture, to taken example from this students who have enthusiastically participated, and participate as enthusiastically, thank you.