Advanced Digital Signal Processing - Wavelets and Multirate

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Lecture No. #36
The Polyphase Approach
The Modulation Approach

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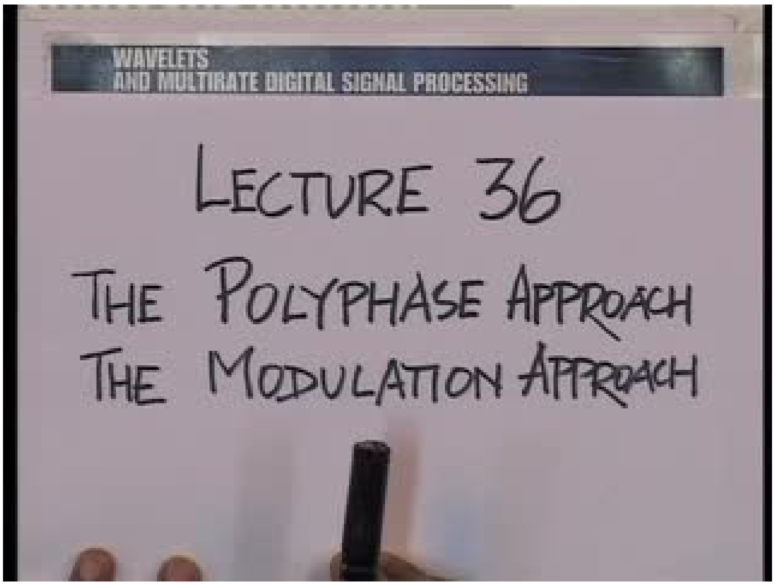

A warm welcome to the thirty-sixth lecture on the subject of wavelet and multirate digital signal processing. You will recall that, in the previous lecture we had hinted briefly at the idea of a poly-phase decomposition; in fact, we had use that idea of a polyphase decomposition for various reasons, we have been using a time and again to construct different kinds of structures to carry out computation efficiently in a filter bank, but we have not put down formally a whole approach based on polyphase components; moreover we have not quite considered the alternative approaches to the question of perfect reconstruction to date. So, in the lecture today what I intend to talk about is two approaches to the general perfect reconstruction problem namely the polyphase approach and the modulation approach, both of these are essentially approaches to handle the question when does a filter bank give perfect reconstruction of the input.

And what we are going to do today is applicable not only to two band filter banks, but also to general m bank filter banks, where m could be a positive integer greater than 2.

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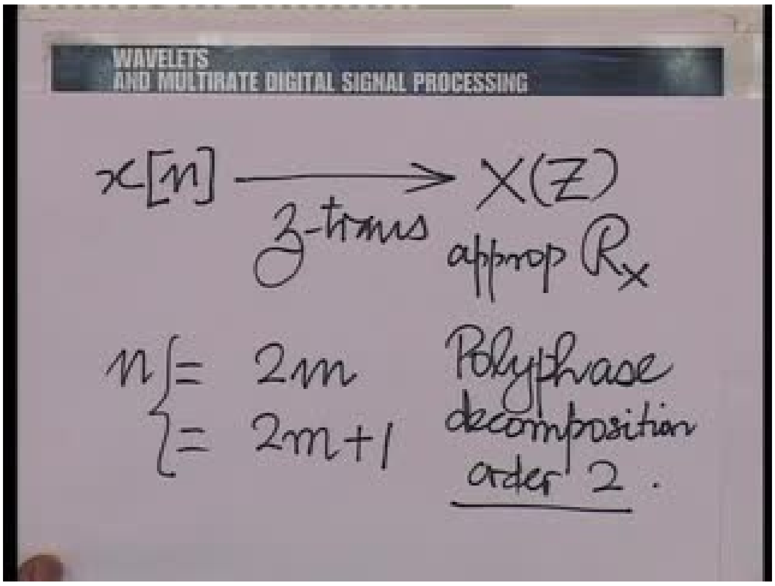

Let us begin with the first of these two approaches namely the polyphase approach. Now, we have already introduced the idea of polyphase components earlier, let me recapitulate that idea. You see when you have a sequence x of n with it is z transform capital X of Z with an appropriate region of convergence, we have seen that we can separate the samples of n which the samples of x n which lie at n multiple of 2 and lie away from a multiple of two; in other words, we can consider the index n as broken down into 2 m and 2 m plus 1 for all integer m, so when we let m very over all the integers here then all possible integers n would be covered. So, what we have talked about here is what is called a second order polyphase component or polyphase component polyphase decomposition of order 2.

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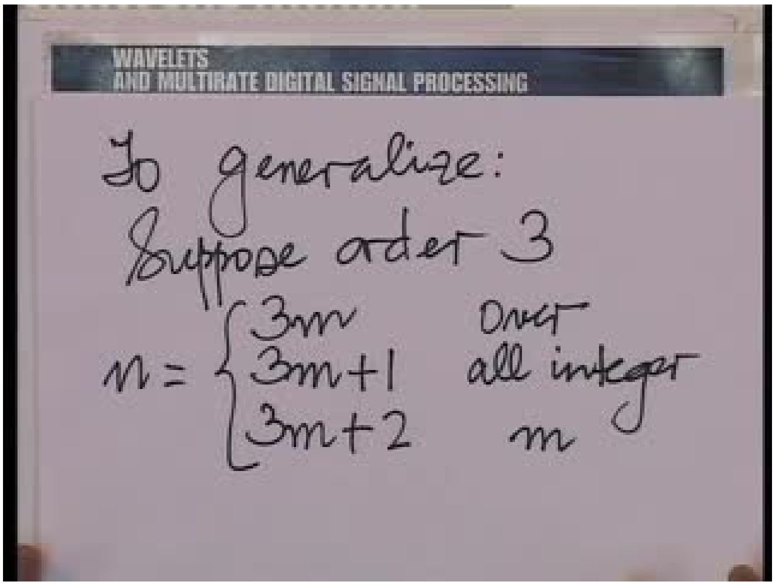

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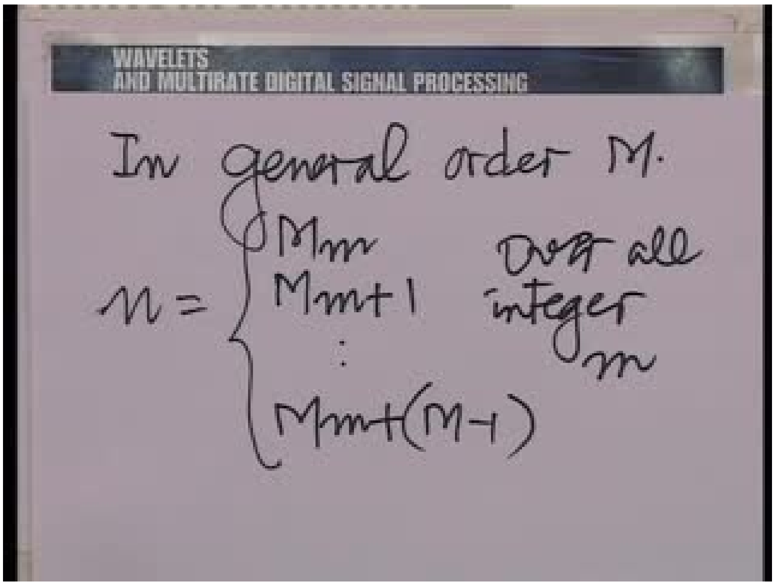

What we want to do first is to generalize this; so, suppose for example we wanted polyphase decomposition of order 3 what we are then going to do is, ask that integer n the one of these three possibilities 3 m, 3 m plus 1, and 3 m plus 2, overall integer m, and when m varies over all possible integers then we have covered all possible integer values of small m here; in general, suppose, one talks of capital M, we are then referring to a decomposition of the index n as follows, capital M times small m, capital m times small m plus 1, and so on up to capital M times small m plus M minus 1.

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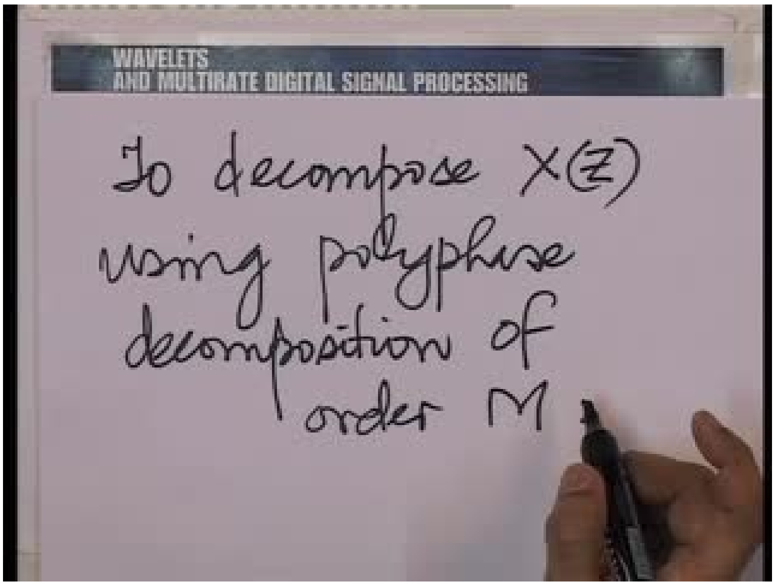

Over all integer small m, and on so doing on letting small m vary over all the possible integers we have actually spanned the entire set of integer m; so, now, let us put down explicitly, the mechanism for decomposition of x z the z transform into the z transform of the polyphase components of order m, to do that we shall essentially split this index capital M in the way we just prescribe; so, what we saying effectively is, to decompose X Z using polyphase decomposition order capital M, essentially we write down capital X of Z in terms of the sequence.

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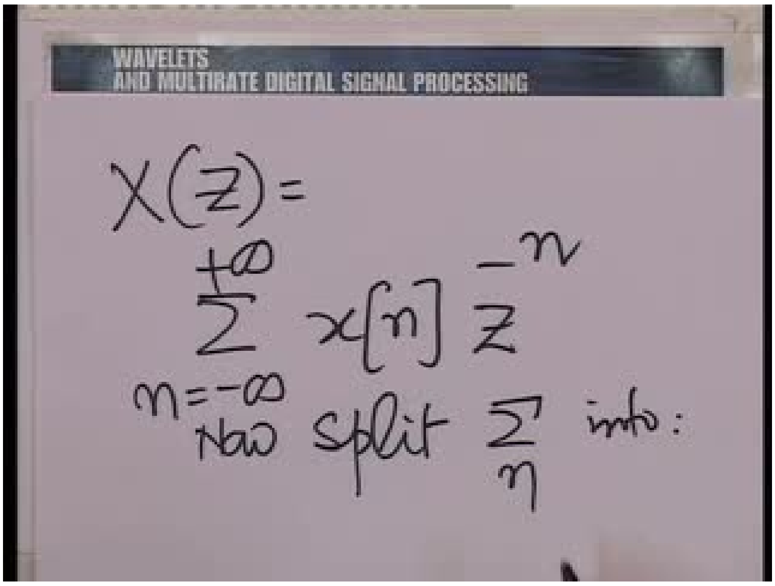

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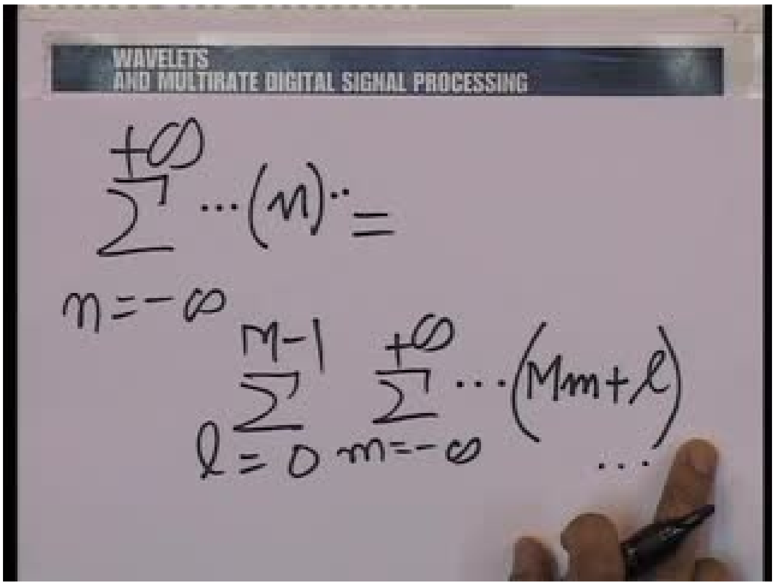

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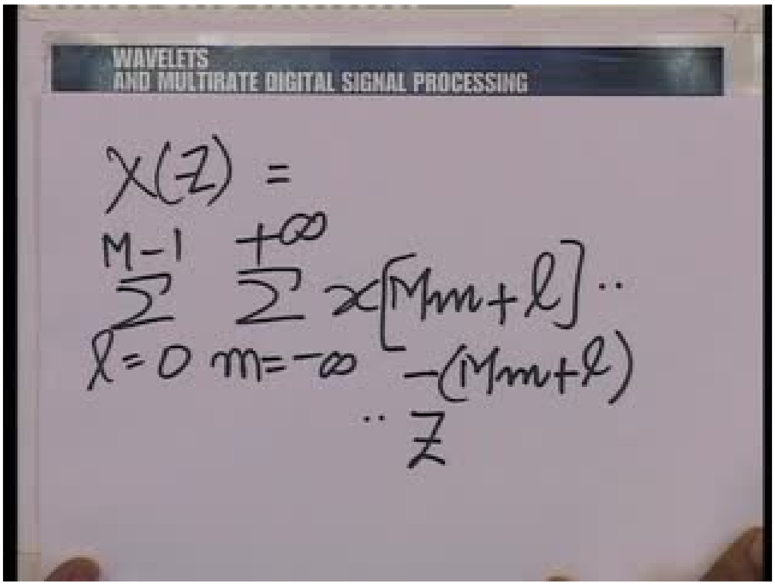

Now, decompose the summation or split, the summation on n into summation n going from minus to plus infinity is the same as summation l going from 0 to M minus 1 and summation m going from minus to plus infinity, and where here you would have a variation as a function of n, here you would have variations as a function of M capital M times small m plus l, so we just change the nature of the variable, the specification of the variable, let me illustrate concretely, so capital X of Z; in other words, becomes summation l going from 0 to capital m minus 1 summation m going from minus to plus infinity x capital M times small m plus l times z raised the power minus capital M times small m plus l; and now, we can split this exponential into two parts, Z raised the power capital M times small m into z raised the power minus l, and noting that Z raised the power minus l can be brought outside the summation on capital on small m, we could rewrite this as follows, capital X of Z is summation l going from 0 to capital M minus 1 Z raised the power minus l and in brackets, then you have summation m going from minus to plus infinity x capital M small m plus l times Z raised the power minus capital M times small m; now, essentially, this is the Z transform of all those points which lie at multiples of capital M plus small l.

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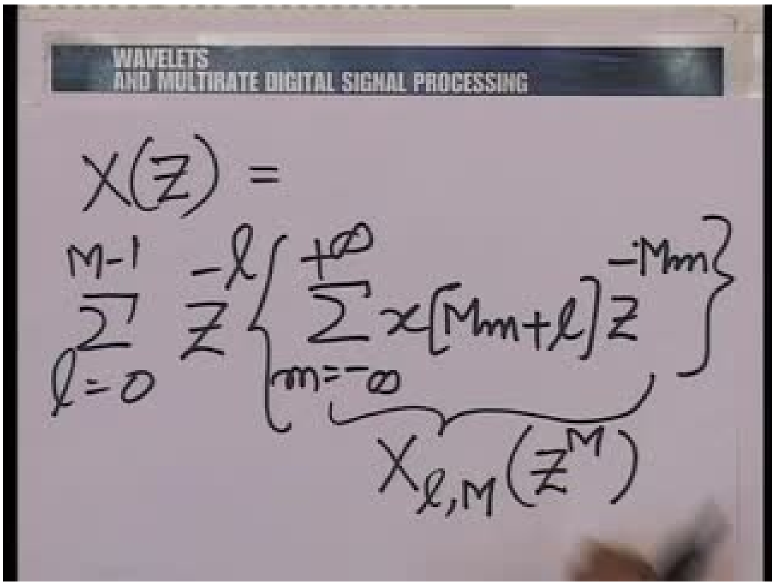

So, for example, when l is equal to 0, essentially this refers to Z transform of all those points which lie at multiples of capital M; when l is equal to 1, this refers to all those points Z transform of all those points which lie at multiples of capital M plus 1 displaced by 1 from multiples of capital M and so on; when l is equal to 2, they displace by 2 steps and so on, and this can only go up to capital M minus one steps, because when you go to capital M you come back to the original case of l equal to 0.

So, what we have done here is to break X Z into m parts m disjoined parts in some sense, and we shall use some notation; now, we shall refer to this quantity here as capital X l m Z raised the power M.

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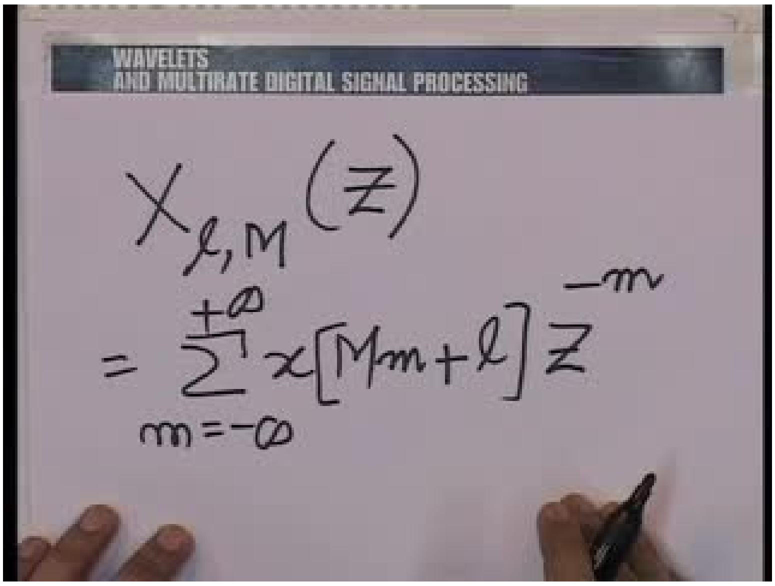

So, what we saying here is, you have the n th order polyphase component and the l th of those components with the argument given by Z raised the power m here; let me write down this explicitly, capital X small l capital M Z raised the Z just Z is essentially summation M going from minus to plus infinity x M small m plus l Z raised the power minus m note here.

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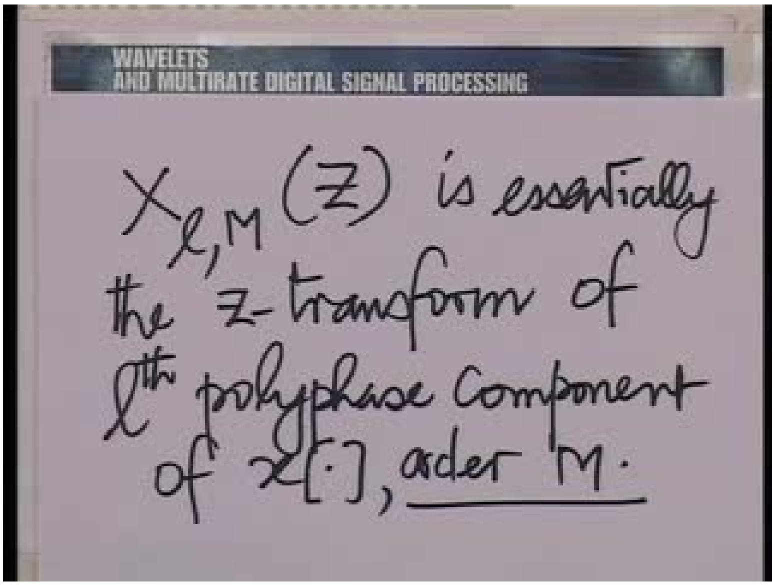

I do not have a capital m occurring here, because I have replaced Z raised the power m by Z, and this is essentially the Z transform of the lth polyphase component of the sequence x order M. So, here the two things are important in general the polyphase decomposition, the order of decomposition and the component number, there would be as many components as the order; so, when capital M is equal to 2, they are going to be two components, 0 and 1; when capital M is equal to 3, they are going to be three components, 0, 1, and 2, and so on.

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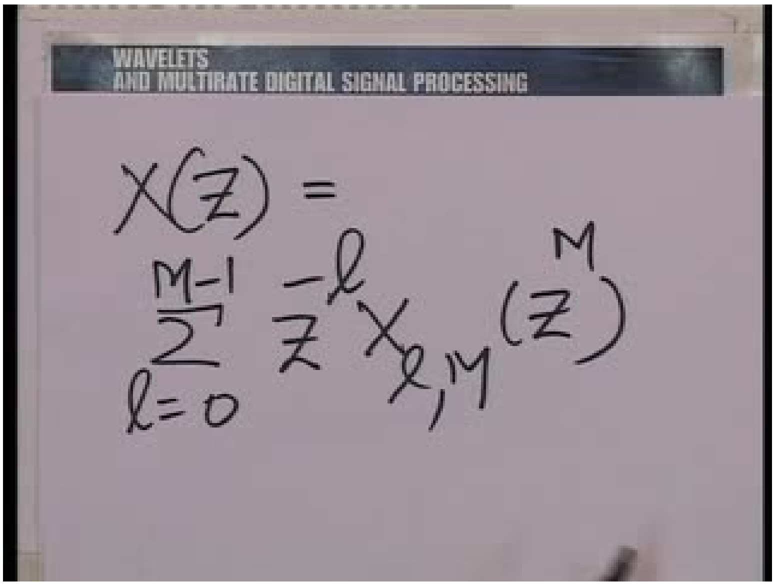

In fact, we can write down a very simple relationship between the Z transform of its original sequence and the Z transform of it is polyphase components; so, capital X of Z becomes summation l going from 0 to M minus 1 Z raised the minus l x l M Z raised the power M, this is the manifestation of polyphase decomposition in the Z domain.

Now, this is just to explain define the idea of polyphase decomposition in general; what we would now like to do is to see how this polyphase decomposition works when you have an analysis and a synthesis site, we would like to write down a general relationship for analysis synthesis polyphase components and how they interact to give perfect reconstruction.

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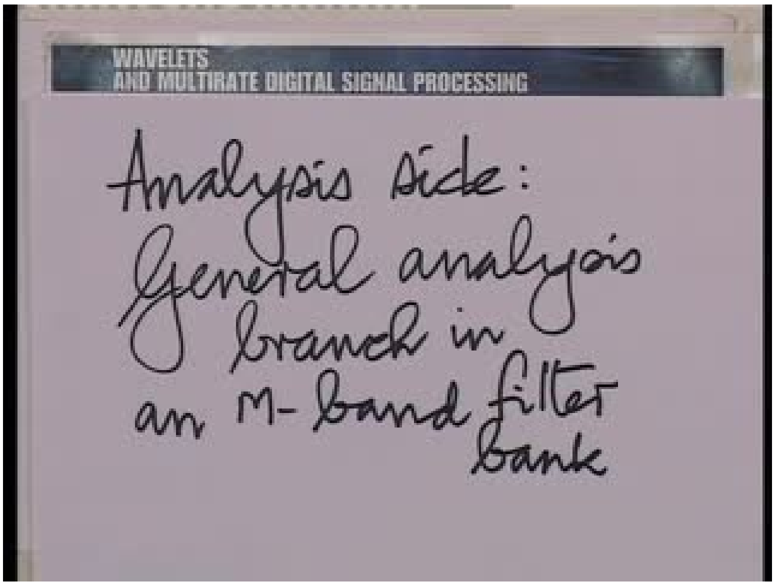

So, you see let us look at the analysis side. In fact, now, for some variety and also to generalize let us look at the general case, general analysis branch in an M band filter bank, now you so far we have been talking just band and slowly my objective in this lecture is to go from two to M band, so I would like to bring in m band first and then put M is equal to 2 as a special case.

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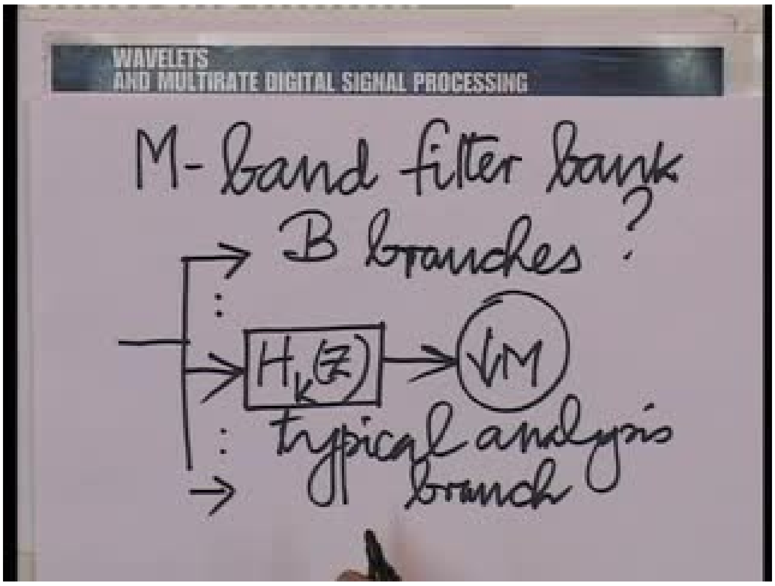

So, what is firstly the m band filter bank, essentially it is an analysis synthesis structure with the following nature on the analysis side you have a typical let us say l th branch where you have a filter or let us use kth branch to avoid mixing up with the poly-phase component index.

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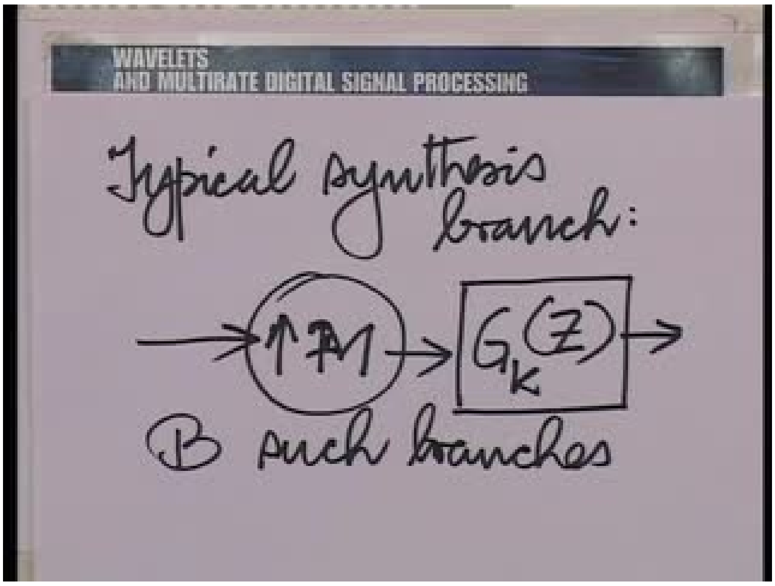

So, capital H K Z followed by a down sampling by M, this is a typical analysis branch; and in general we could have B branches here; now, the number B could be different from the number capital M, that should be stressed, they do not have to be the same; a typical synthesis branch would look like this, you would have an up sampler, we earlier had up samplers of factor two, now you would have up samplers of factor M in general followed by the so called synthesis filter the Kth search, so G K Z.

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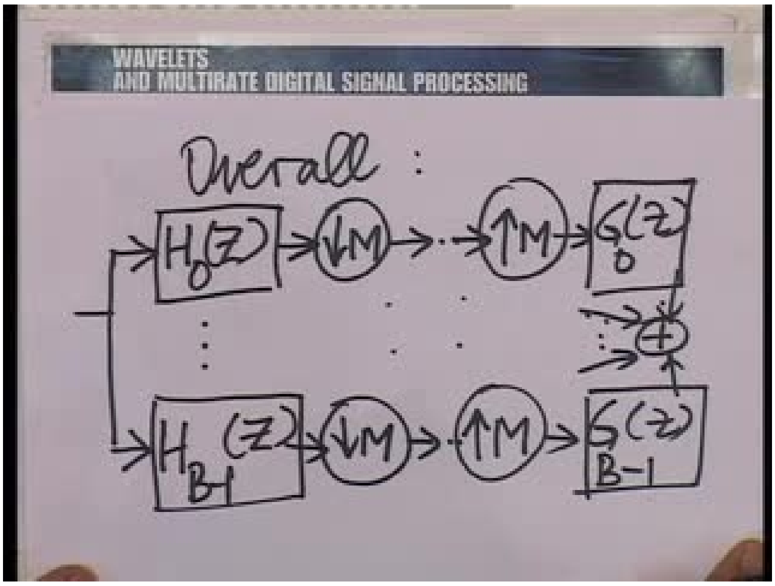

And there could be b such branches too; again I must stress that B need not be the same as capital M, needless to say in an given m band filter bank the number of analysis branches, and the number of synthesis branches must be the same, in fact there is a 1 to 1 correspondence, when there is an analysis there is also synthesis on the same branch, so let us draw the overall structures.

So, this is the structure we are talking about; the notice that you have the same B on the analysis and synthesis side, but B is different from M, and there are therefore three possibilities, B could be equal to M, B could be less than M, B could be greater than M, let us write down these three possibilities here, what is that an analysis filter is always followed by a synthesis filter.

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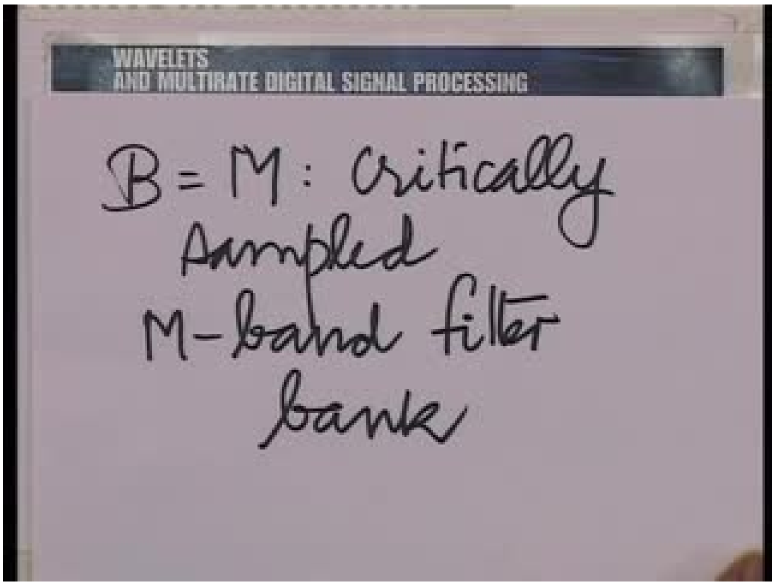

So, you have an analysis and synthesis filter coming in cast deal with the down and up samplers between; so, B could be equal to M, such a filter bank such an M band filter bank is called a critically sampled - M band filter bank.

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B could be less than M, in which case we call it an under sampled M band filter bank; and on other hand B could be greater than M, in which case we call it an over sampled M band filter bank; whether the filter bank is critically sampled or under sampled or over sampled the essential mechanism of analysis whether by the polyphase approach or the modulation approach does not change, the essential idea remains the same.

What needs to be checked is on analysis whether we get the desired condition, whether we get perfect reconstruction, whether we get something else; sometimes we may not want perfect reconstruction, we might want something else from a filter bank, we might want the overall filter bank to perform a certain over all filtering operation that is possible, it is not necessary that we always want perfect reconstruction; whatever it is the overall analysis of what a filter bank does can be done by the polyphase approach or by the modulation approach and that holds whether or not it is critically sampled that is whether it is critically sampled under sampled or over sampled, this is the point which I want to make before I begin to discuss the approaches in depth.

Now, the polyphase approach says decompose the filters both analysis and synthesis into polyphase components and decompose the input and the output also into it is polyphase components of order m, note the order of the polyphase components is the same as the down and up sampling factors.

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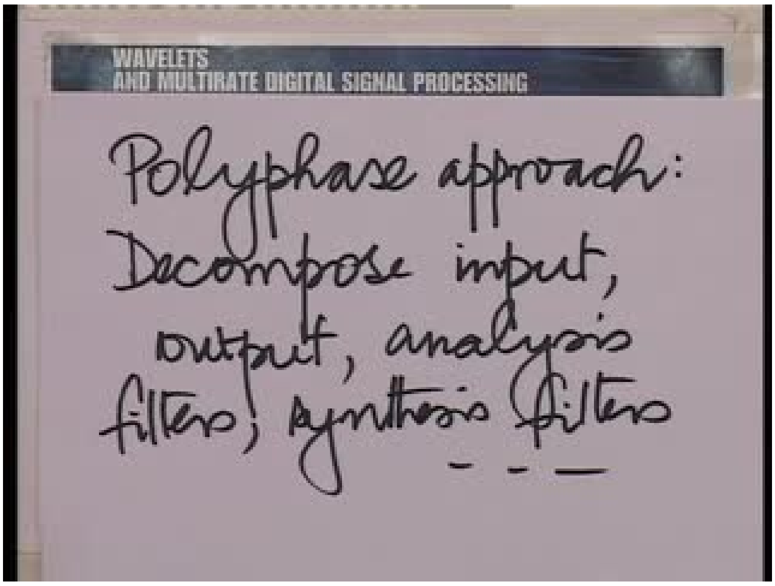

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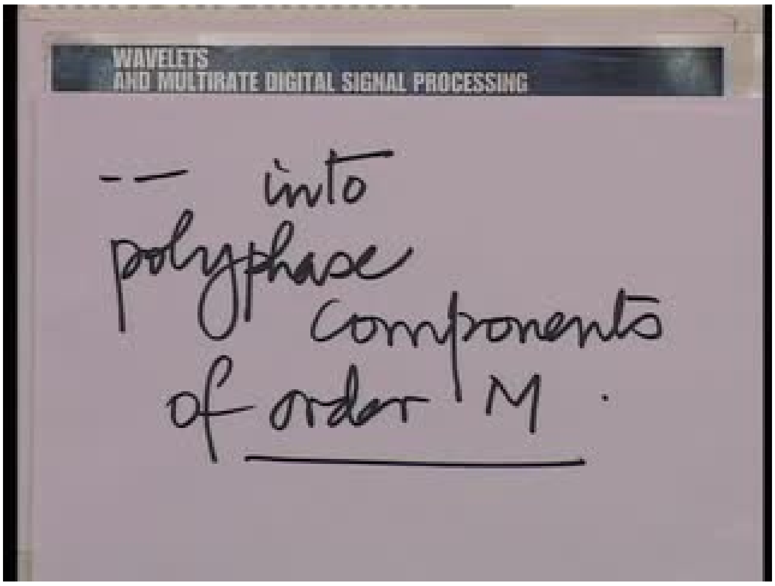

So, the polyphase approach is let us write it down clearly decompose the input, output, analysis filters, and synthesis filters into polyphase components polyphase components of order M, and what we shall do here is to take one of those B branches and see what we get, and once we understand what happens on the K th of the B branches we shall understand what happens overall.

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So, consider the Kth branch, so we have the input subjected to the action of H K Z followed by down sampling by M followed by up sampling by M and then subjected to the action of G K Z, and we use to analyse what this is doing in the poly-phase domain.

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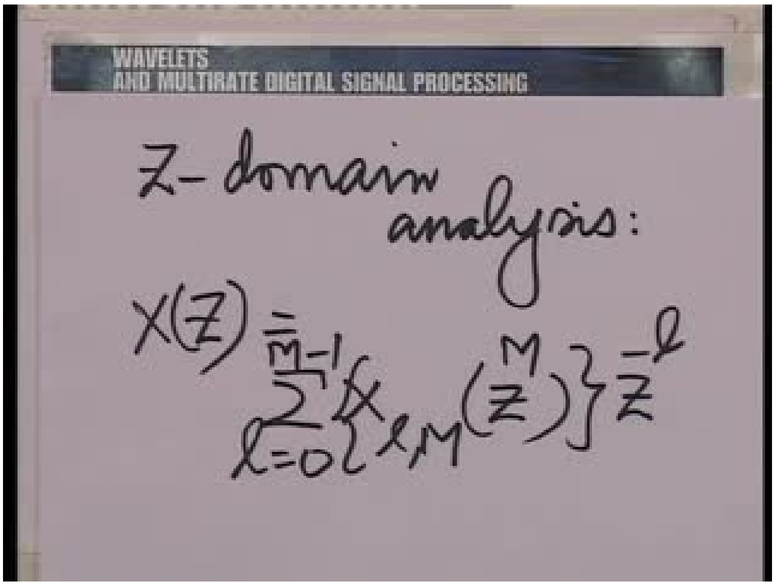

So, let us write down the input Z transform capital X of Z here, and let us see what Z transform emerges here; so, you see X Z can be literal in terms of it is polyphase components and that gives you summation l going from zero to M minus 1 capital X l capital M Z raised the power M multiplied by Z raised the power minus l, where you have capital X l M as the lth poly-phase component of order M of the sequence X similarly we can take the Kth filter here and decompose this as well into it is poly-phase components of order capital M, so we could write now here you will have a 3 fold index, so filter number poly-phase component number order of the poly-phase component Z raised the power M, and now let us multiply.

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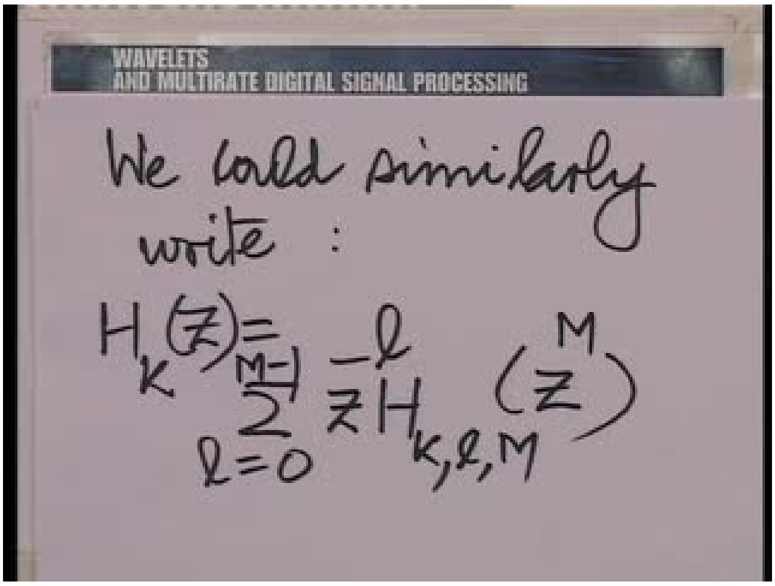

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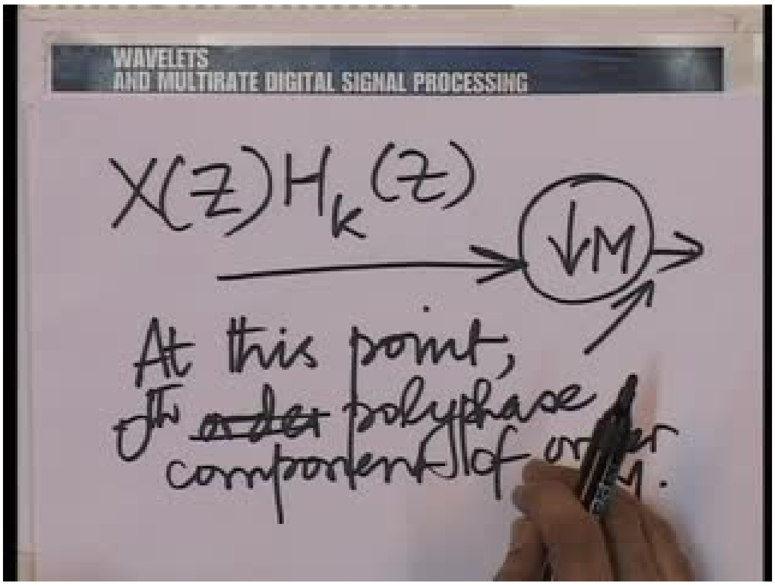

So, X Z H K Z is what emerges after the filtering operation, and we not really interested in X Z times H K Z, so let us short circuit the solution to the first step, what we are really interested in is this subjected to down sampling, we not interested at this point, we are interested at this point.

So, what we want to get at this point is only the 0th order poly-phase component sorry 0th poly-phase component of order M; we not interested in the other poly-phase components the 1th up to the M minus 1th, after we have taken this product which is wish to understand or bring out or pull out that component which is of order 0, and how will you get that.

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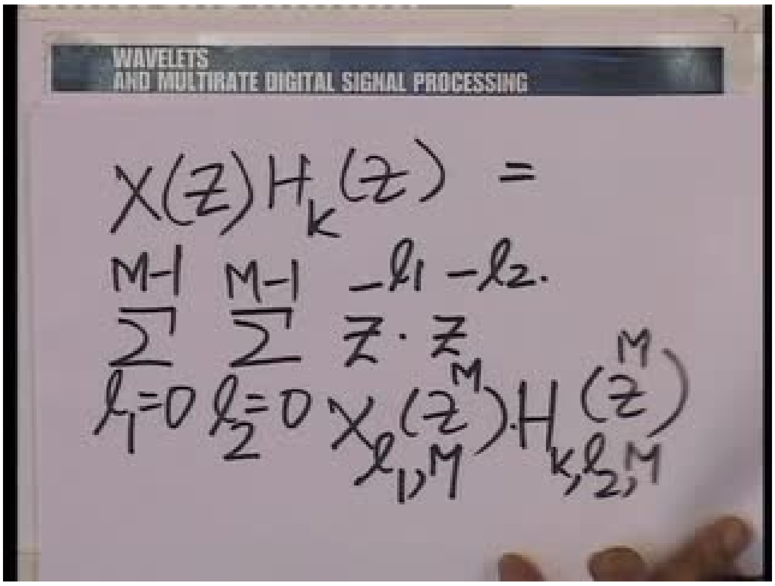

So, we have this product X Z times H K Z is of the form summation l 1 going from 0 to M minus 1 summation l 2 going from 0 to M minus 1 Z raised to the minus l 1 times z raised to the minus l 2 times capital X l 1 M Z raised to the M and capital H K l 2 M Z raised to the M.

So, as far as pulling out the component of order 0 as concern what we need to see is that, they must only be Z raised the power of M terms, they must not be a hanging power of Z in whatever we choose; when will that happen? That is going to happen when l 1 plus l 2 even this essentially contributes a power of l.

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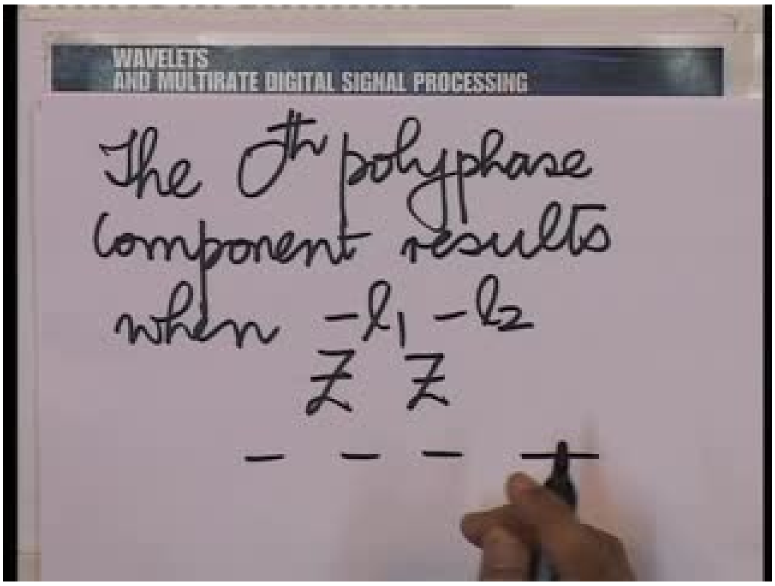

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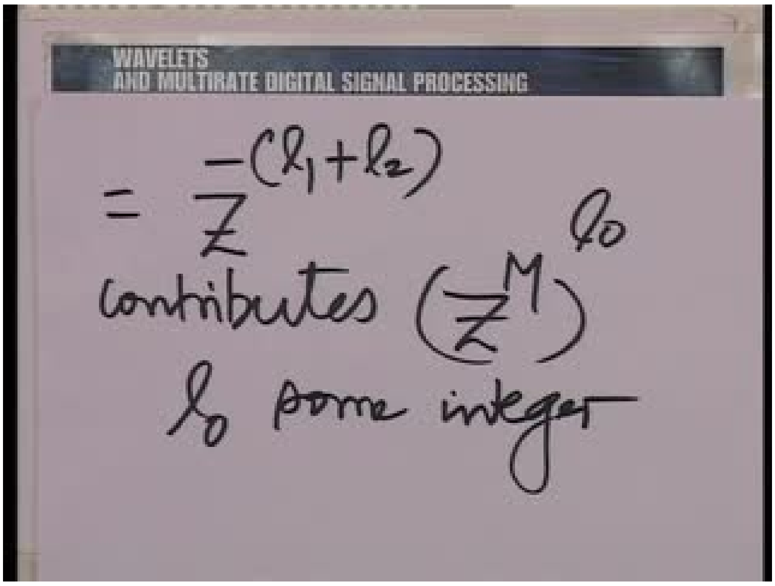

So, what we saying in effect is the 0th poly-phase component results when Z raised the power minus l 1 Z raised the power minus l 2 which is equal to Z raised the power minus l 1 plus l 2 contributes Z raised the power M times some integer, so let us say l 0, and that is easy to document, we can easily see when this is going to happen.

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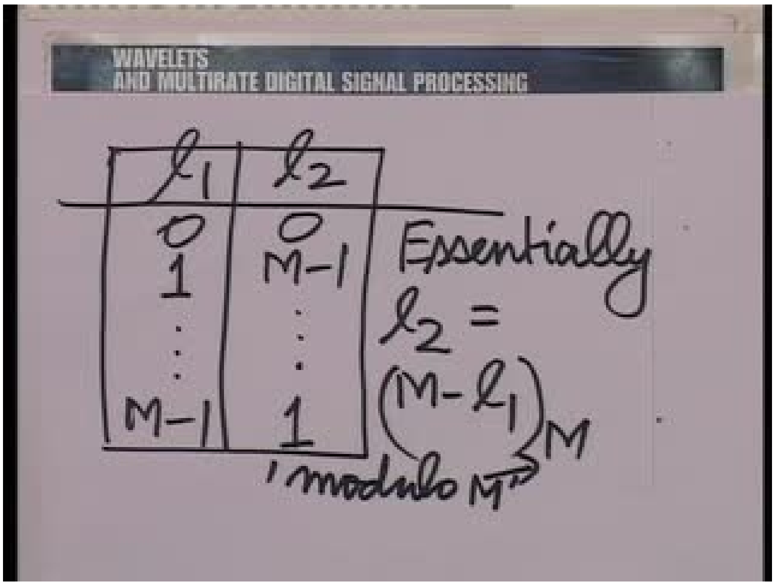

So, let us make a table in fact we can make a table of l 1 and l 2; so, when l 1 is 0, there is no other possibility but that l 2 be 0; when l 1 is 1, there is no possibility other than that l 2 be M minus 1; and finally, when l 1 is M minus 1 there is no possibility other than that l 2 be 1; there is a very easy association, so in fact there exist one layer that needs to go together with each l 1 we can choose only 1 unique l 2, so that we can pull out the 0th order poly-phase component. So, in fact we have a very simple expression for this, you know all that we saying is consider M minus l 1, so when you take M minus 0, but take it modulo capital M, you get 0, M minus 1 will give you M minus 1 as expected M minus 1 gives you 1.

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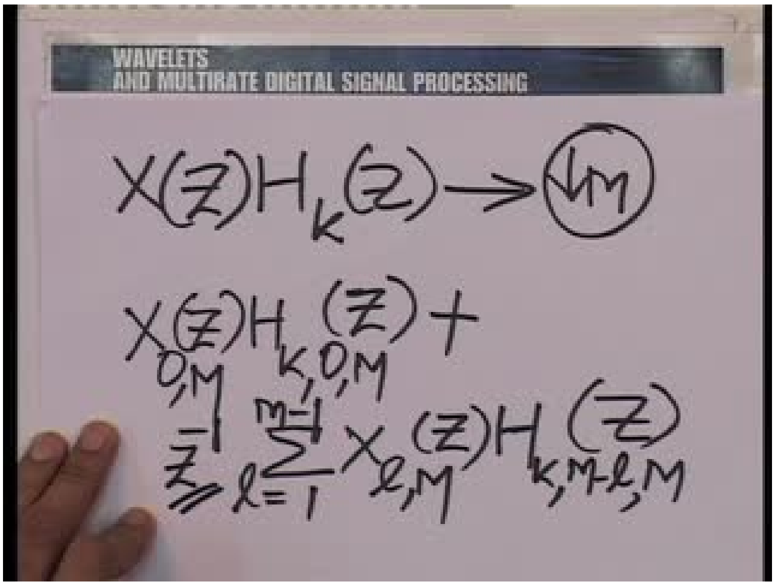

So, in other words, what we are saying here is that l 2 has been chosen as M minus l 1 modulo M, so this is modulo M; so, with this choice what we are saying is X Z H k Z down sampled by m essentially going to give you the following, it is going to give you X 0 Z or if you like X 0 M Z H k 0 M Z plus summation l going from 1 to M minus 1 X l M Z H k M minus l M Z, and there is a multiplication by Z inverse here, please note is a multiplication by Z inverse.

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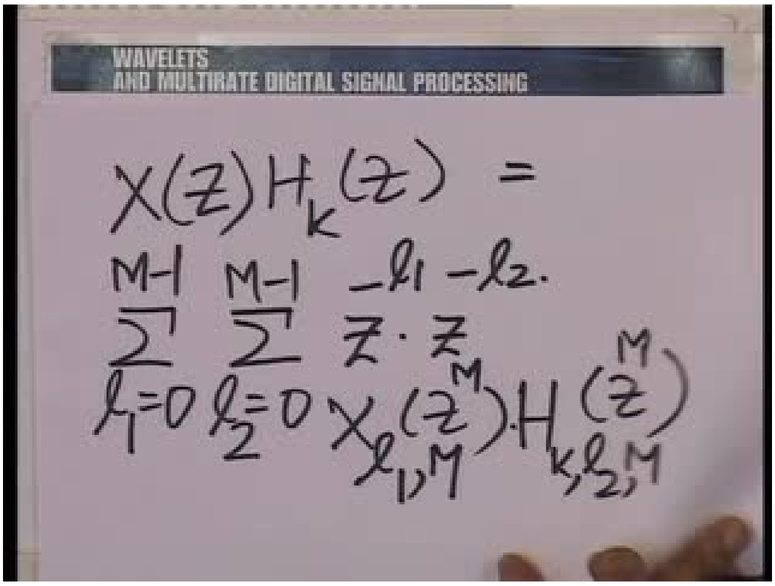

Because except for M equal to 0 or other words except for this case when l 1 is equal to 0 and l 2 is equal to 0, for all the other cases you have l 1 plus l 2 summing up to capital M, so when you down sample by capital M you get a Z inverse power Z raised the power minus M term would have mean originally and down sampling it becomes Z inverse, a simple but elegant and beautiful expression that we have here, so X 0 M and repeat X 0 M times H K 0 M plus Z inverse times summation l going from 1 to M minus 1 X l M H k M minus l M.

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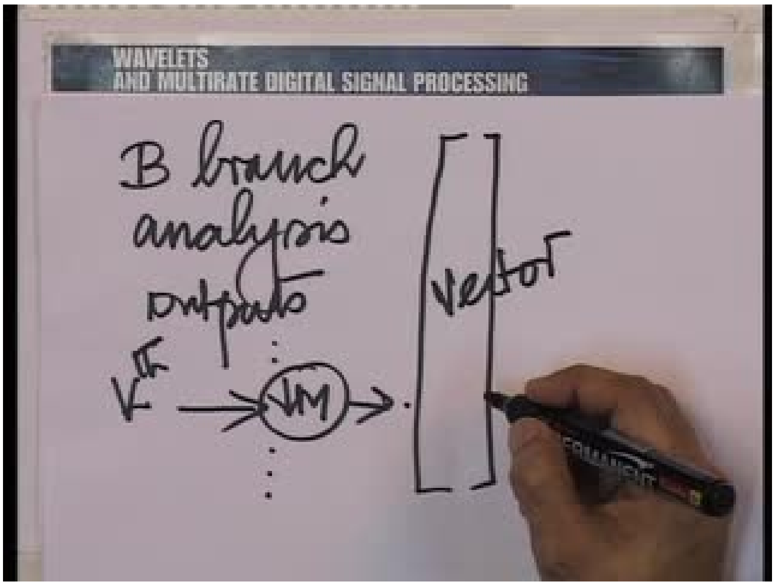

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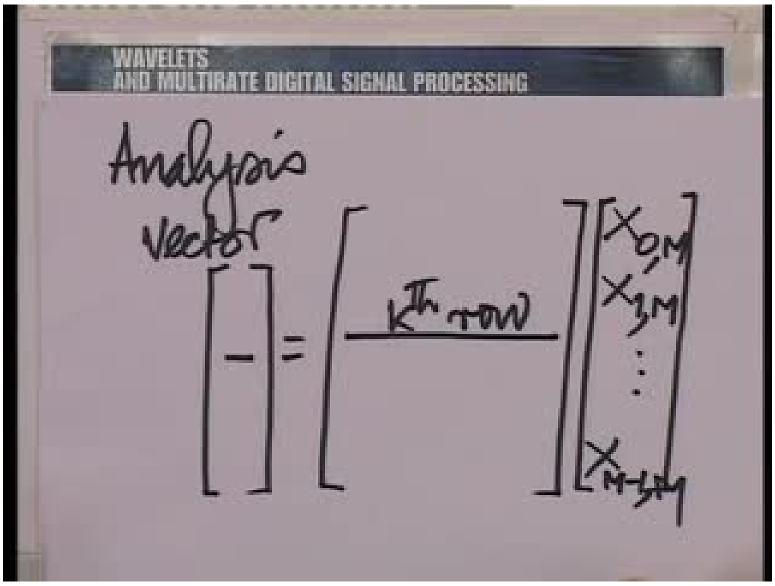

In fact, we can now write down a matrix for each of these branches, so what we are saying in effect is, I can put down the outputs of this B branches, a typical one emerging from the say kth branch at this point, and we could arrange them in the form of a vector, so let us arrange them in the form of a vector, let me write down the analysis vector.

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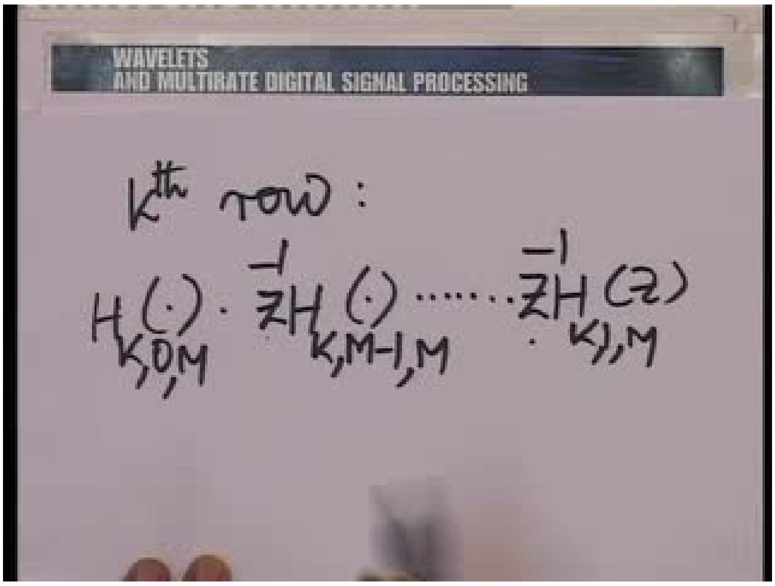

Let me find out the typical kth row of the matrix here, so typical element in the analysis vector would correspond to the k th row times a vector of poly-phase components of X, so let us write that down capital X 0 M capital X 1 M up to capital X M minus 1 M, and I will write down the kth row, the k th row would look like this.

So, note in the kth row we have taken here the Z inverse here, we have taken care of the fact that here if you go back to the matrix vector product the number or the expression in Z we should get multiplied by X 1 M is going to be H k M minus 1 M, the number getting multiplied by X M minus 1 M is H k 1 M.

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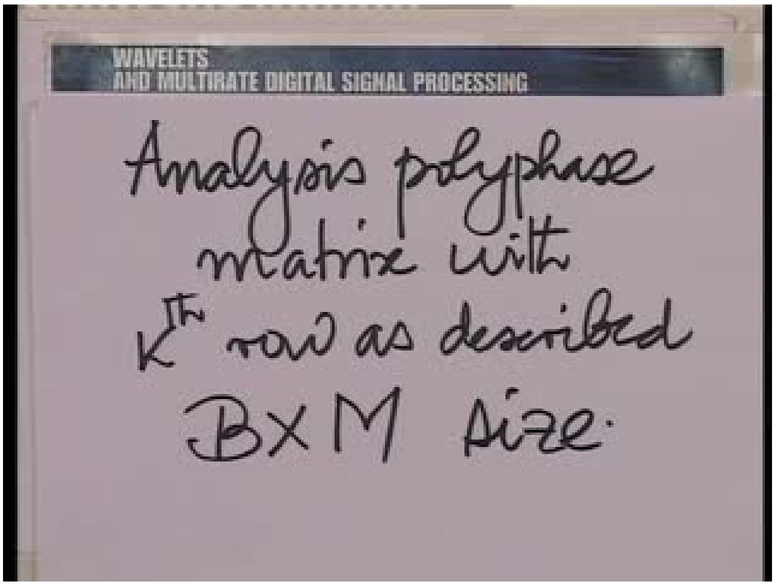

So, says that so called invergent here; now, this structure which we obtain here where you have B rows and the vector of poly phase components of X is called the analysis poly phase matrix; so, in general we have what is called the analysis poly phase matrix with kth row as described, and the side of the analysis poly phase matrix is obviously going to be as many branches times the order of poly phase decomposition.

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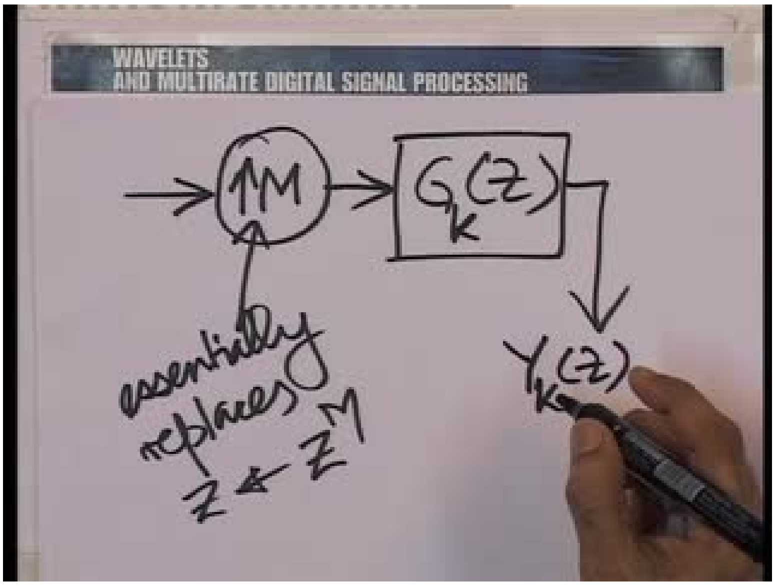

Now, after up sampling every Z is going to be replaced by Z raised the power M that is simple; so, let us consider what happens on the kth branch after up sampling by M followed by filtering by G.

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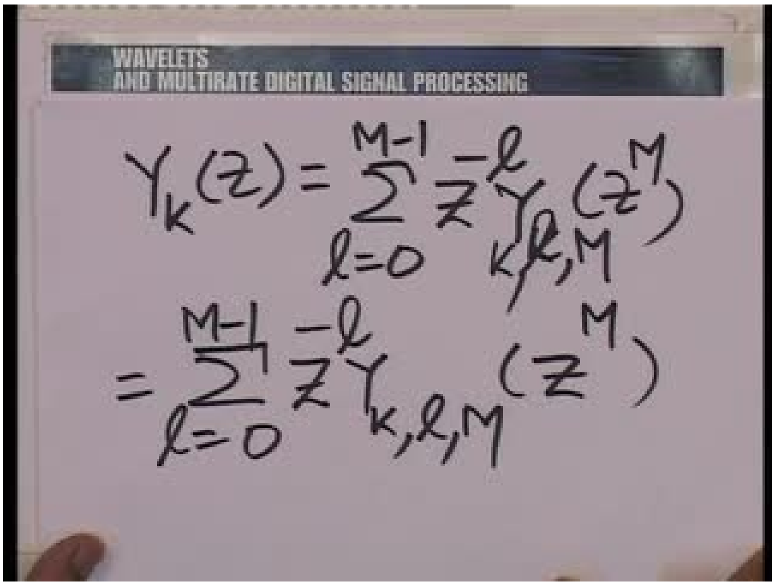

So, you know, essentially, up sampling would replace Z by Z raised the power M, and in filtering with G k Z would again produce M poly phase components here, so let us call this output capital Y k Z, and if happen to decompose capital Y k Z into it is poly phase components, that is we have to write capital Y k Z is summation l going from M minus 1 Z raised the minus l Y k M Z raised the power M, then it is very easy to see that, you know, if you look at it here what comes here is anyway the function of Z raised the power of M.

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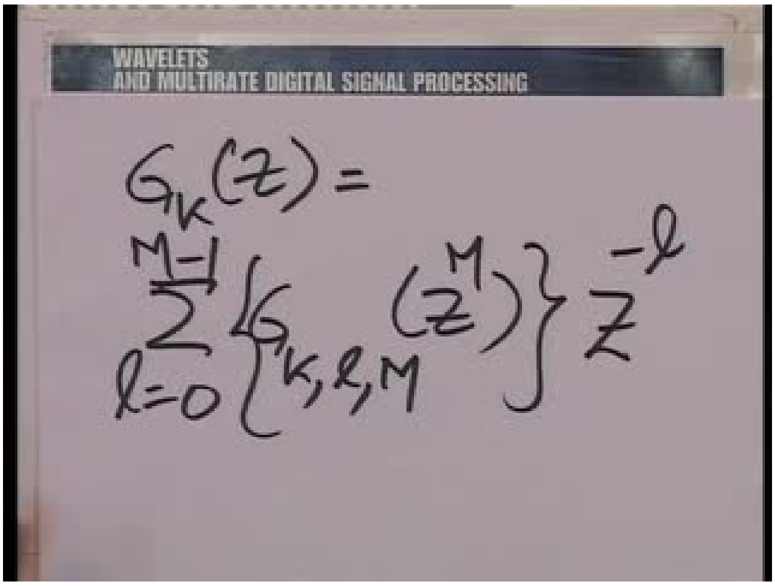

So, to get hanging powers of Z inverse you must take the corresponding hanging powers of Z inverse appearing from this filter; so, in fact, if you take the l th search, well, am this should be l here little correction, so you have Y k, well, actually you should write Y k l, because there are three indices, so I can rewrite this summation l going from 0 to M minus 1 Z raised the power minus l capital Y k l M Z raised the power M, the three indices; and we can make a very clear specification of what Y k l M shall look like, what we are saying is, G k Z would itself be decomposed as summation l going from 0 to M minus 1 G k l M Z raised the power M multiplied by Z raised the power minus l.

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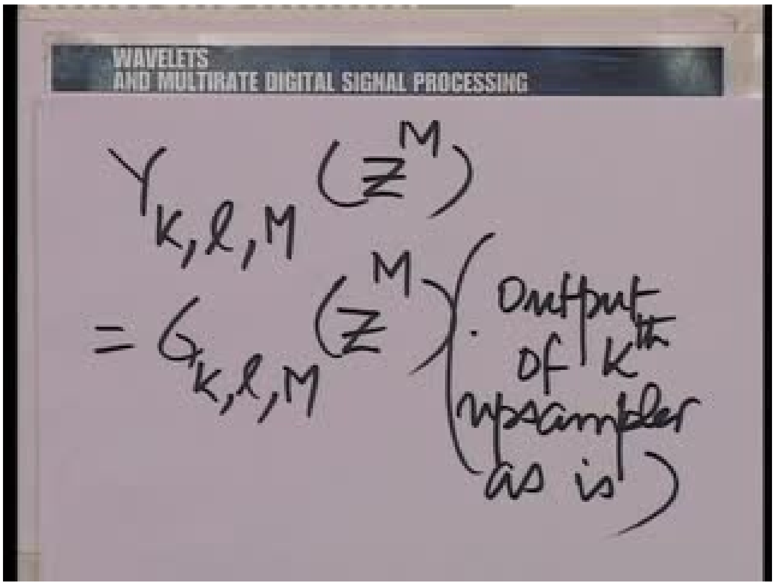

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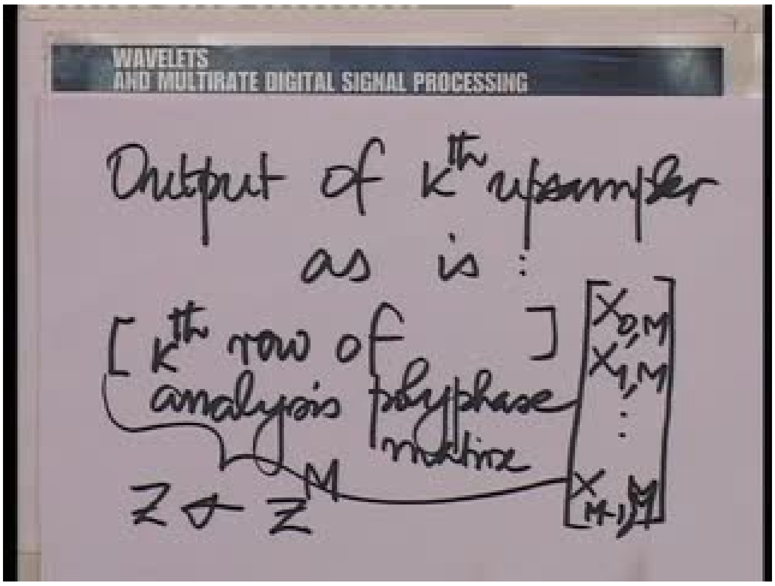

Essentially, Y k l M Z raised the power M is G k l M Z raised the power M times, essentially the output of kth up sampler as is because already in terms of M, in fact we can write down explicitly what that is; the output of the k th up sampler as is, essentially the k th row of analysis poly-phase matrix multiplied by the input poly phase vector; so capital X 0 M capital X 1 M and so on up to capital X minus X M minus 1 M here, the only catch is Z has been replaced by Z raised the power M, that is the only change; in all this which are all functions of Z Z has been replaced by Z raised the power M; now, this is what is called the poly-phase approach to analysing the overall M band filter bank.

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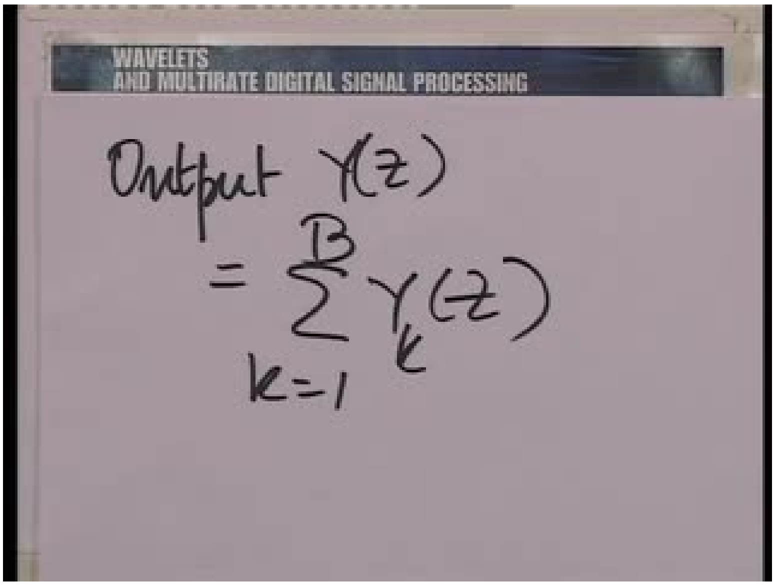

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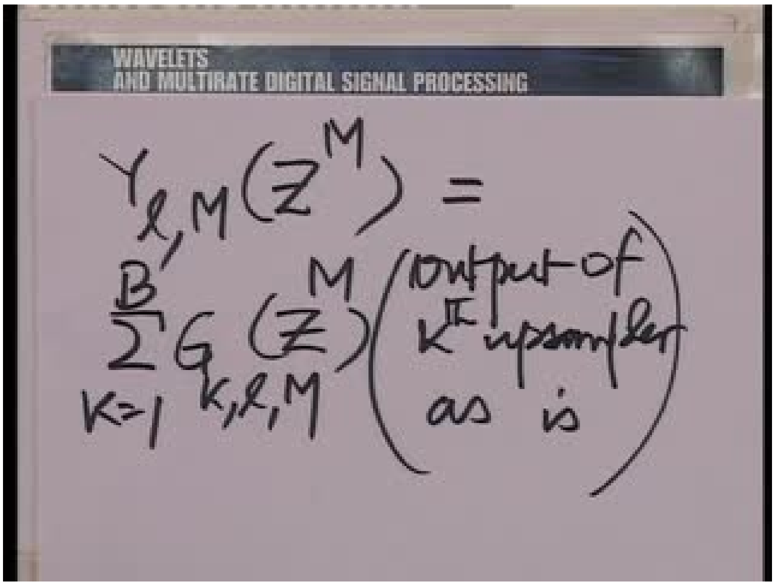

Now, we seen the kth branch, the output is summation k going from 1 to B Y k Z, there are B branches remember, and therefore Y l M Z the lth poly phase component of order M of the output is easily seem to be…, let me just recapitulate for here, you just here the expression here, all that we need to do is for sum over k, so we have summation k going from 1 to B G k l M Z raised the power M times the output of k th up sampler as is…

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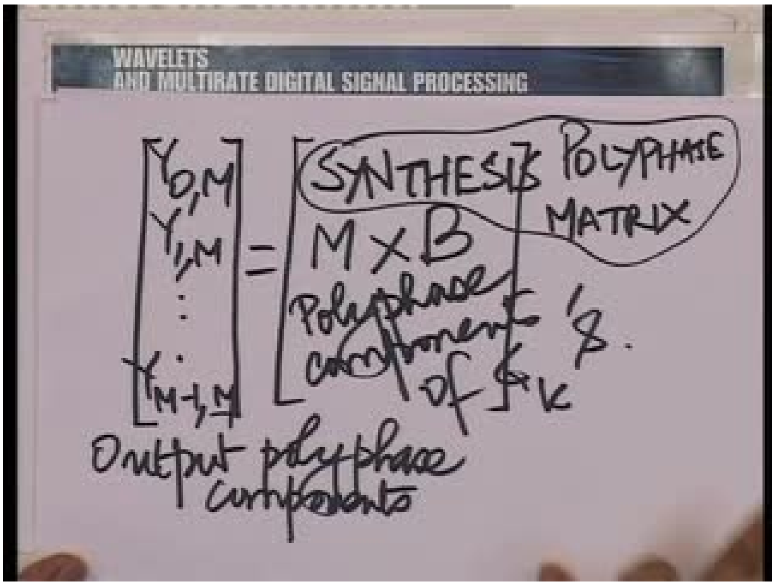

So, now, we could write down a vector of output poly phase components and input poly phase components and relate them; so, we construct a vector of output poly phase components Y 0 M, Y 1 M, and so on up to Y M minus 1 M, and we could note let essentially this can be obtained; now, if I take any 1 of these so take Y 0 M for example, it involves G k 0 M summed over all k going from one to b, so what kind of situation are we talking about, we are talking about a vector of size M here, and a matrix of size M cross B of essentially poly-phase components of that synthesis filter, poly-phase components of the G k's, and I could write down the lth row here, this is called the synthesis poly phase matrix, and let us write down the lth row.

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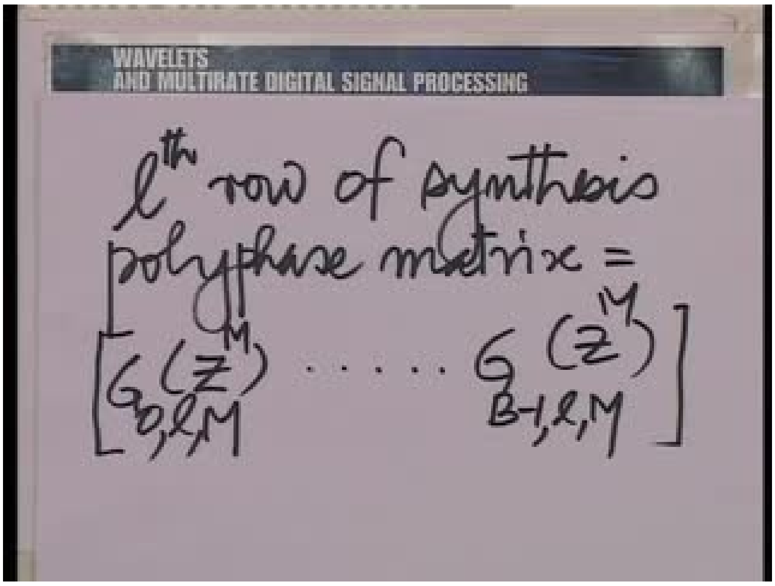

So, not that l goes from 0 to M minus 1, the lth row of the synthesis poly-phase matrix essentially G 0 l M Z raised Z raised the M if that depends on whether it is Z whether the argument Z raised the power of M or Z depends on the argument here, for convenience let us assume the argument here is Z raised the power of M, so Z raised the power of M, and this would go to B such branches, so B minus 1 l M Z raised the power M; so, now, we have M such rows and each row have b elements look at the analyse analysis poly-phase matrix, it has B rows with m elements in each row.

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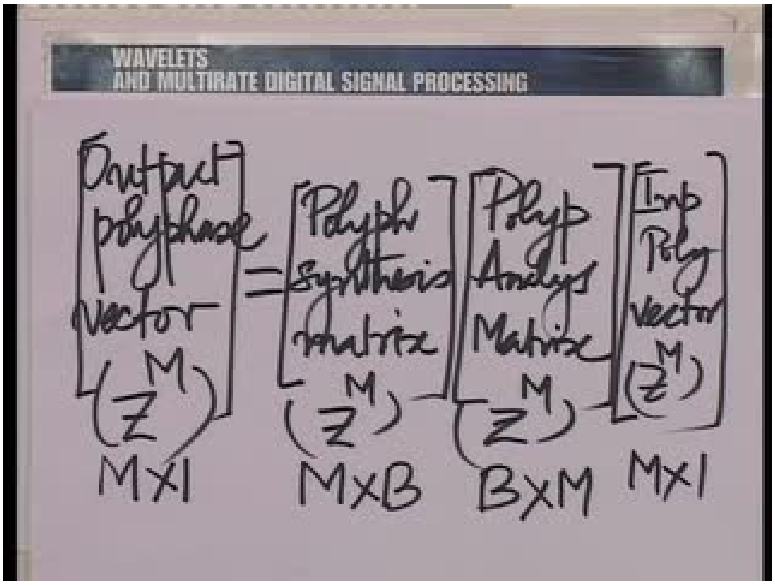

So, over all the output poly phase matrix or poly phase vector as a function of Z raised the power M is equal to the poly phase synthesis matrix as the function of Z raised the power M times the poly phase analysis matrix again as the function of Z raised to the M times the input poly phase vector again as the function of Z raised the M.

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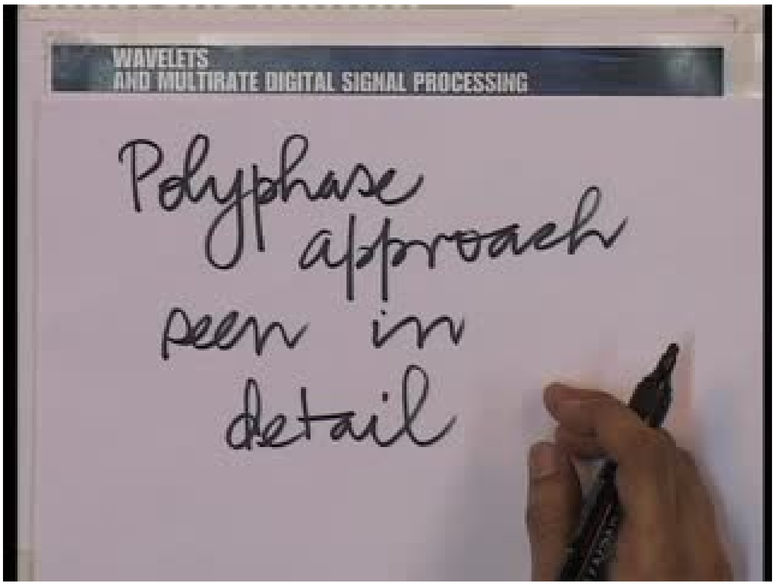

Size is, this is well number of rows M cross 1, this is M cross B, this is built as B and again cross M, and them M cross 1; so, of course, all this is going to be finally M cross 1; now, what we have shown here is an overall structure an overall analysis of an M band filter bank with B branches in terms of the poly phase components; I would now like to spend a few minutes on discussing the modulation approach, but we shall deal with it in depth in the next lecture.

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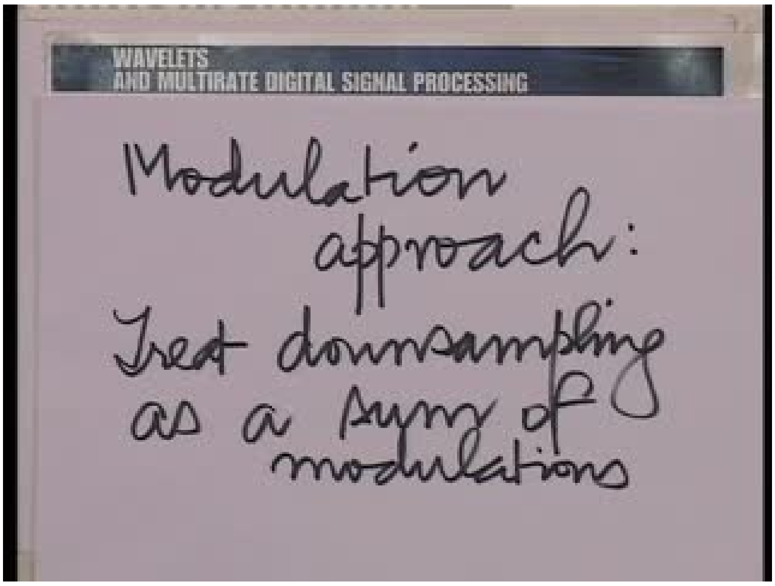

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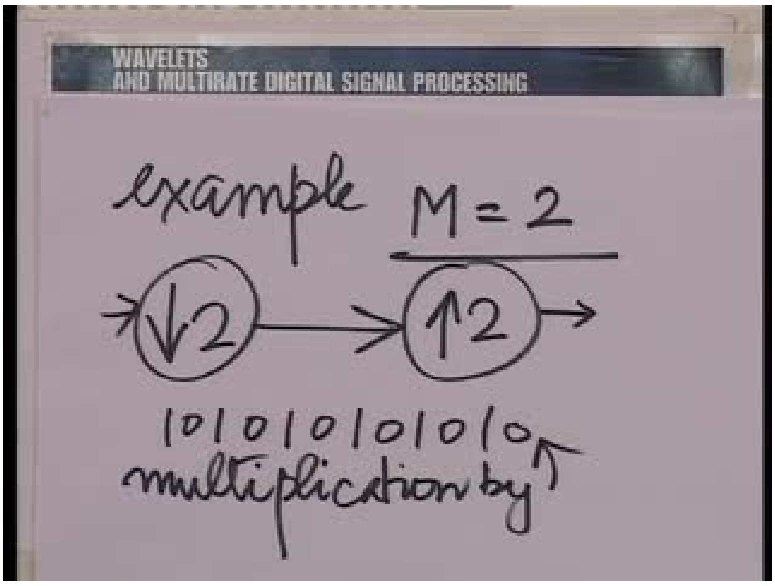

So, we just wish to contrast the modulation approach with the poly-phase approach and leave it at that for this lecture; so, we have seen the poly-phase approach in detail you see in the modulation approach what we do is, as follows treat down sampling as a sum of modulations; so, in other words, let me take the example of M equal to 2, remember that down sampling by followed by up sampling by 2, you know this process of down sampling by 2 followed by up sampling by 2 was treated as equivalent to multiplication by a sequence which is 1 at all multiples of 2 and 0 elsewhere.

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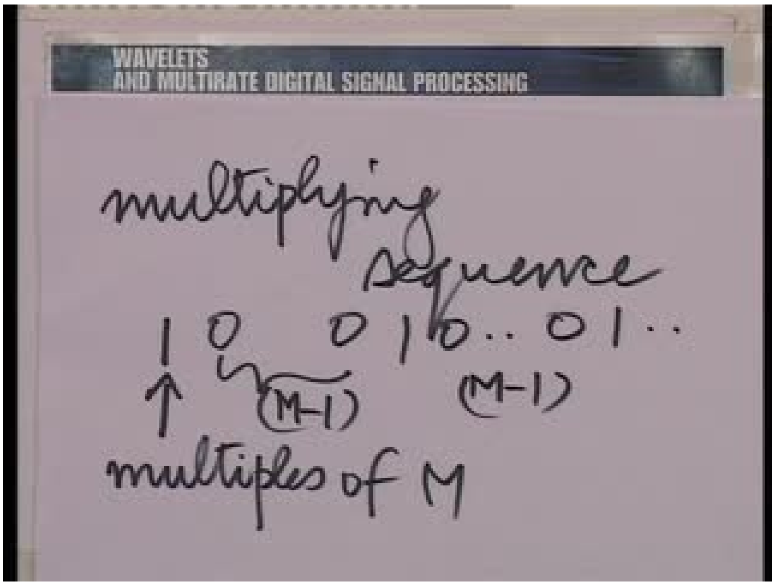

So, you have a sequence which is 1 0 1 0 alternately like this; so, together this was equivalent to multiplication by this. Now, in general, for other M the corresponding multiplying sequence would be 1 at all multiples of capital M followed by M minus 1 0s in between again 1, and then N minus 1 0 is and so on.

So, this periodic sequence, in the modulation approach the idea is instead of decomposing the sequence in time we essentially treat the sequence as sum of modulates, and we combine the down and up sampler when we treated thus as the sum having done so we put down different kind of matrix for the reconstruction treating it as the sum of modulates. In the next lecture we shall go further and in deeper into the modulation approach and contrast it with the poly-phase approach, bringing out the differences and the similarities between the two, and then we shall proceed to establish conditions for perfect reconstruction based on both of these approaches. Thank you.