Advanced Digital Signal Processing –Wavelets and Multirate

Prof. V.M.Gadre
Department of Electrical Engineering,
Indian Institute Of Technology, Bombay.

Lecture No. # 34
Constructing the lattice And its Variants.

A warm welcome to the 34th lecture on the subjects of wavelets and multimate digital signal processing this lecture intense to build upon the structure that we started discussing in the previous lecture let us recall in a few words what we are trying to do in the previous lecture we are trying to bring a modular structure to realize the orthogonal analysis filter bank.

We noted that in building a modular structure we would have an inductive approach and approach where we would construct the smallest order orthogonal filter of length 2 and we demonstrated how the hear analysis filter bank can be. So, construct it and then to expand the length by 2 every time we would meet to introduce an expansion module an inductive module we introduce that inductive module and we prove by mathematically induction that you get an orthogonal filter bank of length 2 more every time we introduce1 instance of that module.

Now, this was constructive in the sense at we assumed that we were drawing a structure of this kind we were putting together module of this kind and we noted that we had the conjugate quadrature filter relationship or relationship in which the orthogonal filter bank conditions on the analysis filter side were obeyed, but what we wanted to do was to go the other way.

Given an orthogonal filter bank we needed to construct such a lattice structure which could realize the orthogonal filter bank. So, for example, if you had a dhobis filter of length four you would want to construct a lattice structure to realize it and it was that direction that we wrote down the recursions that governs the modular structure and you are trying to use those recursions to go1 step back to peel of1 layer as it where let us build on that peeling of process further today.

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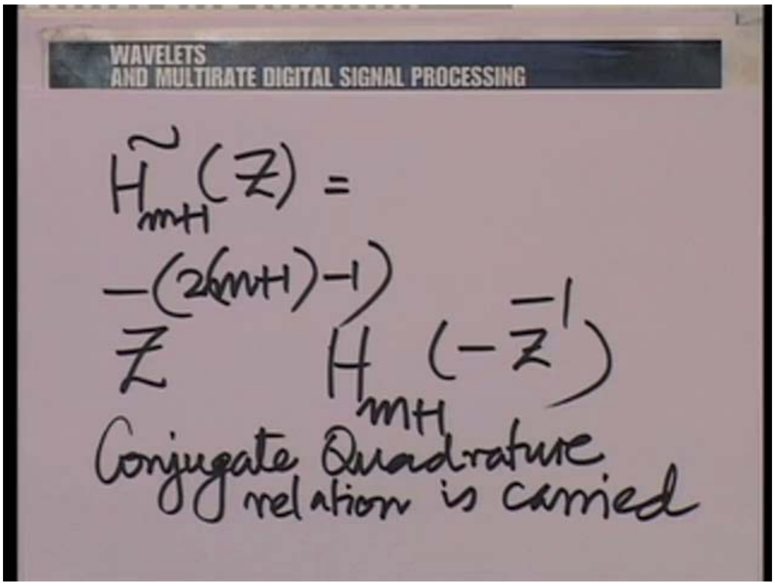

Now, what we said was something like this we said you have1 modular stage like this here you had the system functions H m Z come up to here H m tilde Z come up to there and assuming that we had taken the down sampler well beyond the down sampler is at the end after all the modules what appears next here is as Z to the power minus 2 following that we have the m plus once lattice coefficient as it where which we called k m plus 1 here and the negative of the same coefficients here minus k m plus1 and this brought as to the next system function which we called H m plus 1 Z and H m plus 1 tilde Z where we noted that the conjugate quadrature relation between H m and H m tilde is preserved in H m plus 1 and H m plus 1 tilde what we mean by that is given that H m tilde Z is Z raise the power minus 2 m minus 1 H m minus Z inverse we have H m plus 1 tilde Z is indeed the Z raise the power of 2 m plus 1 minus 1 H m plus 1 minus Z inverse. So, the conjugate quadrature relation is carried it is carried pass the module.

That is what we are saying now what we want to do is to go the other way.

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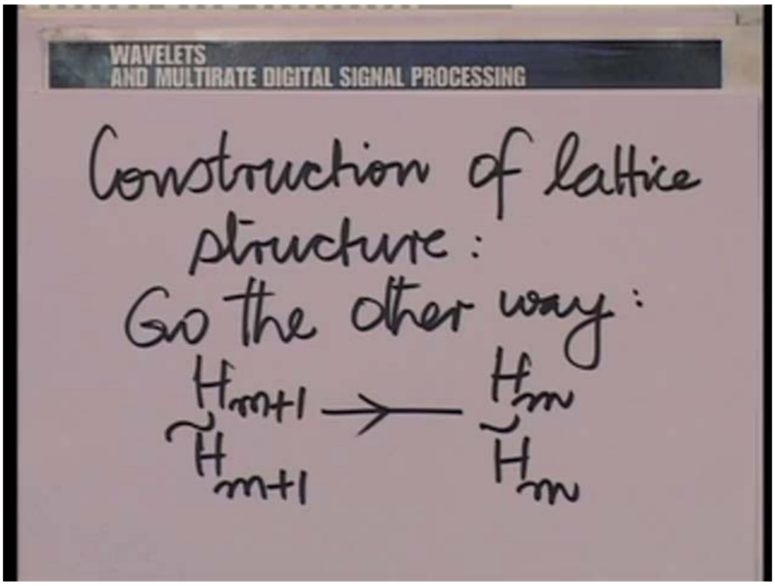

So, are construction construction of a lattice structure essentially means go the other way from H m plus 1 and H m plus 1 tilde to H m and H m tilde now towards the objective we need to write down the recursive expression the inductive expression again that would give us a clue how to go back words.

So, let us write down the inductive relations once again inductive or recursive lattice relations.

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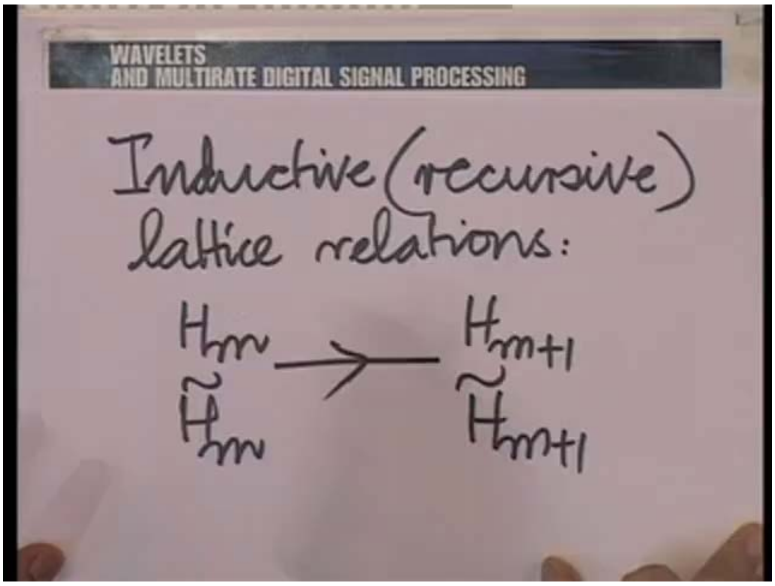

Take you from H m and H m tilde to H m plus 1 and H m plus 1 tilde day and the relations are H m plus 1 z is H m Z plus K m plus 1 Z to the power minus 2 H m tilde Z and H m plus 1 tilde Z a Z raise the power minus 2 H m tilde Z minus k m plus 1 H m Z.

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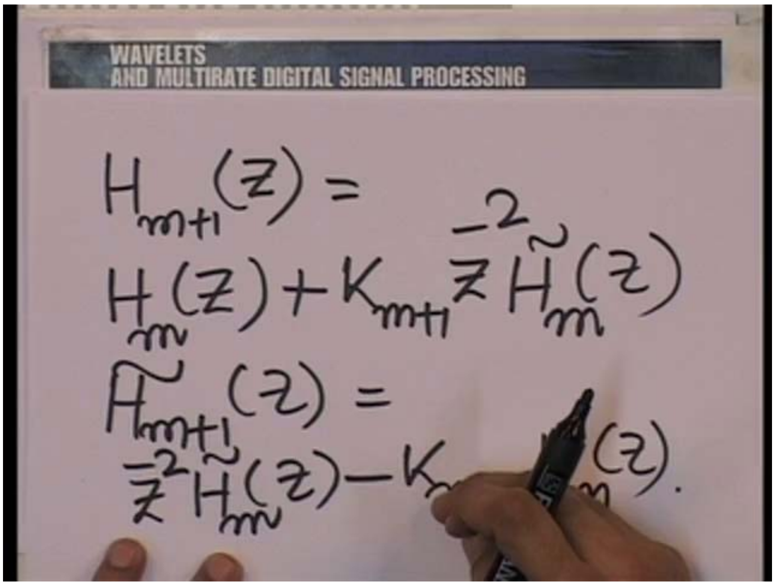

Now, if you observe these 2 relations carefully and we wish to extract H m in terms of H m plus 1 and H m plus 1 tilde all that we need to do as we see is to cancel out this term and that is easily done by multiplying this expression by minus K m minus 1.

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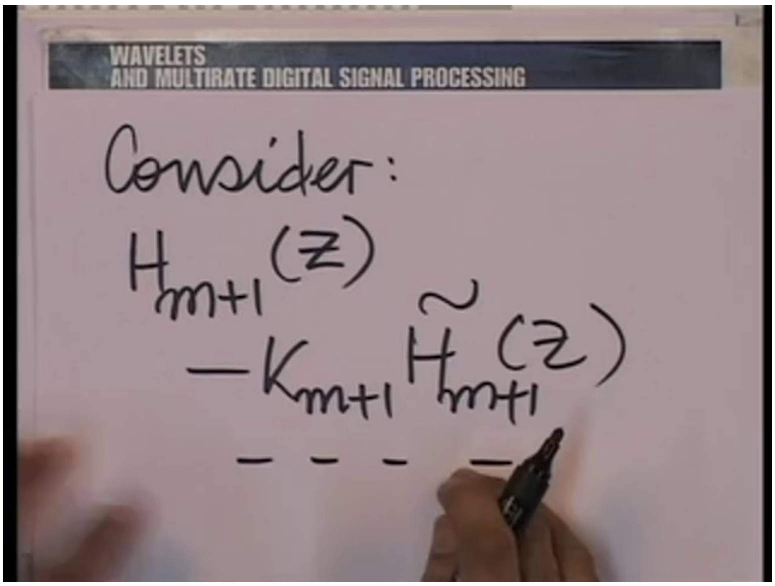

If we multiply this equation by minus K m plus 1 and added to this this term would vanish what you are saying in a effect is consider H m plus 1 Z minus K m plus 1 H m plus 1.

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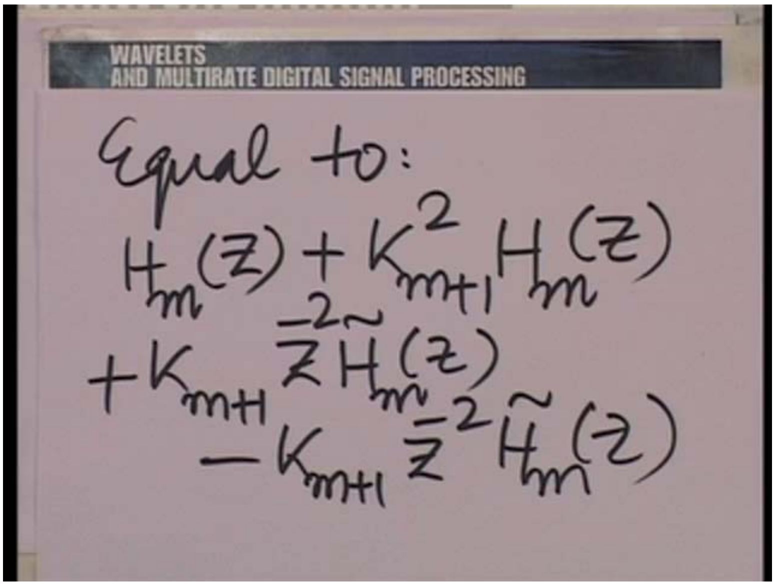

Tilde Z very clearly this would be equal to H m Z plus K m plus 1 squared H m Z plus K m plus 1 Z raise to the minus 2 H m tilde a Z minus K m plus 1 Z raise the power minus 2 H m tilde Z and which of course, very easily seen to be 1 plus K m plus 1 squared H m Z. So, we are eliminated H m tilde.

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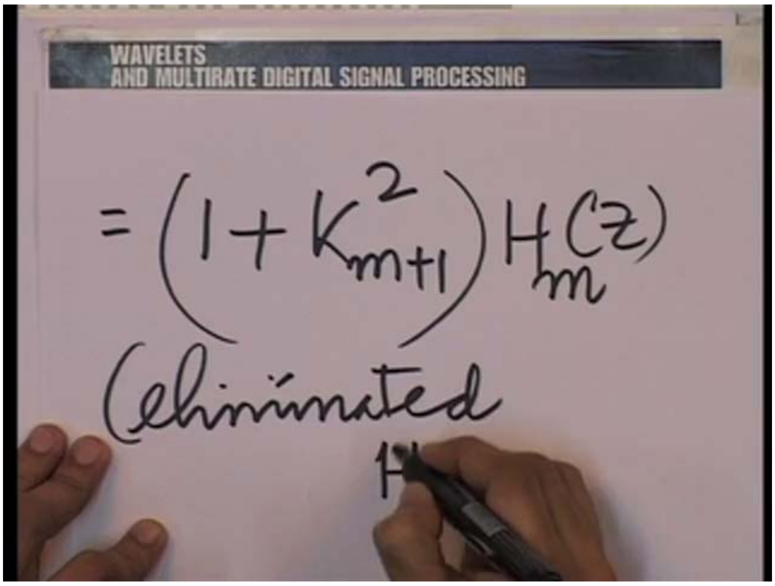

So, we an easy way now of obtaining H m from H m plus 1 and H m plus tilde now you know what we are doing here in terms of real computations or actual realization starting from a higher order filter and going1 step lower in the lattice and what we just shown is if you know the higher order filter you know of course, it is conjugate quadrature filter once you know H m plus 1 it is easy to construct H m plus 1 tilde all that you do is to replace Z by minus Z inverse in the argument and then multiply by Z raise the power are sufficient negative power. So, as to make it (( )) we shall illustrate this specifically for length four in short while, but let us get the algebra complete first.

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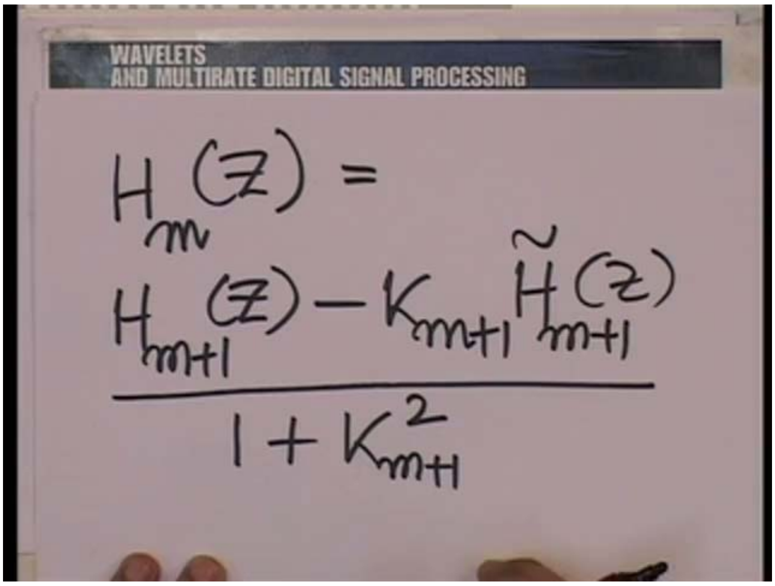

$$H_m(Z) = \frac{H_{m+1}(Z) - K_{m+1}\tilde{H}_{m+1}(Z)}{1 + K_{m+1}^2}$$

So, what we are saying in a effect is H m Z is H m plus 1 Z minus K m plus 1 H m plus 1 tilde Z divided by 1 plus k m plus 1 squared now let us look at the validity of an expression like this validity means is this expression computable does it make sense now of course, this definitely make sense there is no problem, but we do not know what K m plus 1 we need to reason that out and that will also require little bit of reasoning. So, if I new K m plus 1 this is. In fact, very easy to compute and.

So, is the denominator then. So, all that we have to do is to reason out have to find K m plus 1 and. In fact, that we shall do by looking back at the forward recursion once again, but you know as for this denominator is concerned here the validity of division by 1 plus K m plus 1 squared cannot be questioned as long as K m plus 1 is real if K m plus 1 is real we have no problem of validity at all because this is definitely going to be a non negative strictly non 0 quantity.

So, all that we need to do obtain K m plus 1 how and that is done very easily by looking back at the forward recursion. In fact, here again will need to do a bit of inductive reasoning.

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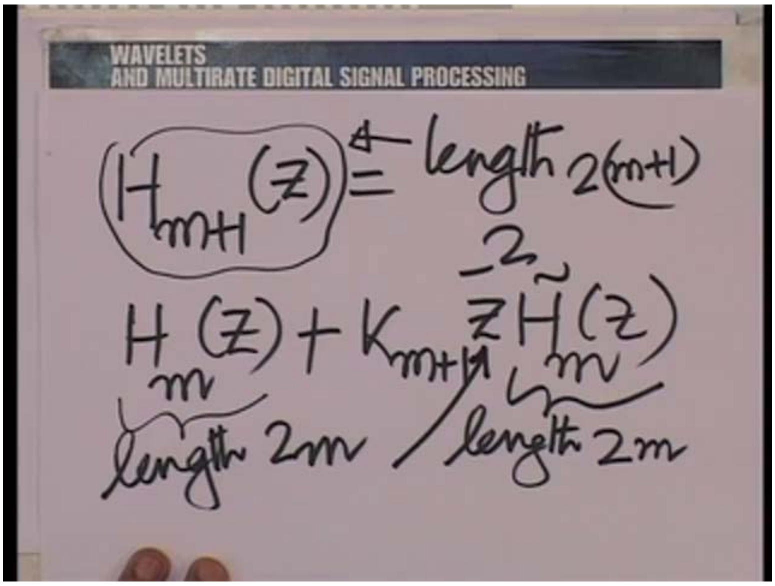

So, let us write down the main step of forward recursion once again H m plus 1 z is H m Z plus K m plus 1 Z raise the power minus 2 H m tilde M now notice that H m plus 1 is going to be a length 2 into m plus 1 and these are going to be of length 2 n. So, this factor Z raise the power of minus 2 is going to push this length 2 m filer 2 steps forward and that is how you are going to get a length of 2 into m plus 1 you know this pushing forward by 2 is what increases the length by 2.

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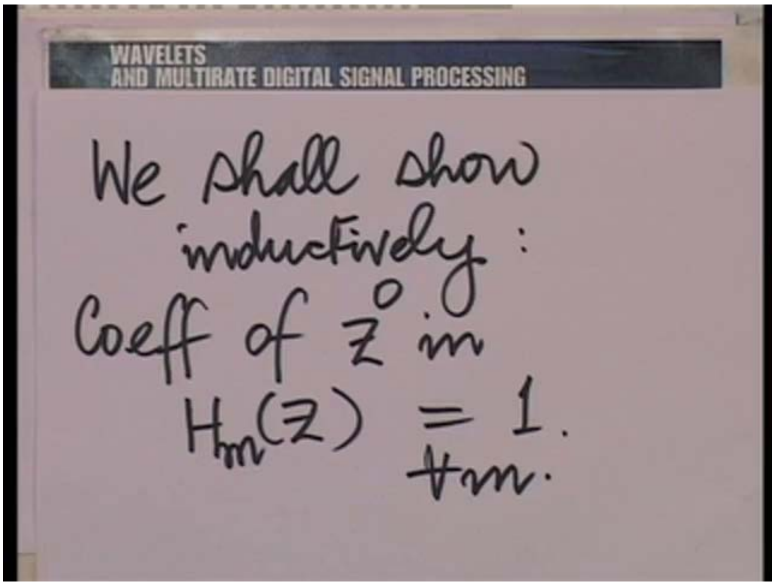

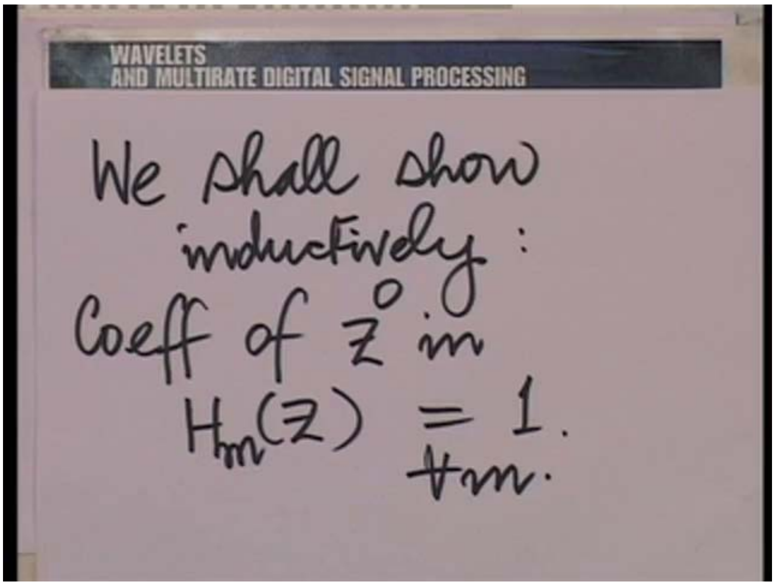

So, this is critical as you can see increasing the length now let us make a simple observation about the coefficient of Z raise the power of 0 we shall show now inductively coefficient of Z raise the power 0 in H m Z is always 1 for all m.

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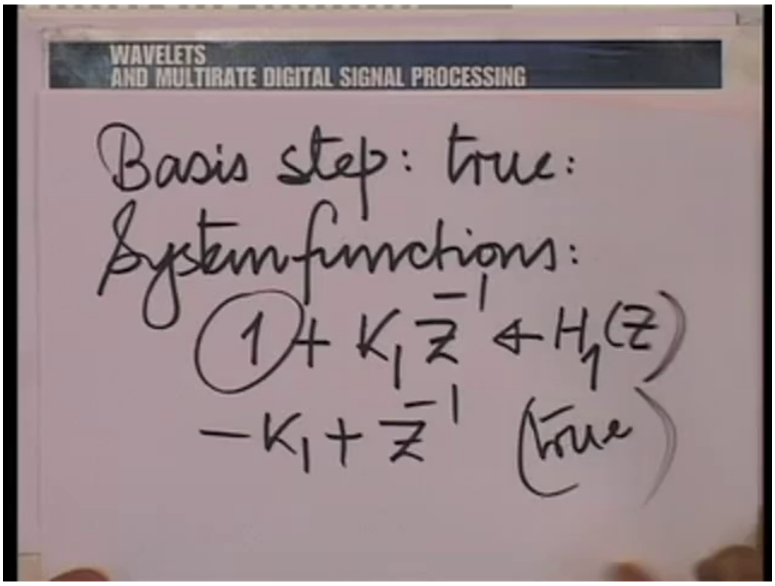

In fact, this is true in the basis step you recall that in the basis step the system functions very simple they were essentially 1 plus K 1 Z inverse and minus K 1 plus Z inverse and here of course, this was the coefficient of Z raise the power of 0 in H 1 Z.

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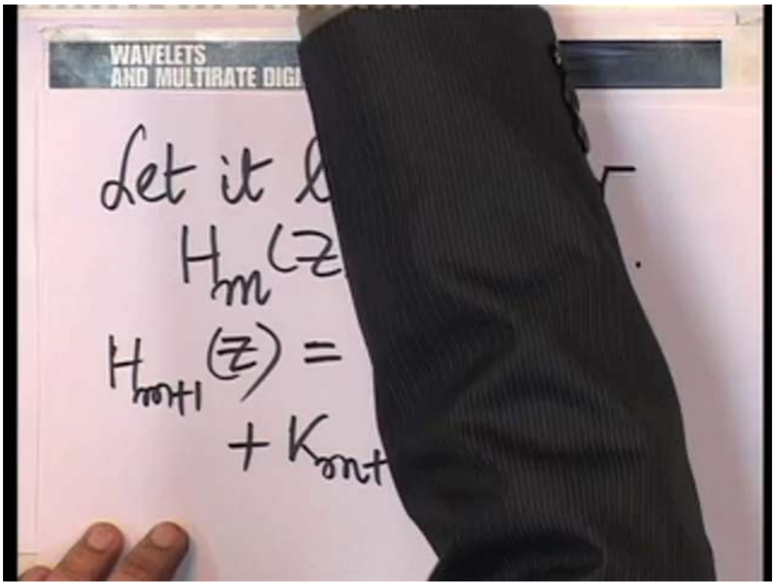

If you please and therefore, it is true now we shall use the inductive step to continue this proof let it be true for H m m greater than equal to 1 now for us H m plus 1 is concerned H m plus 1 Z is H m Z minus K n or other plus K n plus 1 Z raise the power minus 2 H m tilde Z and let us go back a couple of steps in a reasoning when we load this equation you know you can visualize the equation where would the Z raise the power of 0 coefficient here and if any in this term, but please note that in this term you have the Z raise the power of minus 2 common to all the term in this expansion. So, this was the length 2 m and by multiplying by the Z raise the power minus 2 the lowest power of Z inverse in this entire term is 2 now. So, you have no Z raise the power of 0 term here and that has been assumed by induction to be1 and that is carried over here.

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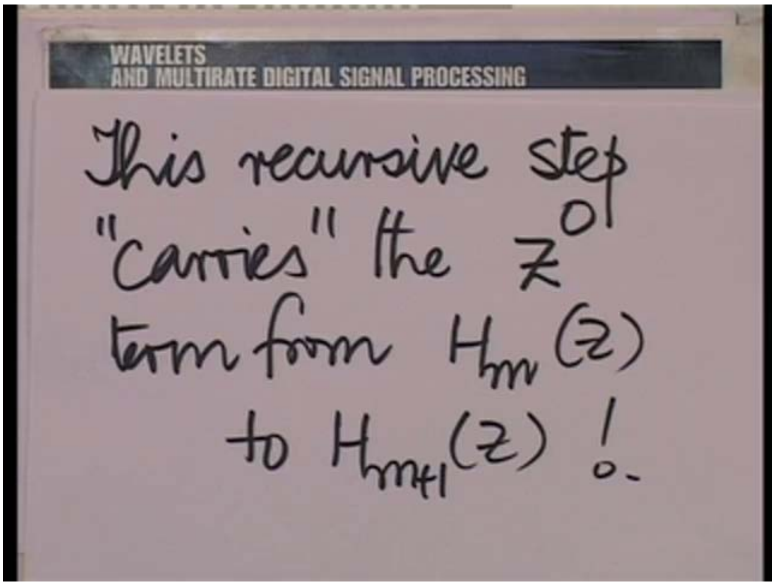

So, it is this equation which carries over the Z raise the power of 0 terms is carried from here to here and of course, by inductive assumption the Z raise the power of 0 term is 1 here it is also 1 there this completes the inductive proof now we know the coefficient of Z raise the power of 0 is always going to be 1 in all the H m s now is very easy to see what k m plus 1 is so.

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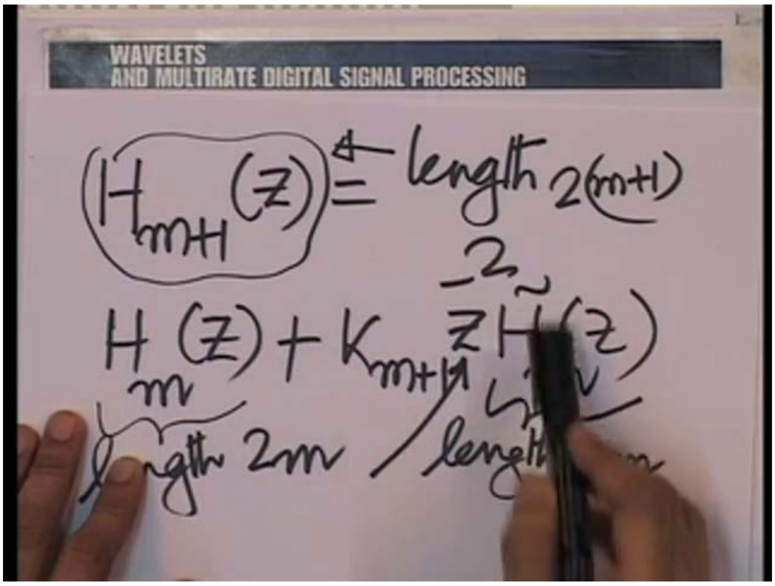

In fact, let us go back to the same equations that we use the minute ago let us look at these equations once again ya H m plus 1 is H m Z plus K m plus 1 Z raise the power minus 2 H m tilde Z now let us understand what H m tilde z is all about. So, H m tilde Z we assumed.

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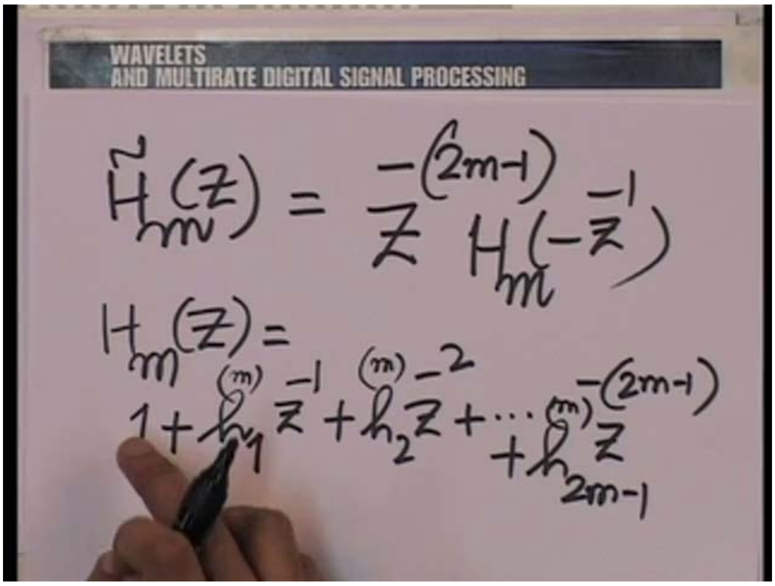

In fact, if shewn inductively that it must be the form Z raise the power minus 2 m minus1 H m minus Z inverse and H m said we know what that looks like. So, m said looks like this it looks like1 plus let us say h1 Z to the power minus1 plus H 2 Z raise the power minus 2 plus H 2 m minus1 Z raise the power minus 2 m minus1 now you know if you want to be very very careful you should subscribe this with m here to emphasize a this corresponds to the mth system function. So, i do that to be careful. So, all of saying is these are the impulse response coefficients are the impulse response points in H m and you know that the coefficient of Z raise the power of 0 is1.

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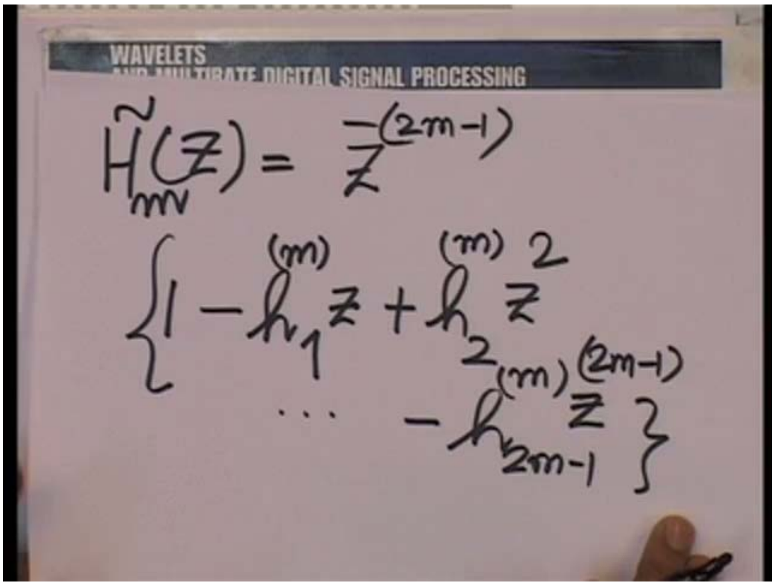

So, this is the form of the system function H m Z as account sequence H m tilde Z is now going to be the following form you know H m tilde Z is first going to have Z replaced by minus Z inverse here. So, to do that we have1 minus h1 m Z plus H 2 m Z squared and. So, on until we come to minus H 2 m minus1 m Z raise the power 2 m minus1 and this whole thing is then multiplied by Z raise the power minus 2 m minus1. So, if you look at what is happening these positive powers are going to be shadowed over by this negative power. So, for example, the H 2 m minus1 m term is going to have Z raise the power of 0 with it now and then you are going to have increasing negative powers of Z as you come down and finally, this is going to correspond to Z raise the power minus 2 m minus1 right there. So, let us write the expansion again.

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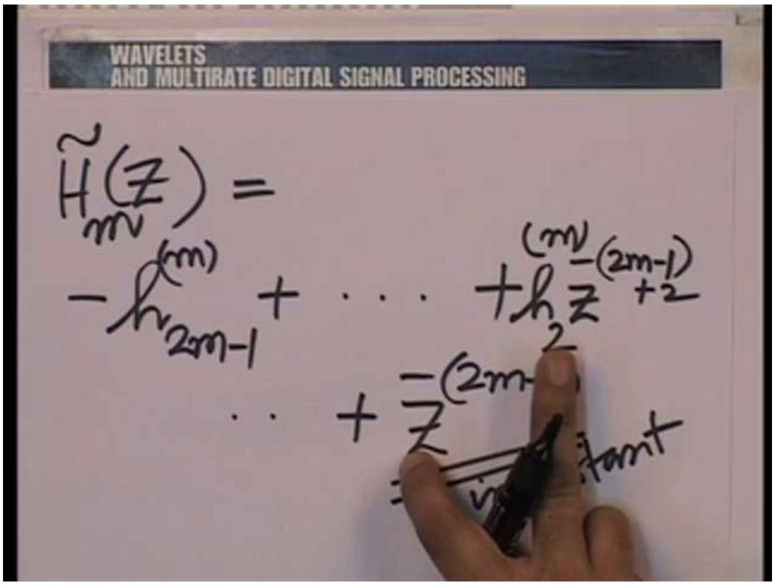

So, H m tilde Z is going to have the form minus 2 m minus1 m plus and. So, on plus H 2 m Z raise the power minus 2 m minus1 plus 2 and as you keep going downwards then you have until you reach Z raise the power minus 2 m minus1 here this is important and now making this observation the coefficient of the highest and of course, odd negative power of Z in H m tilde is1 and we carry that here. So, the coefficient of the highest power of Z inverse inh m plus1 is only going to come from here it cannot come from here the highest negative power here is Z raise the power minus 2 m minus1, but this is of length 2 into m plus1 the highest negative power of Z inverse here is going to be too larger than the highest negative power here and that can only come from here and. In fact, as you can see that must be k m plus1.

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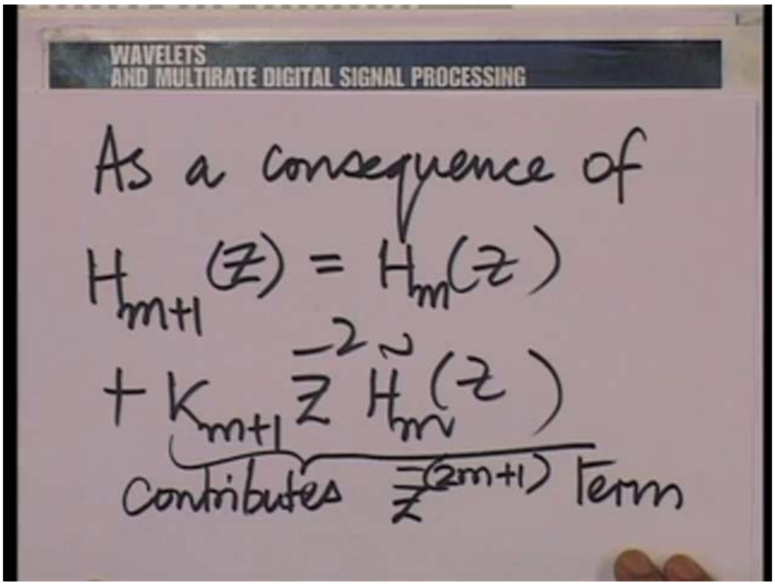

S, what we are saying in a effect is as a consequence of this let us write it down clear as a consequence of H m plus1 said is equal to H m Z plus K m plus1 Z to the power minus 2 H m tilde Z and you know what the form of H m tilde Z this is what H m tilde Z looks like the Z to the power minus 2 m plus1 comes only from here the highest negative power of Z and. In fact, the coefficient of Z raise the power minus 2 m plus1 that is the highest negative power of Z is equal to K m plus1.

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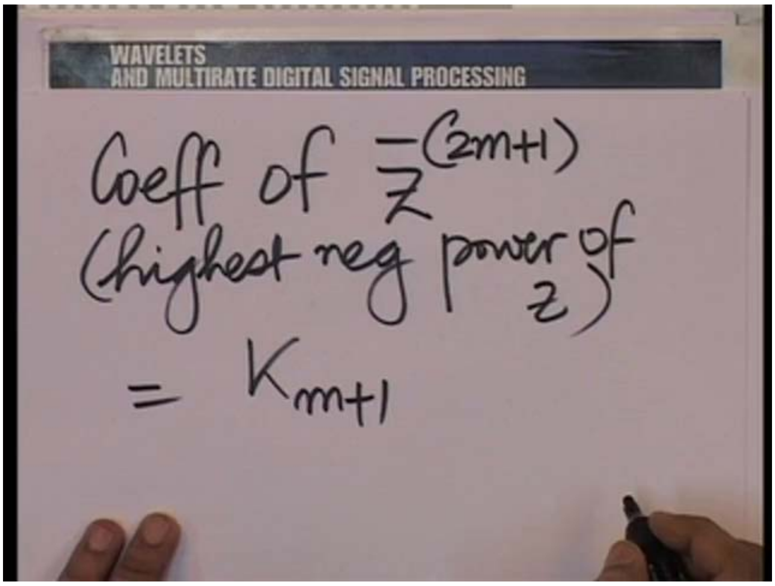

So, now, we have highest negative power of Z. So, once you have the final system function which you are trying to peel of stage by stage look at the coefficient of the highest negative power of Z and that itself is the first lattice coefficient simple.

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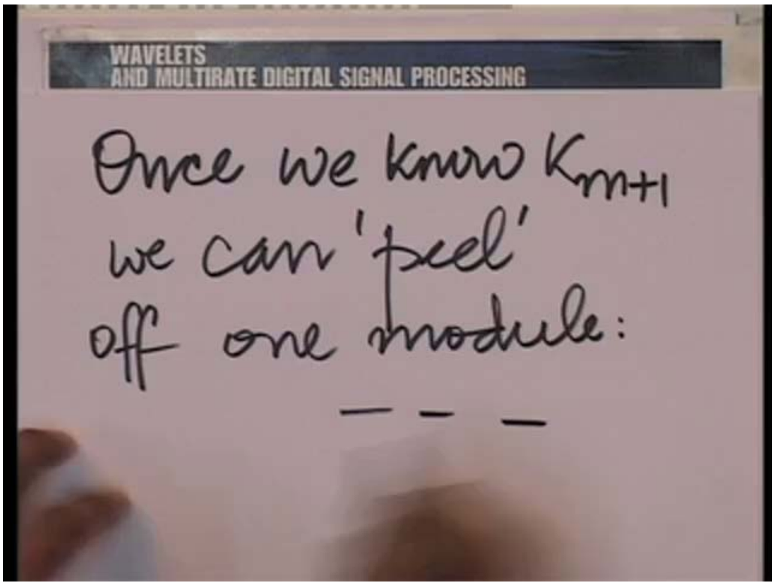

So, once you know k m plus1 you know how to go down descend1 step once we know K m plus1 we can peel off of1 module in other words.

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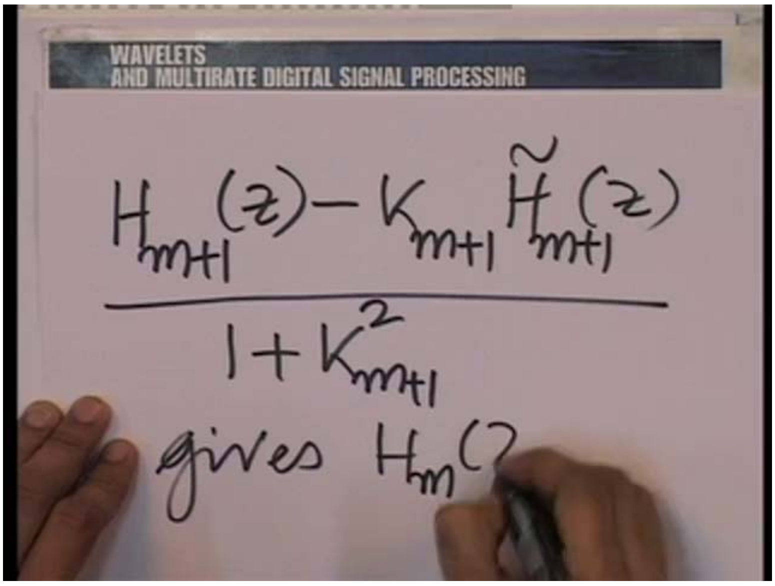

We know H m plus1 we can construct H m plus1 tilde and then H m plus1 Z minus K m plus1 tilde Z divided by1 plus K m plus1 squared gives H m z. So, notionally we have got H m z H m z was the system function after the previous module, but this is notion what we need to worry about is does H m z really have a length of 2 less now instead of again trying to do the algebra generally here as we be in doing for phi. So, this was the general algebra it will be easier for us to take a specific example in see how this (( )). So, let us take of lengthh four orthoonah filter that lengthh four filter could for example, be the dhobash length four filter, but anyway we shall not put specifically the numerical values of the impulse response coefficients, but we do not need to understand how this recursion words you would find it much easier to just carry out the recursion and demonstrate that it feasible.

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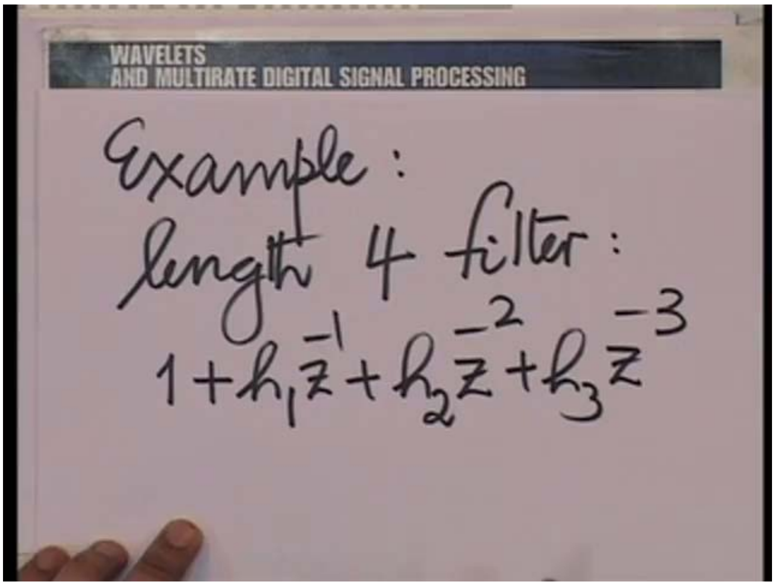

So, let us take an example lengthh four filter. So, if you had a form1 plus h1 z inverse plus H 2 Z raise the minus 2 plus H there z raise the power minus there. So, we wash to peel of1 stage in other words what we are saying is we have this situation we have 2 stages of the lattice here. So, we know the system function here and therefore, also here known at this point and we wish to obtained K 2 and K 1 that was we are saying.

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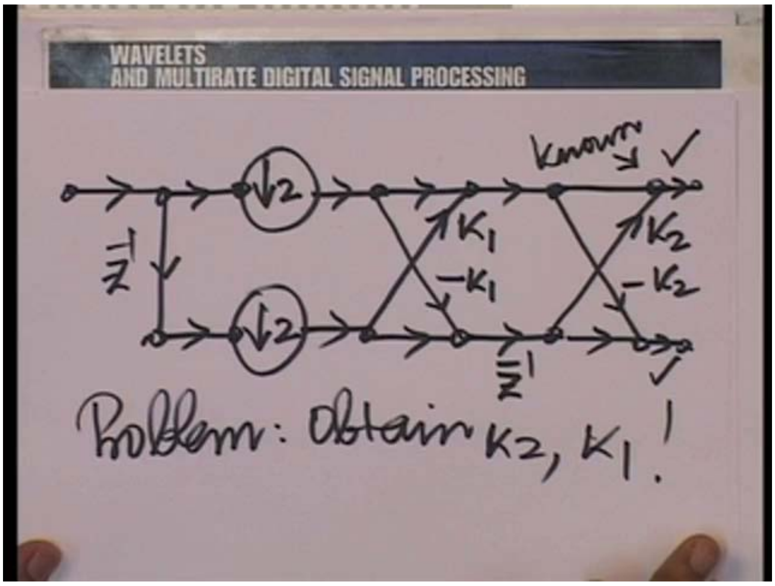

So, the problem is obtained K 2 and K 1 now what we just said makes it very easy for us to obtain k 2 given the length four filter.

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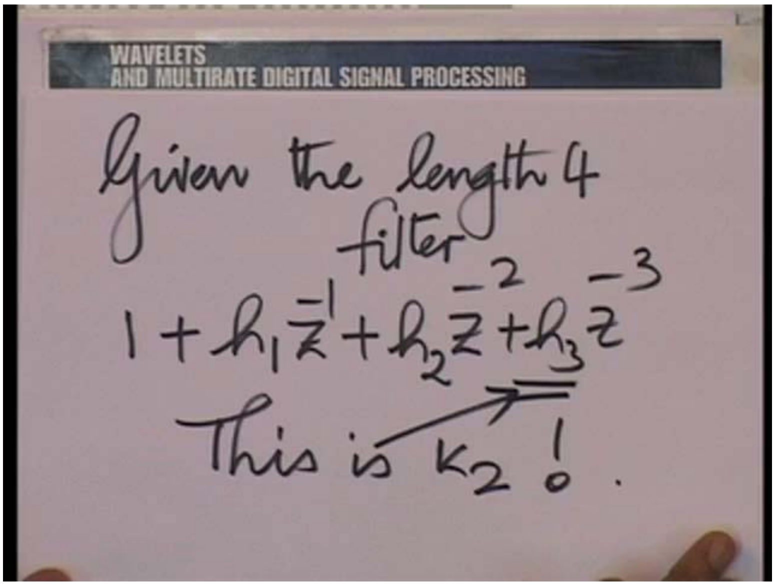

1 plush1 z inverse plus H 2 z raise the power minus 2 plus H three z raise the power of minus three what we discussed a few minutes before this namely that the most negative power of Z reveals K 2 as. So, this essentially is K 2 and having motive that we can obtained1 step lower.

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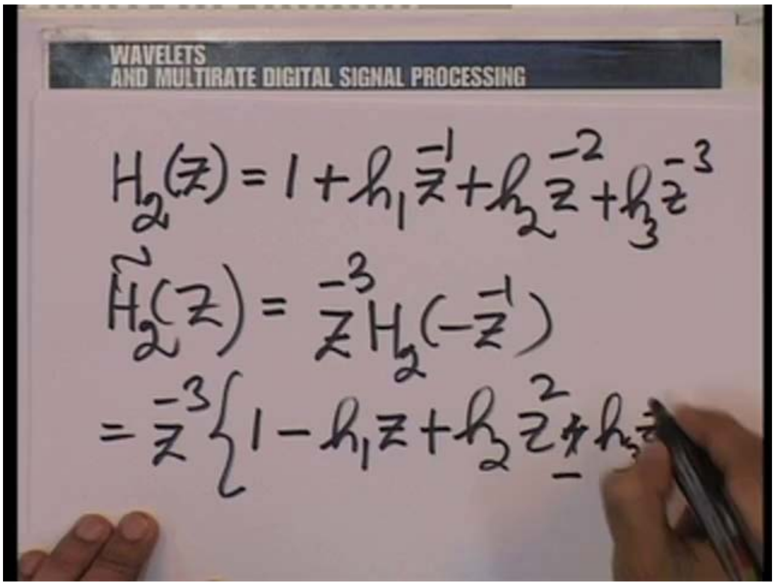

So, we can construct if we call this H Z or let us say H 2 z after the second stage then we can also construct H 2 tilde Z and H 2 tilde Z is going to be very simple it is going to Z raise the power minus three H 2 minus Z inverse which is z raise the power minus three1 minus h1 Z plus H 2 Z squared minus three z raise the power minus three. So, H 2 tilde z can be simplified it is minus H three plus H 2 Z inverse minus h1 Z raise the power minus 2 plus Z raise the power minus three.

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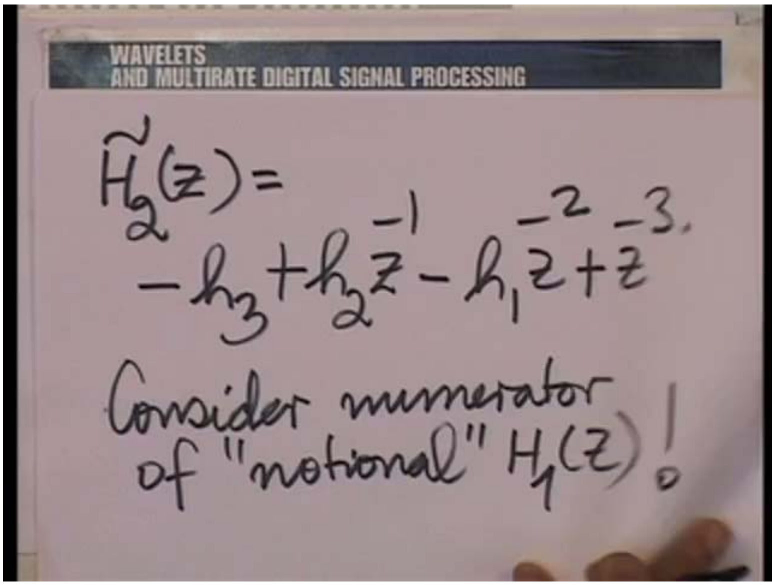

So, all that we now need to do is to consider the numerator of notional h1 Z that numerator h1 Z notionally as you remember was H 2 Z minus K 2 H 2 tilde Z divided by1 plus K 2 the whole squared and if you only k are to expand this.

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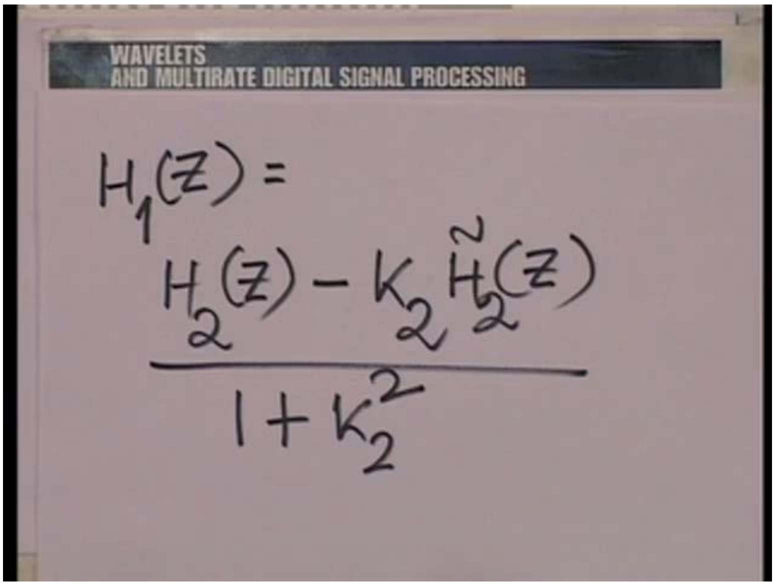

So, you know to expand the numerator it will be easier for me to write down the expansion in 2 lines1 line for H 2 Z and1 line for minus K 2 H 2 tilde Z.

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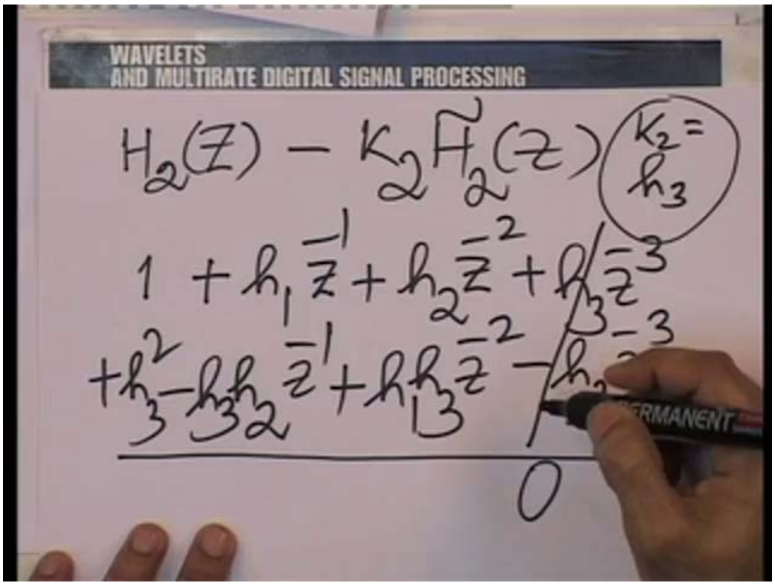

So, H 2 Z minus K 2 H 2 tilde Z and noting that K 2 is essentially H three we have1 plus h1 Z inverse plus H 2 z raise the power minus 2 plus H three Z raise the power minus three minus K 2 H 2 tilde z which is minus H three Z raise the power minus three plus h1 H three z raise the power minus 2 and something else here. So, of course, we just completed its minus H three H 2 z raise the power minus1 and of course, plus H three squared here now this is very interesting. So, let us look at this expression carefully. So, let us look at the coefficient of z raise the power minus three this anyway become zero identically what is interesting is this the coefficient of z raise the power minus 2 it is H 2 plus h1 H three, but let us go back to the basic requirement that we had on an orthogonal filter remember the whole idea in constructing the dhobash filter for example, was that the impulse response was orthogonal to its even translates let us put that condition down again in see what we gives us.

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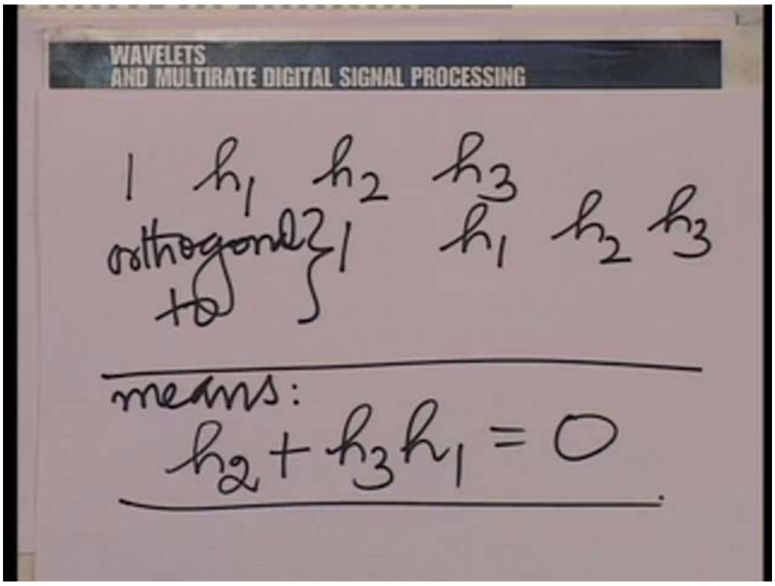

So,1 h1 H 2 H there is orthoonah to1 h1 H 2 H there and; that means, H 2 into1 plus H there into h1 is 0 now go back to the coefficient of z raise the power of minus 2 here that is H 2 plus h1 H there low and be whole that the same as this and it is identically zero and account of the orthogonality to even translate. So, let us make an note of that.

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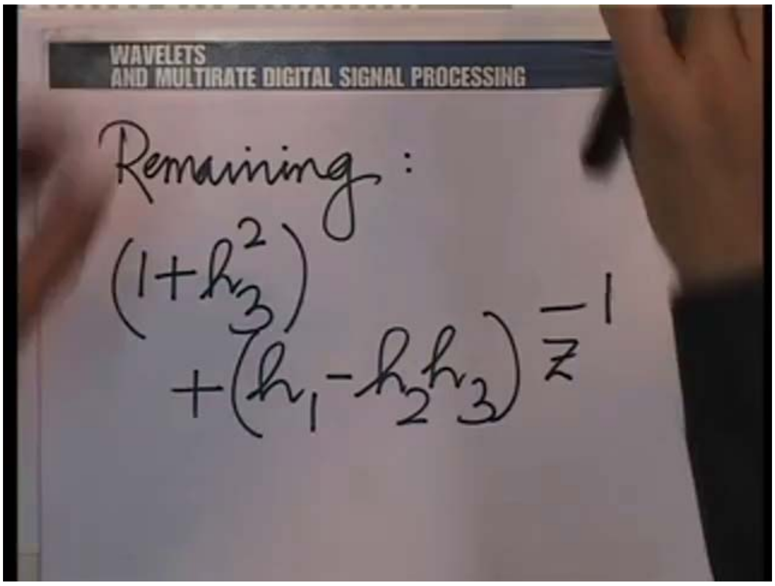

So, this is zero because of this orthogonality to even translates. So, what is left now remaining is1 plus H three squared plus h1 minus H 2 H three times z raise the power minus1.

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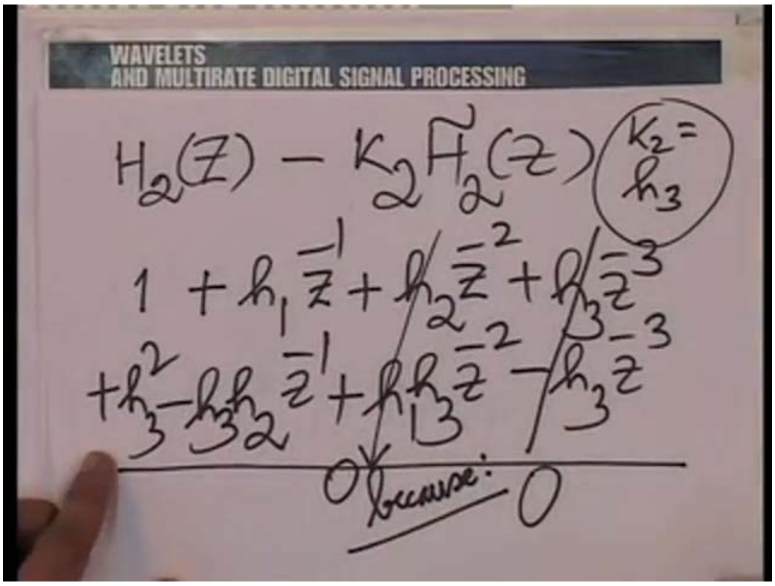

So, we are arrange very good situation as far as H 2 Z minus k 2 tilde z is concerned the length has gone down by 2 as we indeed wanted it 2.

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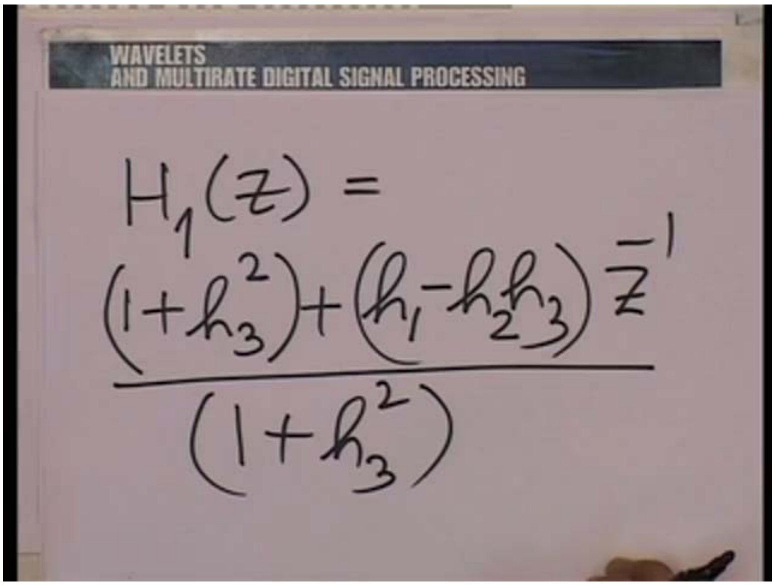

Accordingly h1 z is now going to be1 plus H three squared plus h1 minus H 2 H three into z inverse divided by1 plus H three squared K 2 squared remember.

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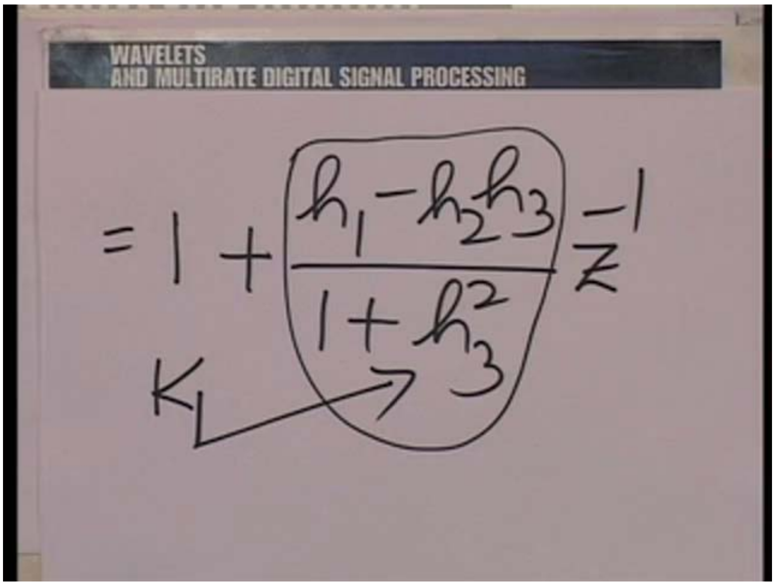

here and that is very simple that is1 plus h1 minus H 2 H there divided by1 plus H three squared times Z inverse and low in behold also reveals to us k1 direct. So, we completed the lattice construction as you can see for the dhobosh case is going to work definitely all that the required is orthogonality to even translates nothing else was needed because of the orhtogonality to even translates when we carried out the numerator part of the peeling process the length decreased by 2 now you know1 must understand where there is something unusual here in going to H m of that matter H m tilde from H m plus1 of that matter H m plus1 tilde there was just a little bit of solution of 2 equations involved that could be an solve for any H m plus1 and H m plus 2 tilde there is nothang very special about the form of H m or H m tilde of that matter in writing down that downwards. However.

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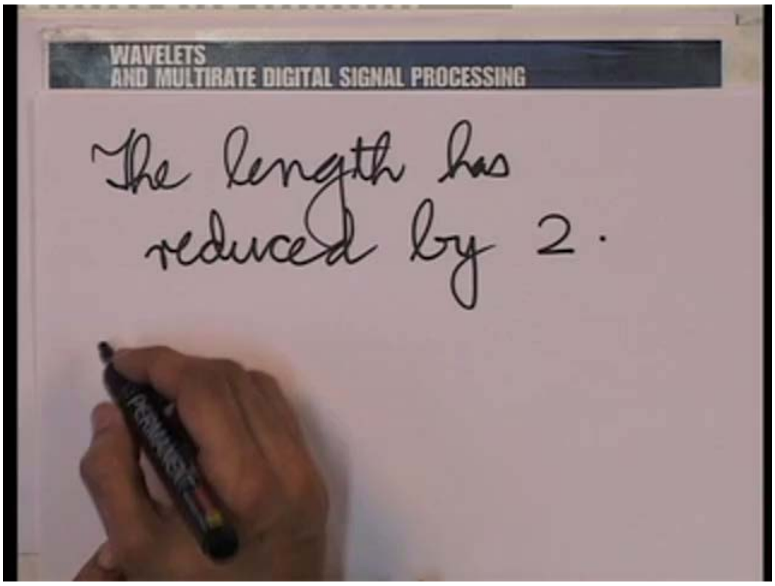

What was important that when you went from H m plus1 to H m you needed the lengthh to go down not by1, but by 2 and this decrease of lengthh by 2 is what brogue Then the requirement of orthogonality of even translates. So, it is only because of the orthogonality to even translates that the lengthh decrease by 2 in the numerator part of the expansion and of course, once your assure at the coefficient k which you had peeled off of the coefficient k by looking at the coefficient of the highhest negative power of Z in H m plus1 gave us yield it clearly revealed the corresponding k of the outer most lattice stage. So, now, we can see how to peel of stage by stage we done it specifically for a 2 stage lattice and its not difficult to extend this to a there stage or a multistage lattice beyond there stages now a few remark are in order here you know what we done now is actually to demonstrate with an example that when you carry out this backward recursion again .

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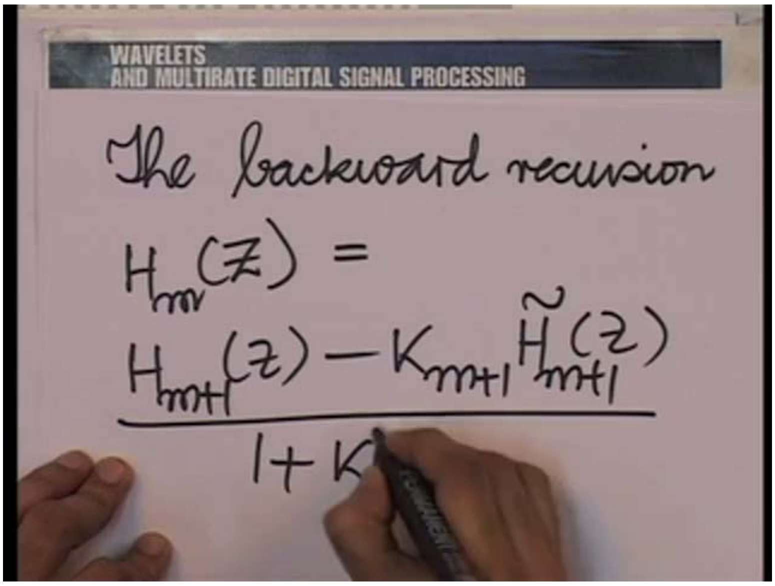

Let us make this observation the backward recursion H m Z is H m plus Z minus K m plus1 H m plus1 tilde Z divided by1 plus K m plus1 squared is effected by the following step.

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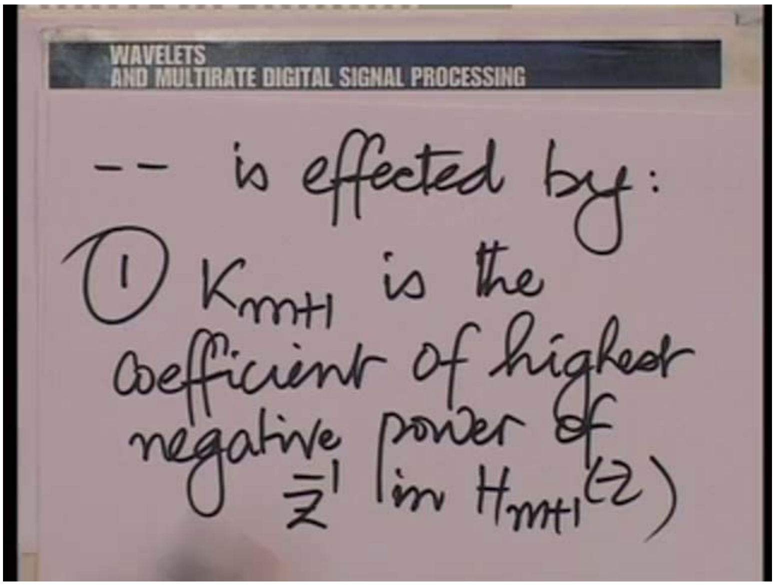

Set no1 K m plus1 is the coefficient of the highhest negative power of Z inverse in H m plus1.

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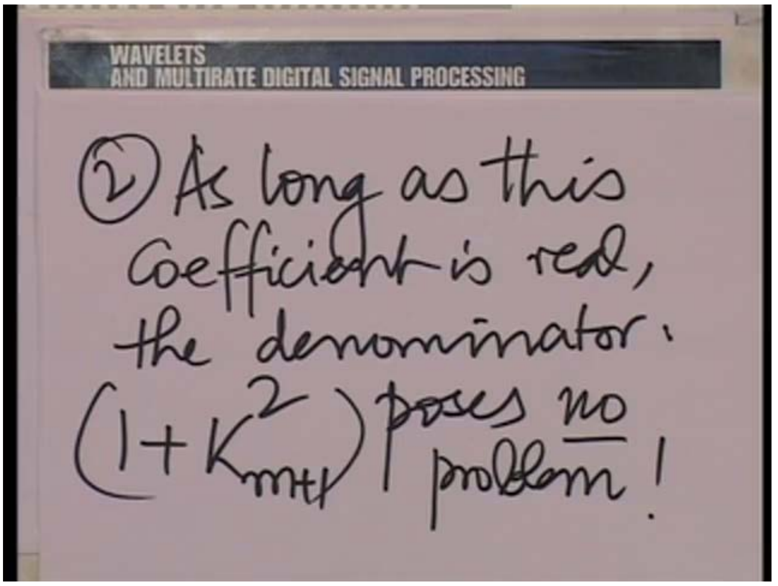

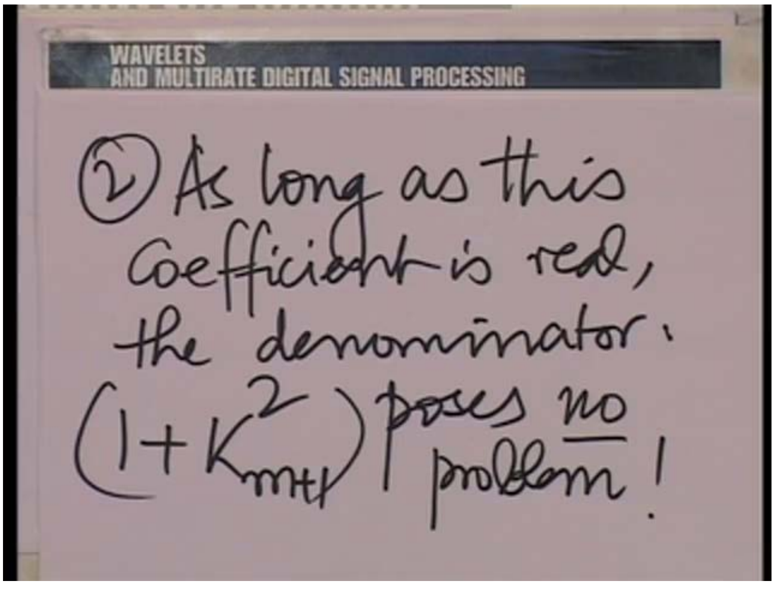

As long as this coefficient is real the denominator which is1 plus K m plus1 squared poses no problem what i mean by that is a denominator can pose a problem if it is 0 you cannot divide by 0. So, you are assure that this is not going to be 0 if k m plus1 is real or in to words the coefficient of the highest power of Z inverse is real you guaranteed this (( )) does not going to division by 0.

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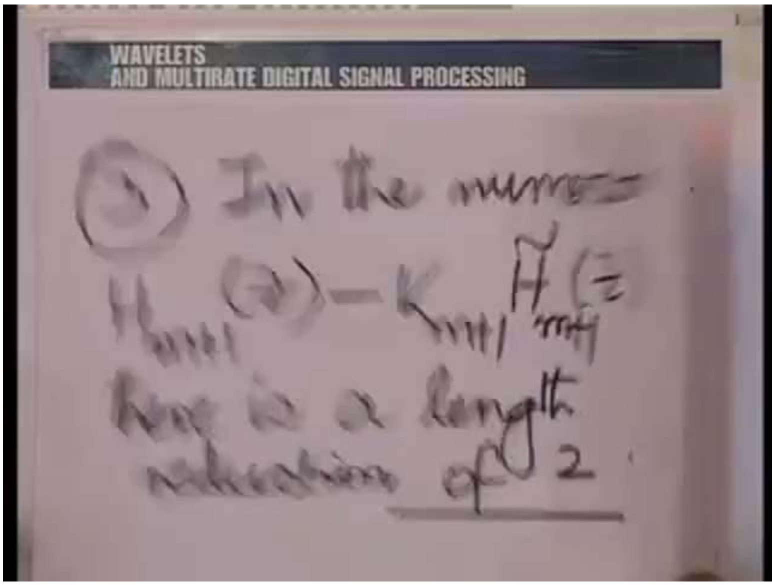

having made that observation now the thard observation is in the numerator namely H m plus1 Z minus K m plus1 H m plus1 tilde Z there is a lengthh reduction of 2 and the justification you see reduction by1 is easy essential.

If you look at this numerator the coefficient of the highhest power of Z inverse here is K m plus1 and the coefficient of the highhest power of Z inverse here withoot this would be a1, but with this its minus K m plus1.

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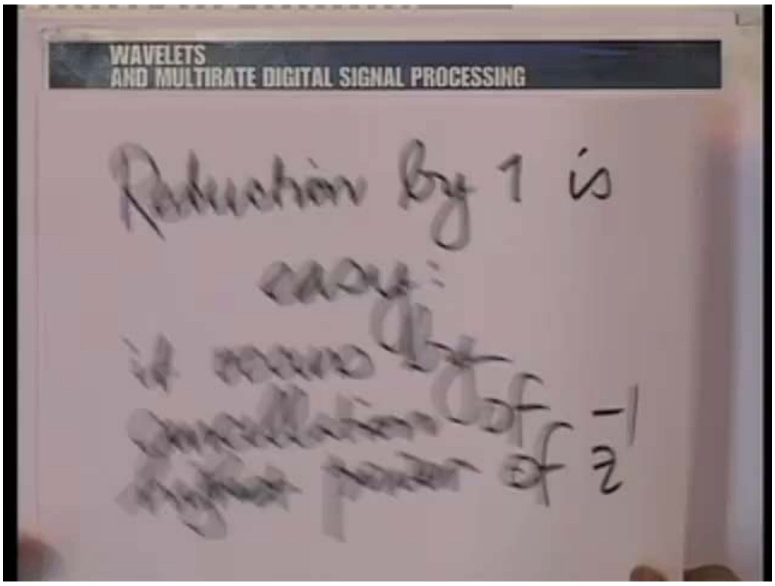

So, that would cancel out. So, essentially reduction by1 is easy it occurs by cancellation of highest power of Z inverse, but the additional1 step reduction.

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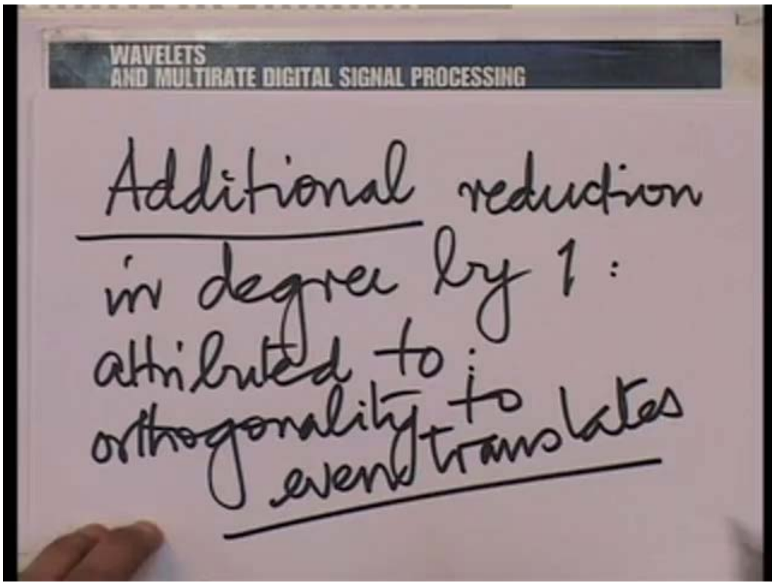

Additional reduction in degree by1 is attributed to orthogonality to even translates this is a very important observation this is where the orthogonality of the filter bank where it not an orthogonal filter bank that we were talking about we would of course, have trouble getting in this additional reduction of technique now finally, we need to complete this lecture by putting in a variant and that variant is essentially to transpose the structure that we construct it in other words how could we use the lattice structure that we have construct it to build a lattice structure for the synthess side let us complete this lecture by discussing this variant.

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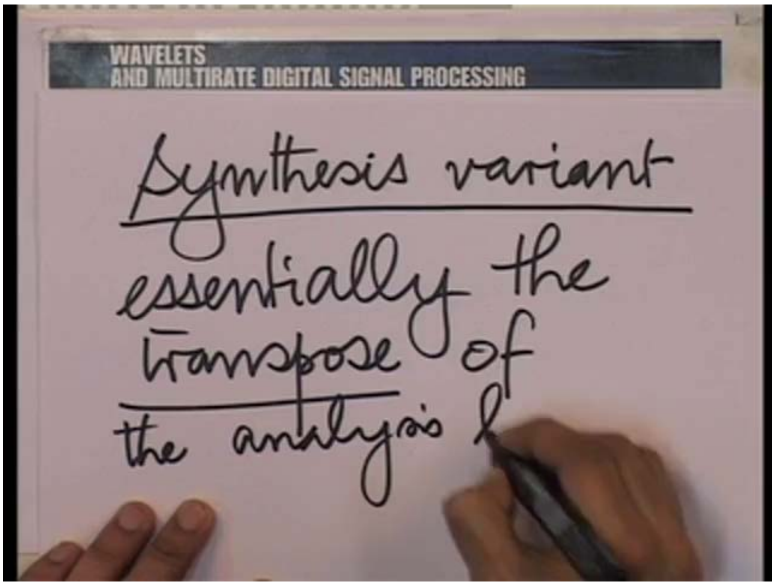

The synthesis variant its very easy to construct the synthesis variant essentially it is the transport of the analysis lattice and what i shall do is to construct the synthesis variant first for a length 2 and then for length four and that would illustrate and how do we do it in general.

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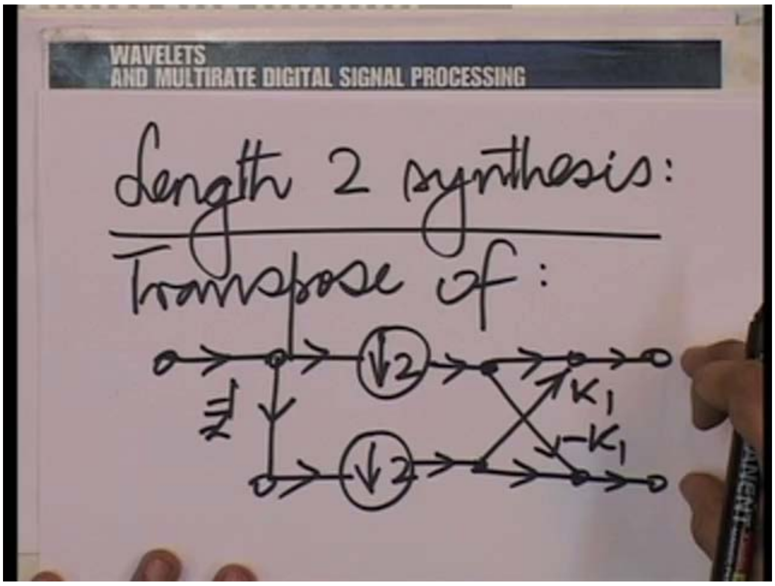

So, length 2 synthesis lattice is essentially the transpose of this and you can visualize what the transport will look like let us just over it orally first and then draw it we would have the transport with all these arrows reversed here you would have up sampler at this point these arrows reversed the Z inverse as it is this now becoming a summing point inside of a branching point and then resulting in the output here let us draw the transpose.

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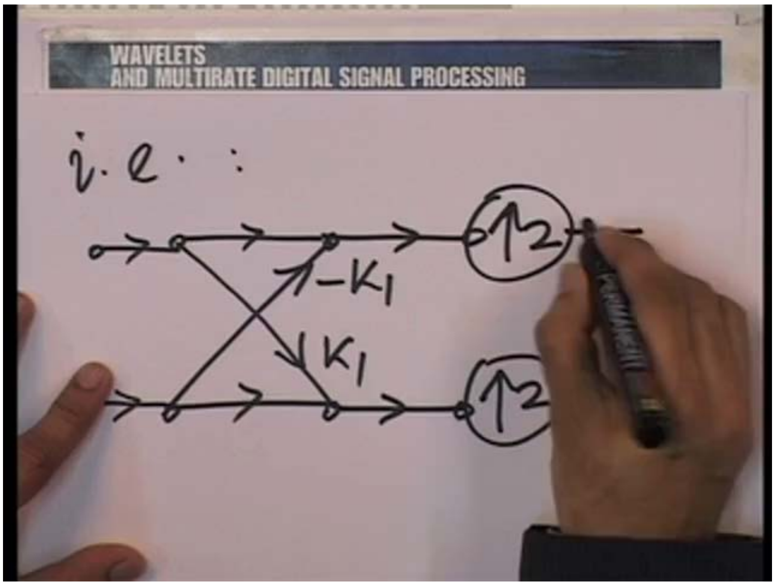

They we are this is the transpose of length 2 stage now all that we need to do to complete this discussion is to show the transpose of the inductive module.

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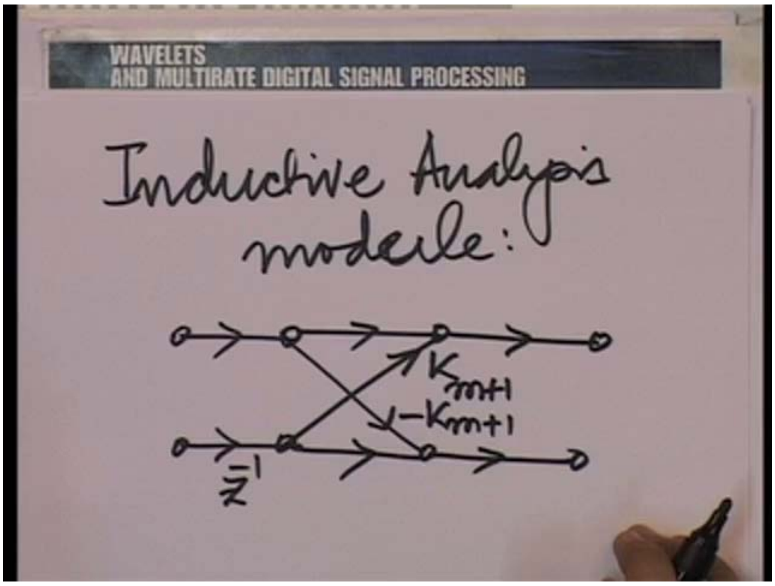

So, the inductive analysis module was essentially this of course, this is after the down sampler let us say K m plus1 here and minus K m plus1 there and therefore, the corresponding senthess inductive module would look like this once we have the inductive module and the basis module it is easy to construct the complete lattice on the senthess side and now I leave it as an exercise for the class at the end of this lecture to work out the same kind of recursions has we done for the analysis side to construct the senthess filter bank leaving that exercise to you we conclude the lecture here to observe that we can a beautifully computationary efficient structure call the lattice to realize an orthogonal analysis and now senthess filter bank thank you.