Advanced Digital Signal Processing – Wavelets and Mutlirate

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Module No. # 01
Lecture No. # 32
'Nobel' Identities and The Haar Wave packet Transform

A warm welcome to the 32nd lecture on the subject of wavelets and multirate digital signal processing. In the previous lecture, we had introduced the idea of the wave packet transform. Essentially, we had hinted at the aim behind the wave packet transform; namely, to be able to decompose the high pass branch or the detailed as well, into sub spaces.

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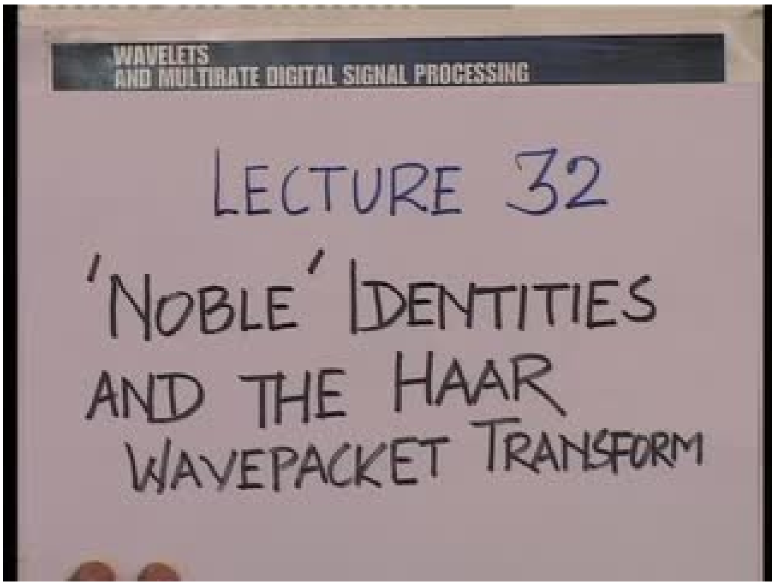

We had made one important observation. The decomposition of the high pass branch proceed somewhat differently from the decomposition of the low pass branch. There are some counter intuitive observations in the frequency domain. However, we had considered only the ideal 2 band filter bank when we talked about wave packet transform the last time. If you wish that we should be able to realize a wave packet transform to implement it then, we must look at how the wave packet transform could operate given realistic filter banks and that is what we aim to do in the lecture today. Therefore, in the lecture today, we intend to talk about some identities pertaining to multirate digital signal processing which are often called as Noble identities.

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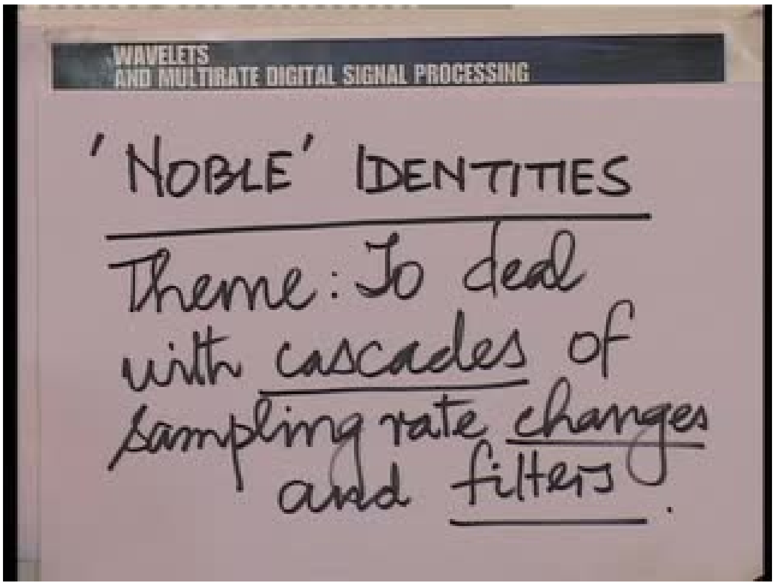

Noble in the sense, that they are extremely important in building multirate structures and in analyzing multirate structures. Further, we will use the noble identities to build the wave packet transform in the context of the haar multi resolution analysis. So, with that little introduction to what we intend to do in the lecture today, let us proceed to talk about the so called noble identities that governs the behavior of multirate systems.

You know the essential idea in the noble identities is actually to look at what happens when we cascade. So here, you know these are called noble identities because, they occur very frequently whenever you to wish to iterate multirate systems. So, for example, in the discrete wavelet transform we have iterations of down samplers and filters one after the other. In fact, on the synthesis side you have up samplers and filters.

So, the noble identities try and deal with this the situations, how can you aggregate together different down samplers and filters or different up samplers and filters? Let us first look at the down sampling context - the analysis context. So, let us put down the basic theme, the theme in the noble identities is to deal with cascades - cascades of sampling rate changes and filters.

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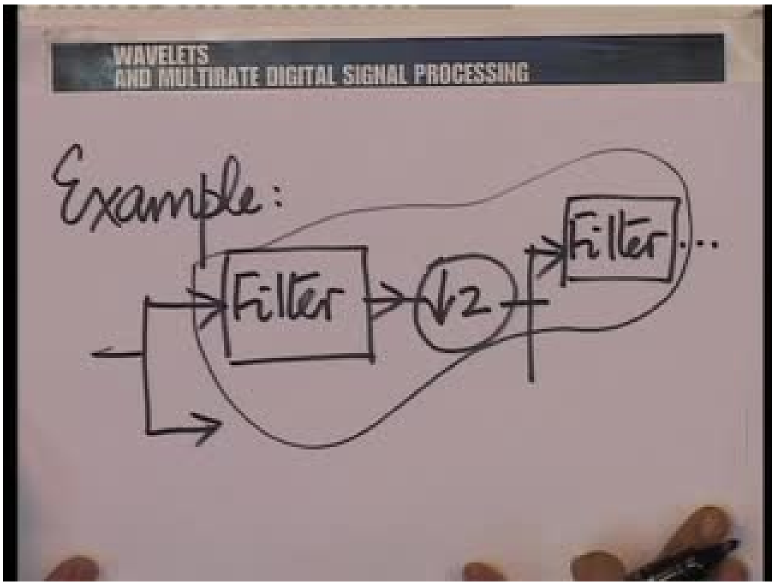

let us take an example. An example could be something like this: what happens in the discrete wavelet transformation? So, you have the filter followed by a down sampler again, followed by a filter in a down sampler and so on.

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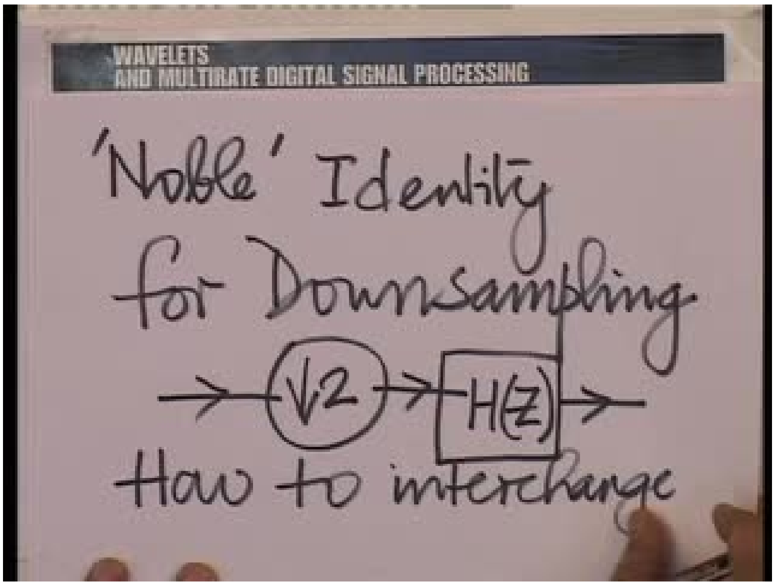

So, it is this kind of cascade that we wish to be able to address and therefore, let us consider the noble identity for the down sampling first. Now, the noble identity for down sampling emerges from the following consideration: how can we represent a down sampler followed by a filter? Let us call that filter H of Z; so we describe the system function of the filter as the H of Z. A noble identity essentially answers the question how to interchange.

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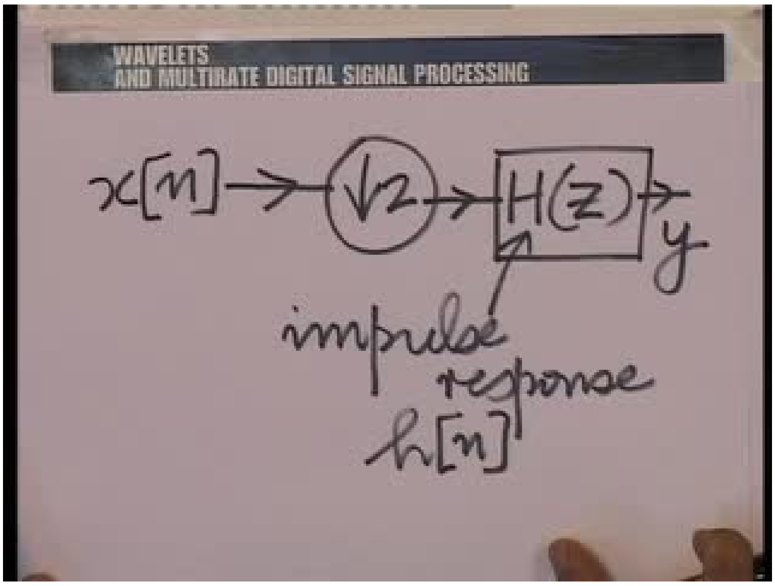

Can I replace this by some other filter followed by a down sampler? That is the question that this, so called, noble identity tries to address. Now, you know we could answer or we could address this problem by assuming that a sequence x of n is fed to an down sampler, followed by this filter H Z. Let us assume that this filter has the system function H Z and the corresponding impulse response is small h of n. Let us call the output y; here y of n I mean. Now it is very obvious what y of n is. Essentially, y of n is the sequence here convolved with the first impulse response h n. The sequence here has at every sample location the value of this sequence at twice that sample location. So, for example the sequence at this point for example, the sequence at this point, would have at the point n equal to 3, the value x 6. For that matter, at 0 it would take the same x of 0 at minus 1; here, it would take the value x of minus 2. So, if we call this x 1 here and what we are saying in effect is, x 1 of n is x of 2 n for all n belonging to the set of integers and therefore, y of n which is essentially x 1 convolved with h evaluated at n can be described as follows: Y of n is obviously, summation K running from minus to plus infinity either x 1 K times h of n minus n minus K have or you it could do it the other way we will write both the expressions and see which is convenient.

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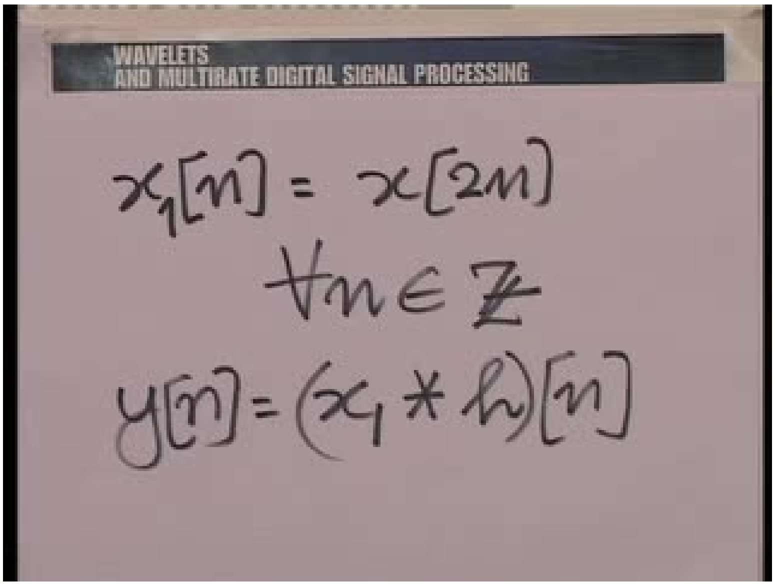

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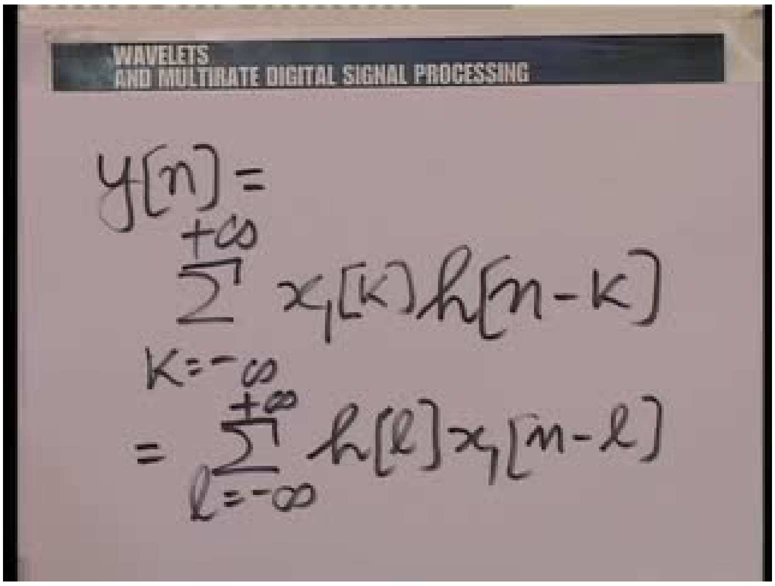

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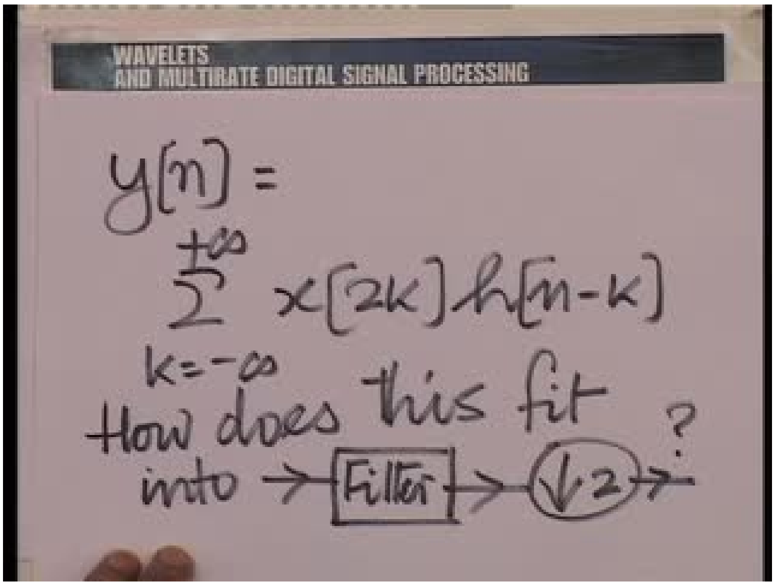

Now, x 1 K is easy to write; so, x 1 k let us takes the fast first of these 2 so it is very clear that y n becomes summation K going from to minus to plus infinity x of 2K times h of n minus K. Now, what you are saying here is in effect that we are doing almost what a convolution does. Remember, our objective is to interchange the position of the down sampler and the filter. So, from this expression here we are trying to get an equivalent expression for a system comprised of a filter followed by a down sampler in which the operation should give the same expression as here. So, the question that you are asking is, how does this fit into filter followed by down sampler?

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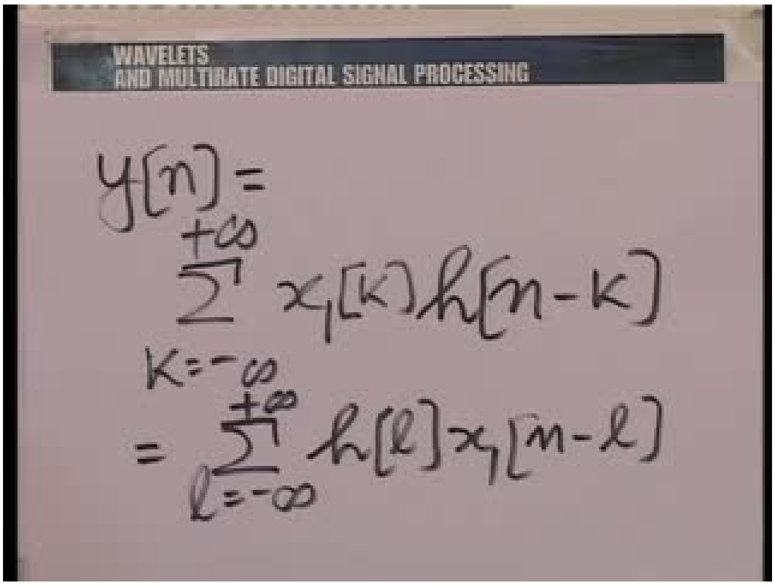

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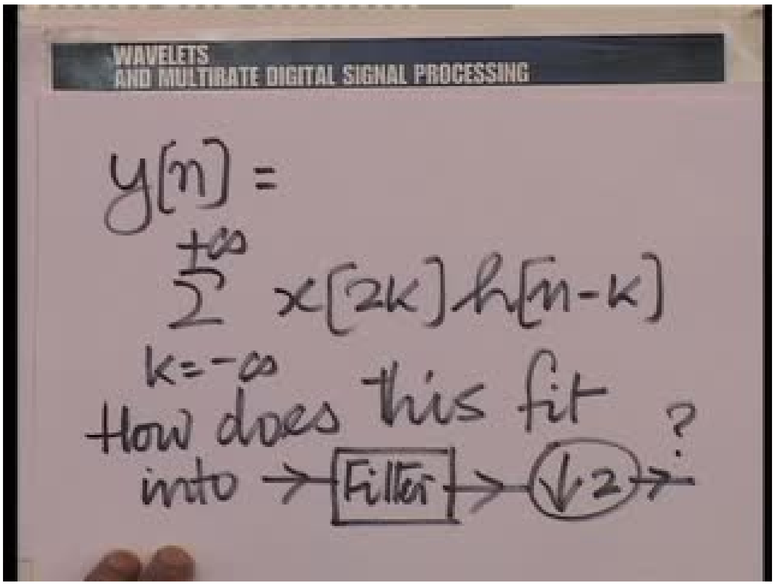

Now, you know let us see whether it is easier to answer this question by using this expression or by using the expression here. Please remember that only when we have a down sampling operation we multiply the index by 2. So, suppose we want to use this expression here and note that x 1 of n minus l is actually x of 2 n minus 2 l then in some sense that we already taken care of the requirement of 2 n write right there. So instead of using this expression let us use this expression to arrive at an answer to this question.

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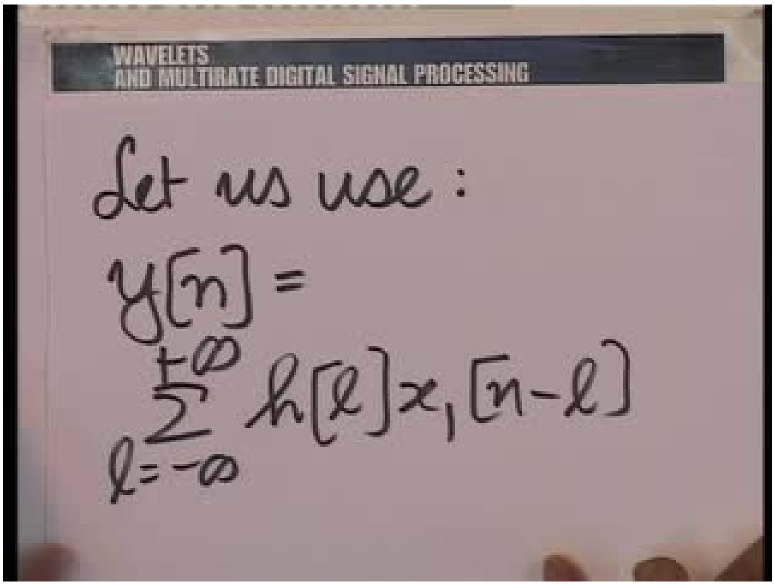

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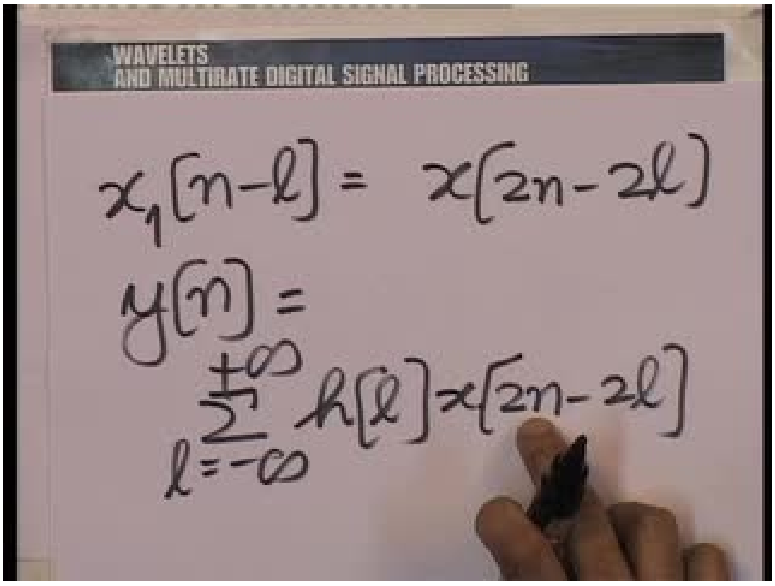

Let us use y of n is a summation l going to from minus to plus infinity to plus infinity h of l x 1 of n minus l and note that x 1 of n minus l is in fact x of 2n minus 2l; where upon we have y of n is summation l going from minus to plus infinity h of l, x of 2n minus 2l and now we are almost there in answering the question. You know, if you look back at the drawing that we had here, this down sample by 2 has been taken care of in this. So, we already have a 2n here. So, what we could do is to regard this 2n index has being before the filter, you know, the 2n has been created better for the down sampling operation. If I replace the 2n here by n, I would get what appears just after the filter.

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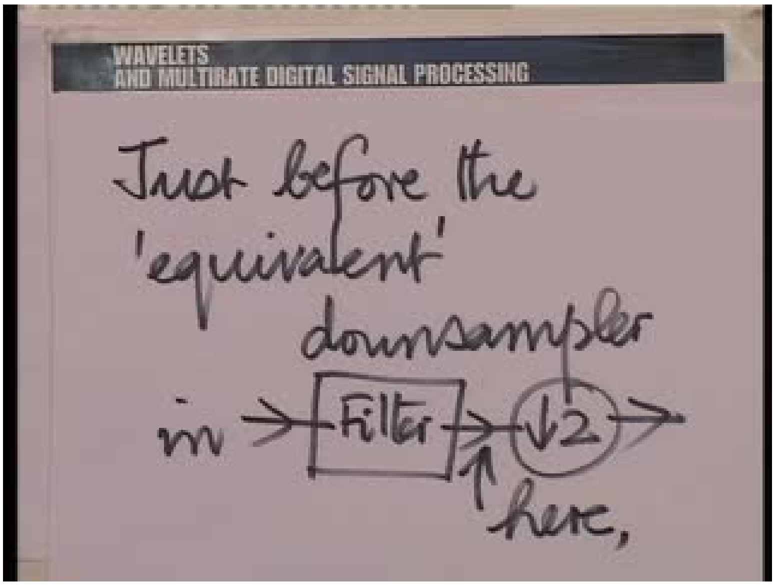

In other words, just before the equivalent down sampler, in filter and down sampler. So, at this point we have summation l going from minus to plus infinity h l x n minus l n minus 2l. Now, it is very clear what the filter should be you know this is almost like a convolution; except that it appears as if you are taking only the even points in the impulse response.

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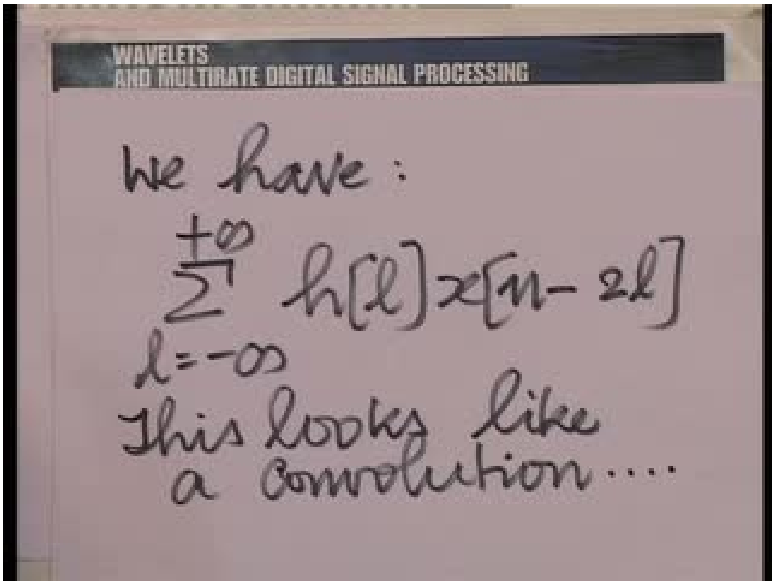

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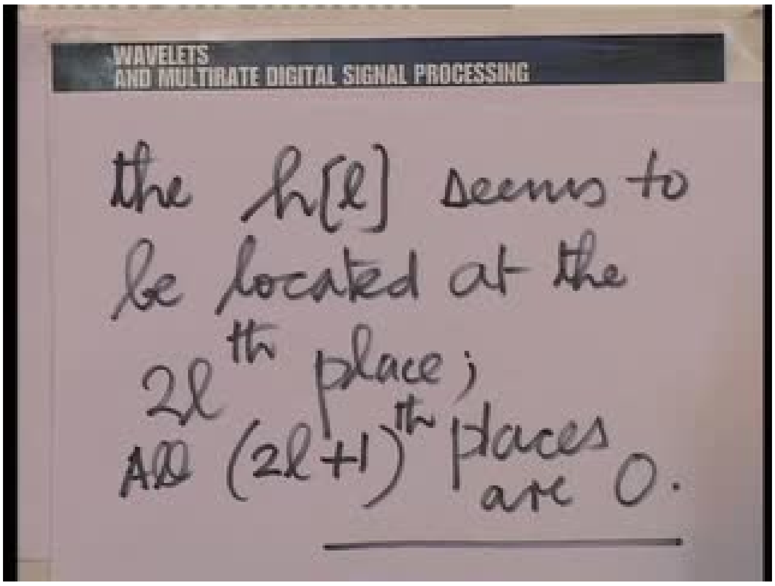

So, you are allowing only the even points in the impulse response to operate and the odd points have been analyzed this looks like a convolution but the h l seems to be located at the 2l th point.

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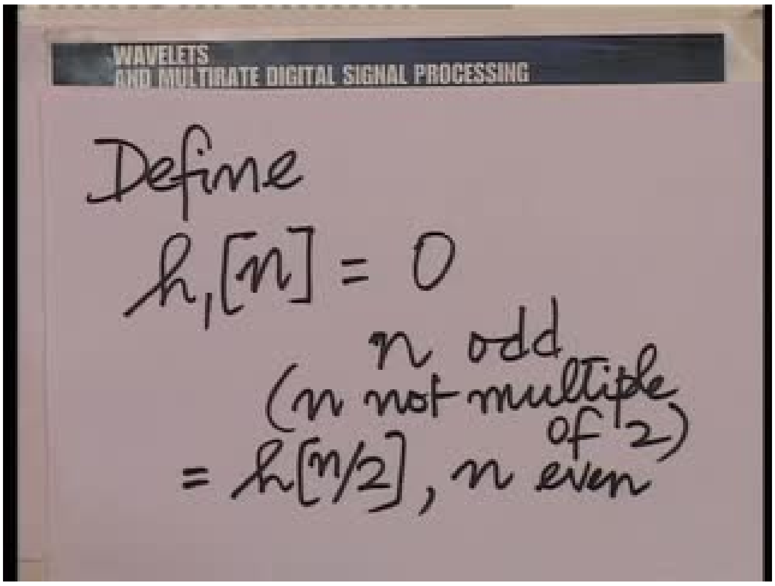

All other places so 2l plus 1 th places are 0; that is the effective h that we have here. It is as if this came from the 2l th place and at the 2l plus 1 th place we have all zeros. In other words, if we define h 1 n in such a way so h 1 n is equal to 0 when n is odd or in other words n not a multiple of 2 and equal to h n by 2 when n is a multiple of 2.

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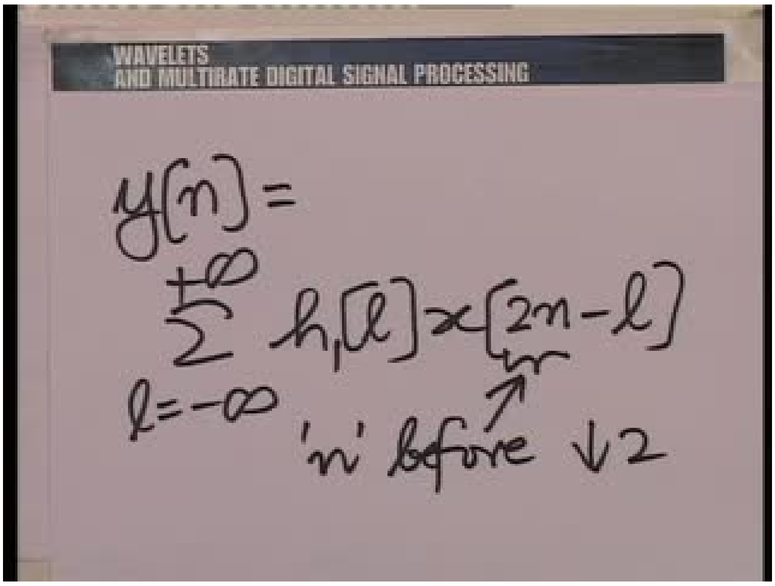

In that case, what we have here then is that y n is simply summation l going from minus to plus infinity h 1 l x 2n minus l. Once again, before the down sampler this is n so it is very clear what is happing here. The filter that operates before the down sampler is essentially 1 with the impulse response h 1.

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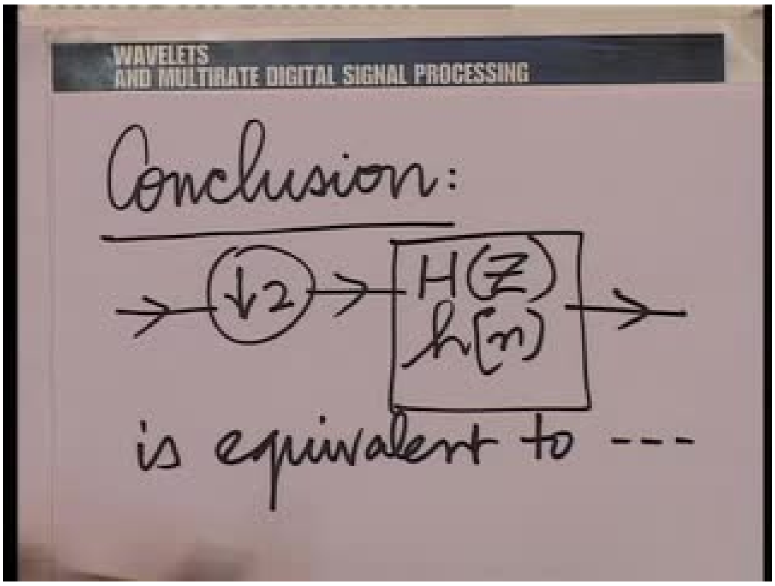

Essentially, the impulse response of the equivalent filter preceding the down sampler is the original impulse response, up sampled by a factor of 2. It is as simple as that; so conclusion is that a down sampler followed by a filter with system function h z or impulse response h n is equivalent to H of Z squared down sampled by 2 filtering by H of Z squared and the down sampling by 2. In other words, the impulse response of the filter preceding the down sampler is the original impulse response up sampled by 2.

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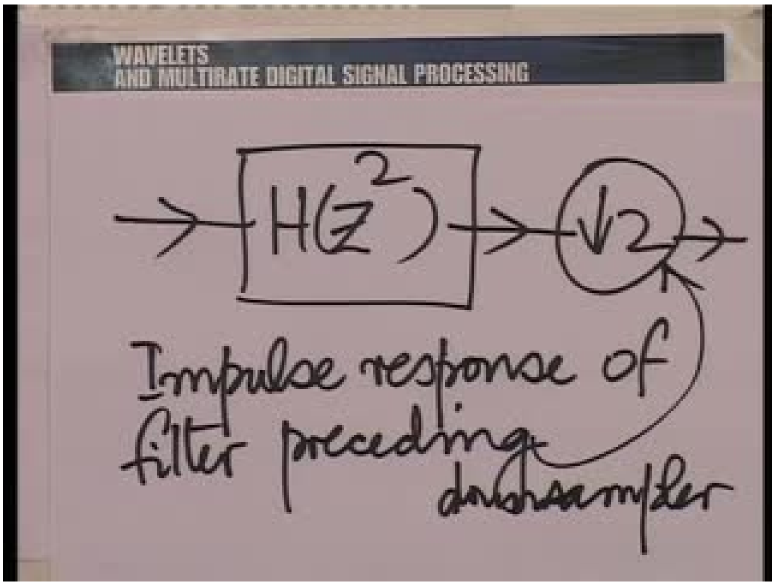

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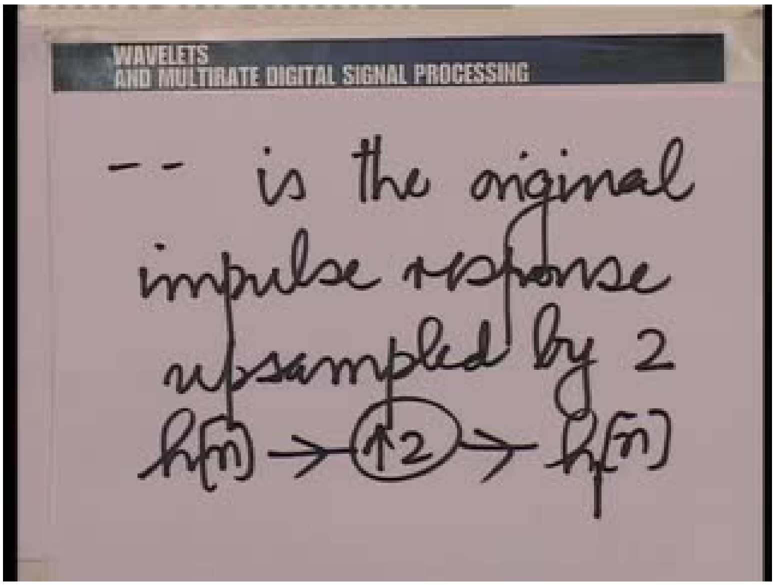

So, h n now this is what is called the noble identity for down sampling. In other words, whenever you have a filter transfer function where the oat odd samples of the impulse response are 0. The even samples could be non-zero; we could replace that filter followed by a down sampler by an equivalent operation where there is a down sampling first and then a filtering operation.

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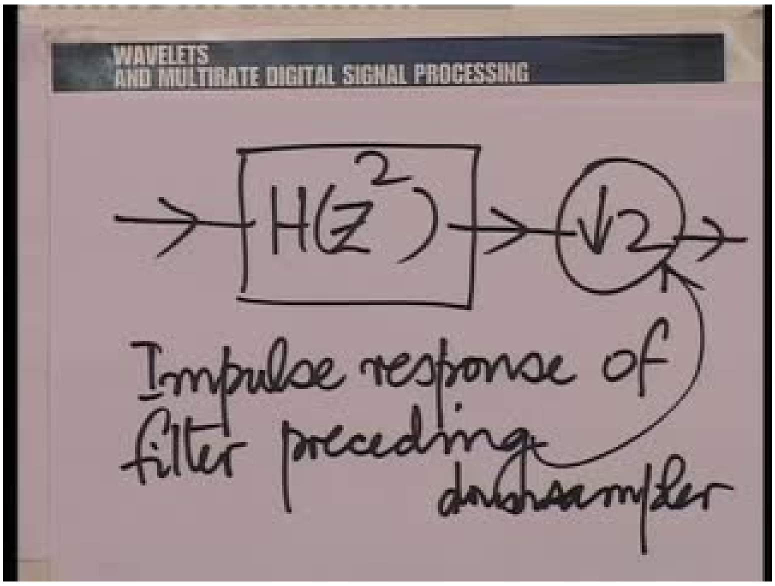

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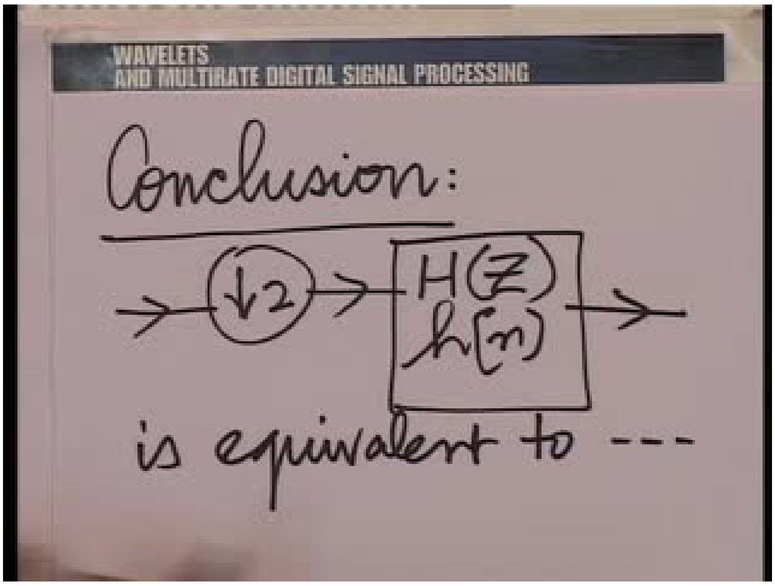

Now, comparatively what is the advantage or disadvantage of each of this? These structures we must ask that question; why should we want, perhaps, to look at this structure as supposed to this?

There are 2 reasons; one is purely analytical. So, you may want to get equivalent structures and today we would like to do that when we discuss the wave packet transfer. But, the other reason is also computational; for example, if you at this structure carefully what you doing is, first killing or removing the sample which are not going to be involved in the computation in down sampling and then you are doing a filtering operation.

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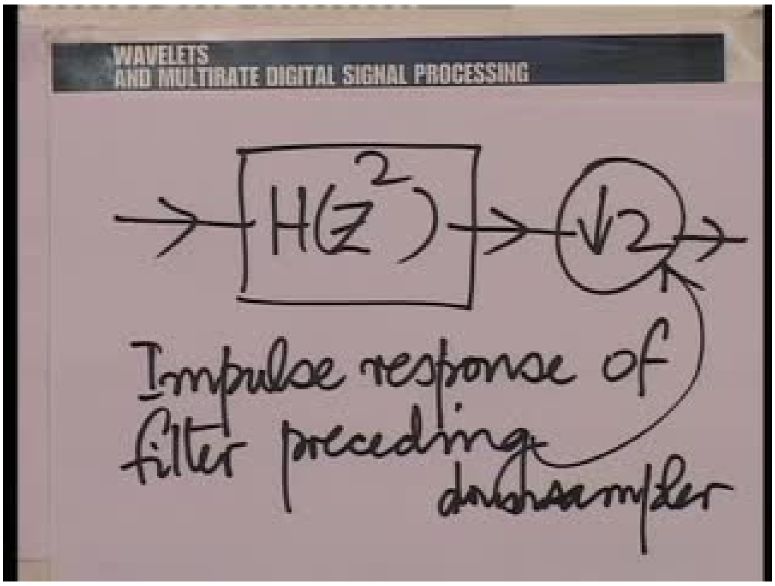

On the other hand, what you are doing in this structure here is first filtering that means, for example, before down sampling you are going to generate many samples here which are ultimately just going to be thrown away. To generate samples at this point you are going to do a lot of work because, you are going to do convolution operation here so this is an inefficient structure.

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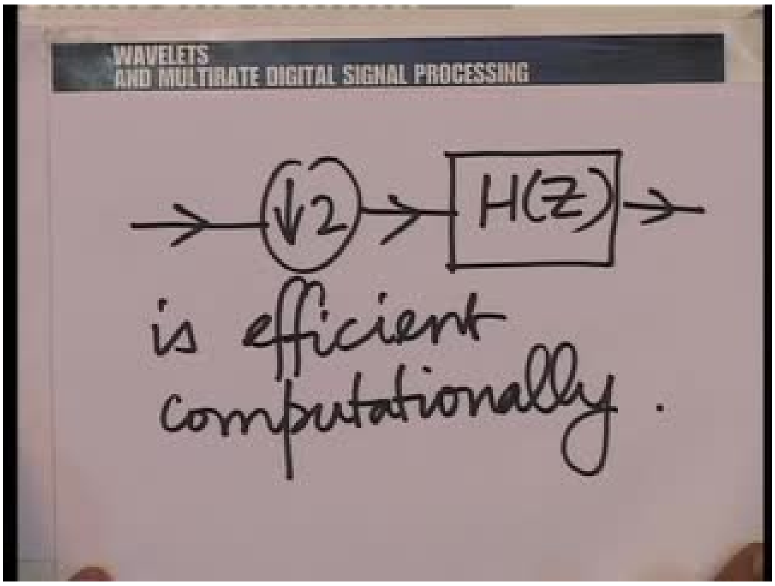

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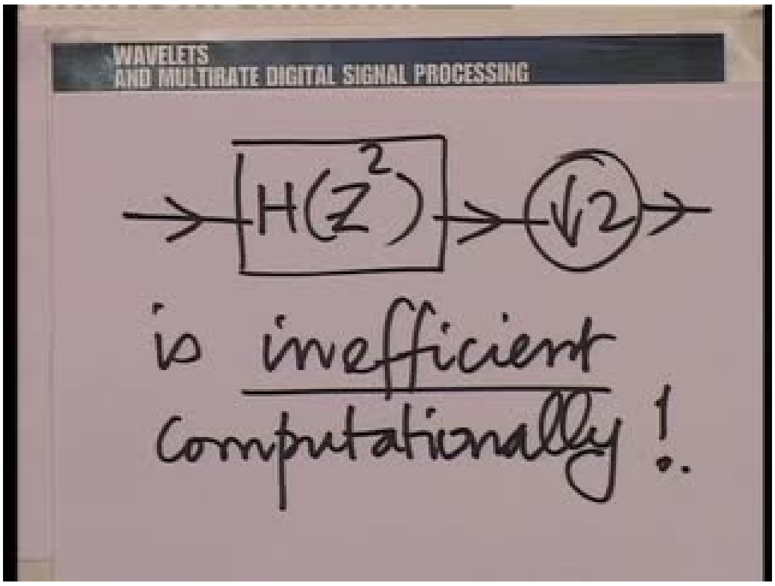

Whereas, this structure is the efficient version of this inefficient structure we should make a note of that. That is the a very important conclusion give we have drawn this structure is efficient computationally that is because we are doing away with samples that do not figure in the computational and this structure is inefficient computationally.

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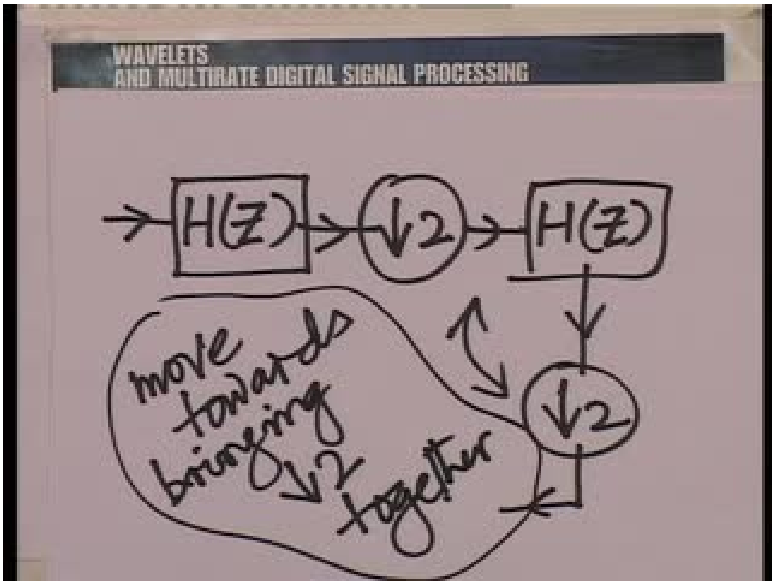

However, when it comes to analysis which is what we need to do. Today, to build the notion of the wave packet transform is going to be the other way around. We are going to go from what seems to be an inefficient structure to an equivalent. So, for example, what we are going to today is something like this we are going to go from a cascade like this so you would have say an H Z followed by a down sampler and another H Z followed by an another down sampler and so on and they we are going to try and interchange. So, we are going to bring this filter here; we are going to put the down sampler past and that means, we are going to move towards bringing the down samplers together. This is going to help us build an equivalent structure when there is a cascade. So, you know either motion - the motion from efficient, so called efficient to so called inefficient or the motion from so called inefficient to the so called efficient structures, are important in the contest of multirate digital signal processing. Now, before I proceed to go to the precise application of this replacement, let me first address the next noble identity and that is the noble identity of up sampling.

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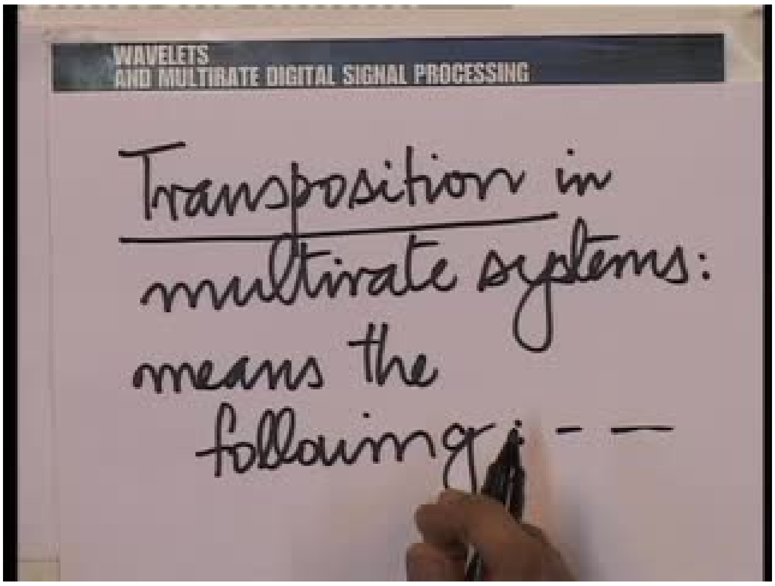

Now, you know one principle that is very useful in many multirate systems and in fact, whenever we wish to build examples and of structures of in multirate systems is what is called the principle of transposition. The word transpose has been used frequently in discrete time signal processing. You will recall that when we did not have up samplers and down samplers, when we had essentially old linear shift in variant operations. If, you had a signal flow graph in which they were only the traditional linear shift in variant signal flow operations mainly namely, multiplication by a constant summing up of several branches, delay by multiples of the sampling time and so on; then, we could define what is called the transpose of a signal flow graph.

The transpose of a signal flow graph is obtained by reversing all arrows in the signal flow graph but keeping the multipliers as they are. When you reverse all arrows what is the branching point could become a summation; what is the summation point can become a branching point. So, the idea of transposition can also be extended to when there are sampling rate; operation sampling rate change operations like down samplers and up samplers.

The only difference is that when we reverse the arrows we was must also reverse the nature of the sampling rate change operation. So, if you reverse the down sampler you must get an up sampler of the same factor with this change. We define the notion of the transposition to get us one more identity when there is 1 in multirate systems. So, transposition in multirate systems means the following operations:

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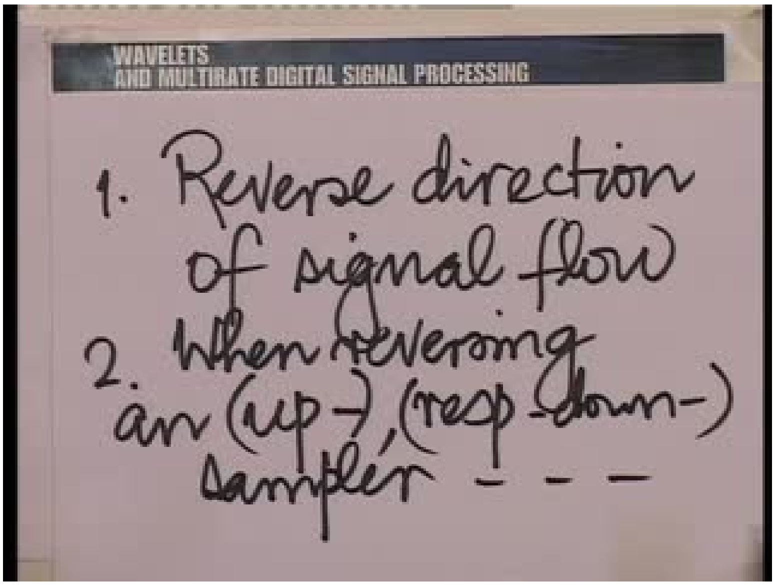

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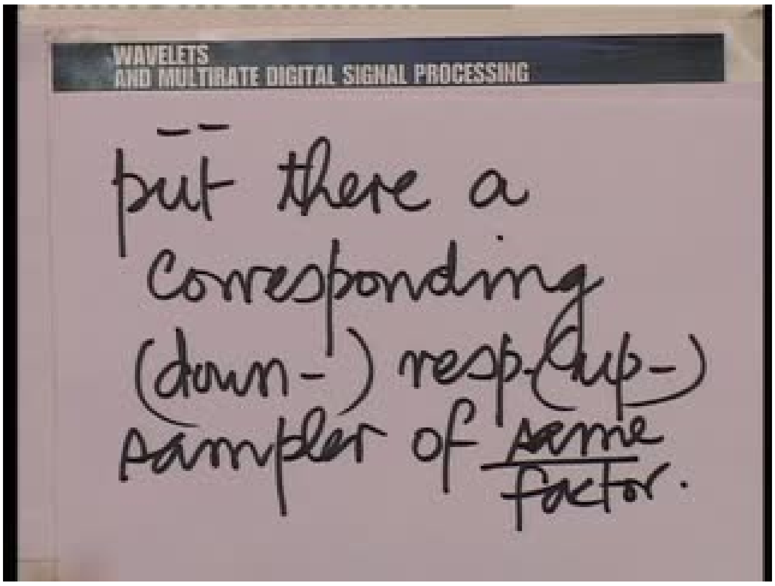

It means, first reverse the direction of signal flow; second, when reversing an up respectively down sampler, I will give a little bit of the caution has to be used, put there a corresponding down or respectively up sampler of the sampler of the same factor; that is, important same factor.

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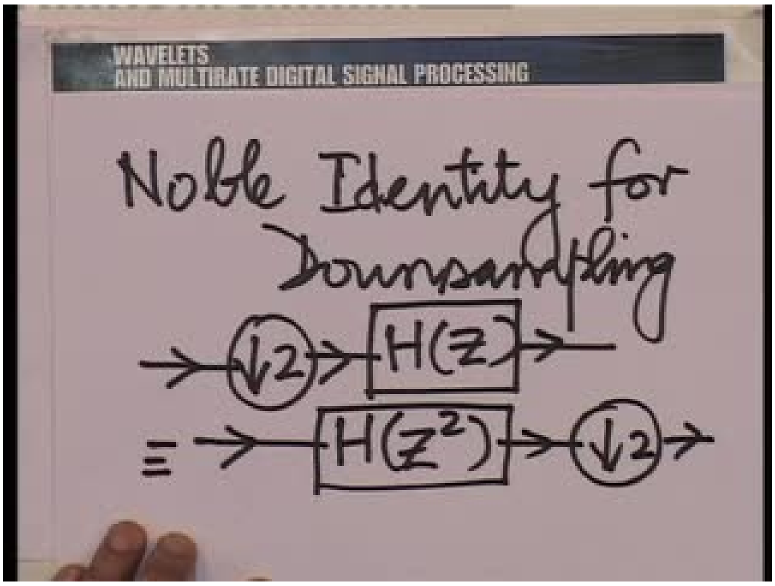

Now, what we are going to do is to use this principle of transposition to arrive at the noble identity for up sampling and of course, then we can prove it. So, let us put down the noble identity for the down sampling once again. Essentially, it says down sample followed by H Z is equivalent to H Z squared followed by down sampling. Let us use the principle of transposition. Transposition would mean that you reverse all the arrows here. Now, when you reverse the arrow here, you are going to start by reversing this you know. So, when you reverse and go across the down sampler in the other direction, you must put there in an up sampler of the same factor. According to the rule of transposition, this is as it is, so this is the discontinuous to be a H Z squared. Again, when you go past and reverse here you have H Z as it is but, then here you must put an up sampler.

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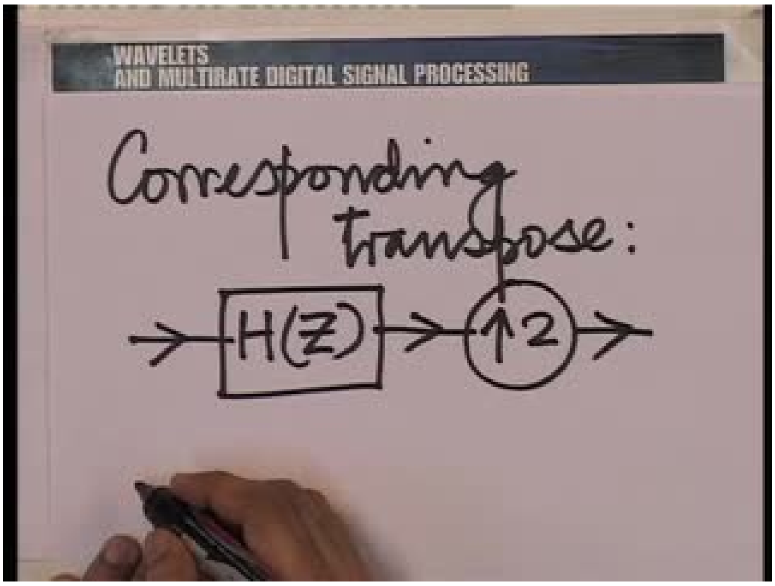

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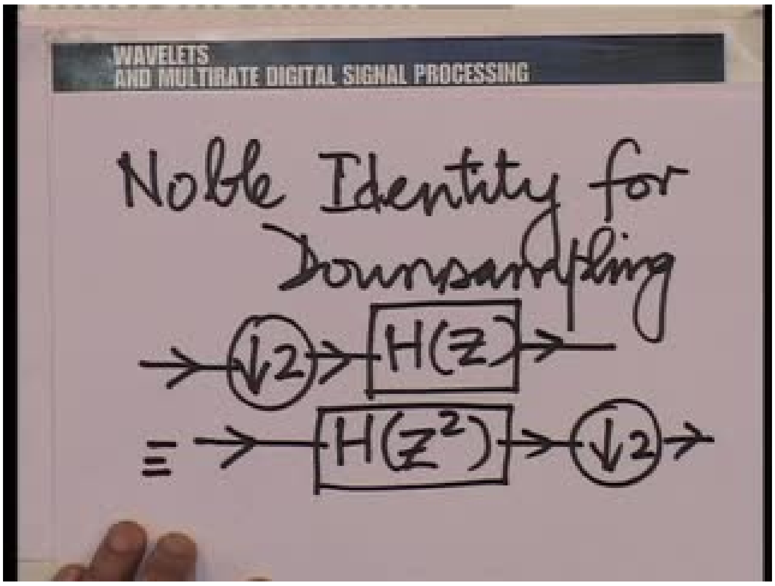

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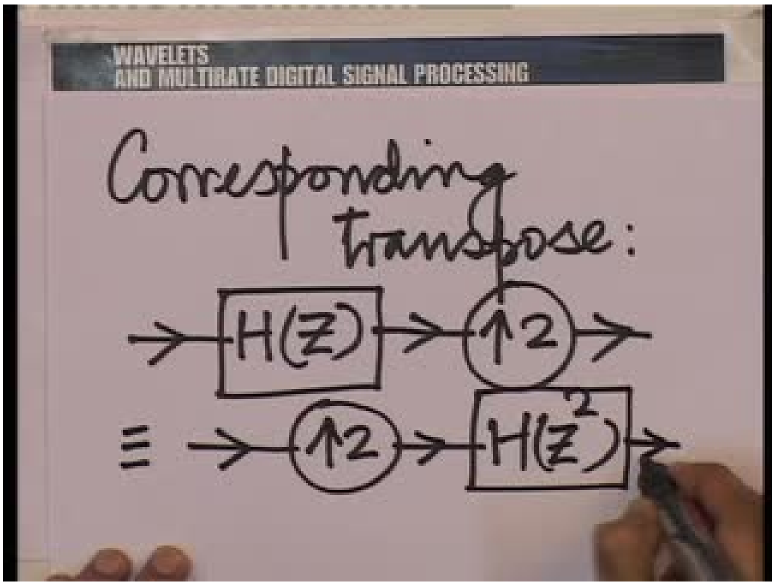

In the place of the down sampler of factor two such so, it has drawn the corresponding transpose. The transpose would therefore, be this; look at this; this is the transpose of this H Z first and then up sample by 2; it is equivalent to up sample by 2 and then an H Z squared (Refer Slide Time: 31:47).

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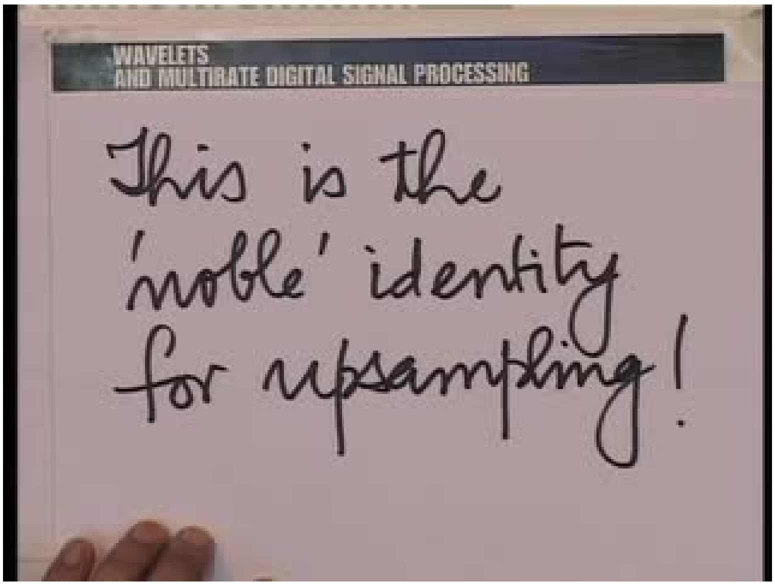

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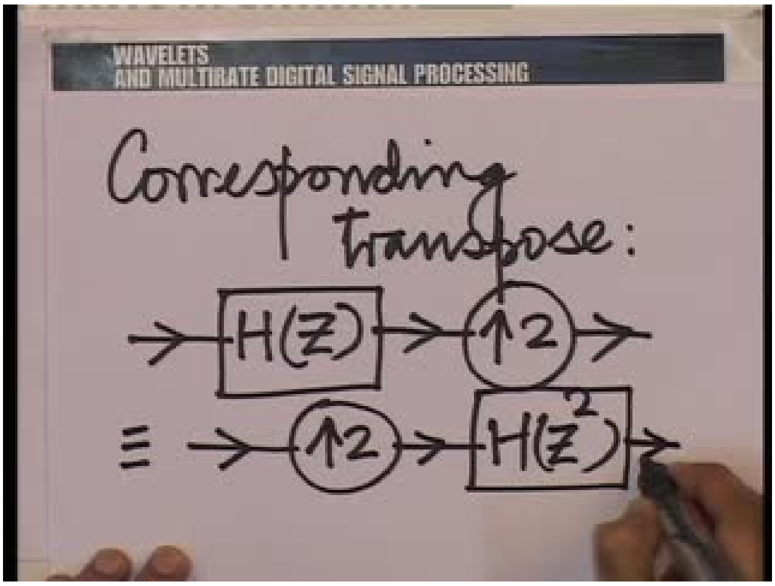

This is the transpose of this (Refer Slide Time: 32:17); up sample by 2 and the other way and then H Z squared and this is essentially, the noble identity for up sampling. So, what is the noble identity for up sampling? Say that if you filter followed by an up sampling by 2, it is the same thing as up sampling by 2 and then filtering by a filter whose impulse response is the up sample version of the impulse response of this filter. Of course, in both cases there is an up sampling involved.

The other way of looking at it is, going from here to here (Refer Slide Time: 33:17),there is a down sampling and going from here to here there is an up sampling. So, you know there are different ways of remerges; remembering this now, what I proved here or rather what I have written down here is, essentially a noble identity for down sampling and up sampling by sample factor sampling rate changes of factor 2.

We can generalize in any very straight forward way, to any other sampling rate change factor. In fact, let me write down in general the noble identities for any up sampling or down sampling factors. Of course, integer factor and in fact, I leave as an exercise for you to do 3 things 1 is just as we proof proved the noble identity for down sampling as by a factor of 2. I ask you as a first exercise to prove the noble identity for up sampling by a factor of 2. Subsequently, in the next exercises I leave it to you to prove the moral; more general versions of the noble identities for down sampling and up sampling. So, let me write down these exercises and leave you to do them and attempt them later.

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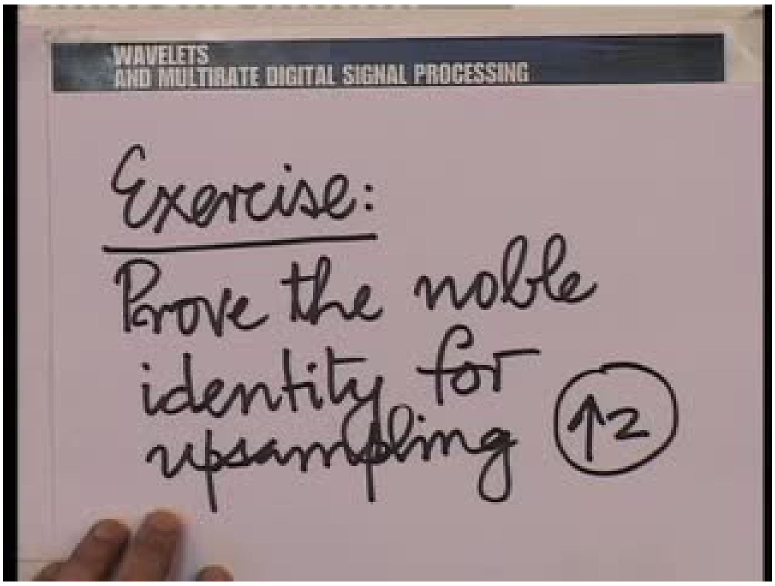

So, exercise number 1: I explain the exercises and I will leave it to you to do that them prove.

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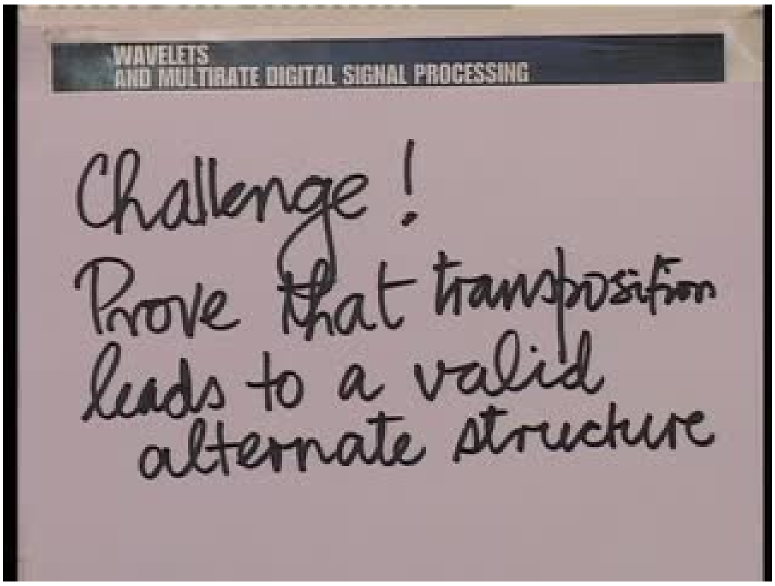

Here, I am talking about up sampling by a factor of 2. Please remember, when I talked about transposition that is not a proof. That is more like a pneumonic and aid to the memory. That was not a proof; if you wish to treat that as a proof then, we must prove in general that transposition leads to a valid structure. In fact, let me put that before you as a challenge. So, challenge - prove that transposition leads to a valid alternate structure; a slightly abstract thing to prove. For example, what we are saying is transposition gave us the ability to prove a new theorem from a previous one here. Now, to show that this is in general true, to show that when we transpose a certain structure or when we transpose 2 equivalent structures we are getting another pair of equivalent structures, this is the challenge. A slightly difficult job, anyway, that is a challenge. That is not what I would call a traditional tutorial exercise but, I do like to put some challenges before the class. Anyway, let me put the second exercise before you.

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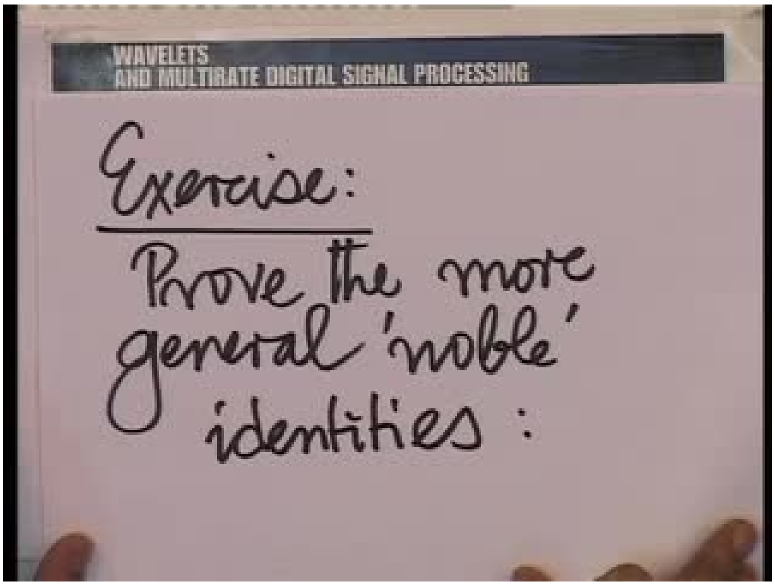

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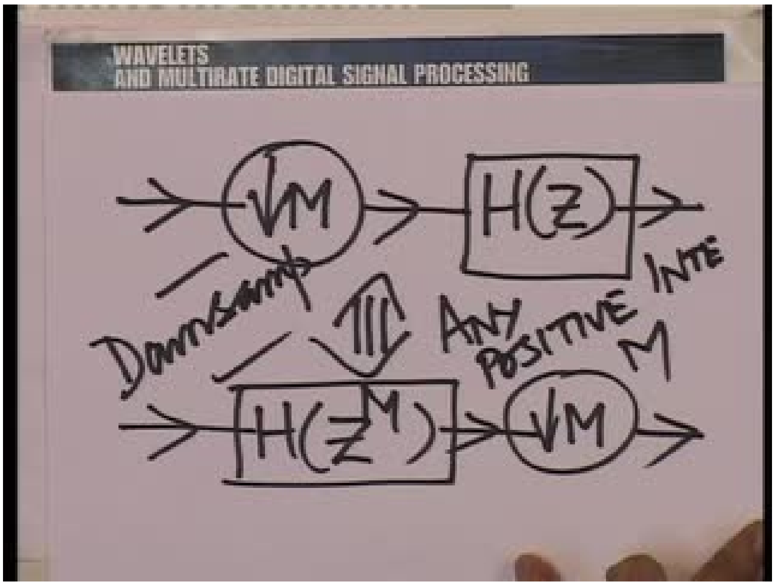

Anyway let we put the second exercise; before the second exercise is proved, the more general noble identities down sampled by M followed by H Z is equivalent to filter by H Z raise to the M followed by down sample by M for any positive integer M.

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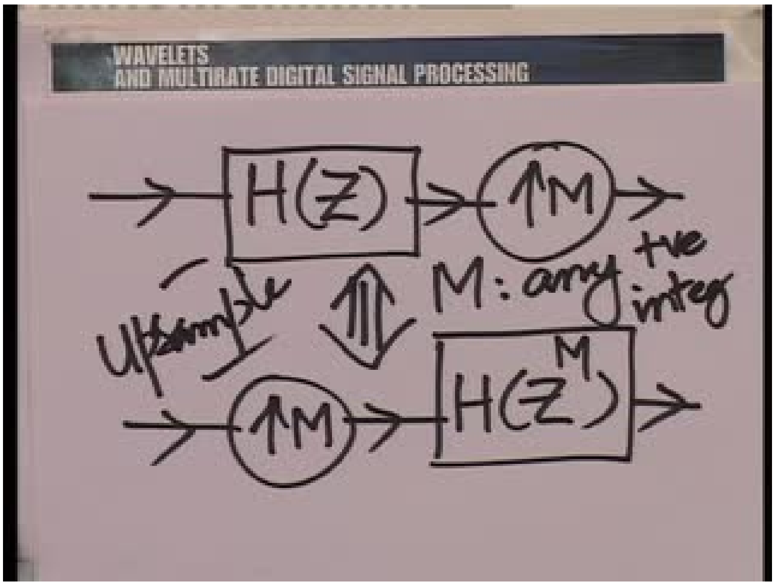

This is for down sampling and correspondingly for up sampling we have M for any positive integer and this is for up sampler. Anyway, with that little background on the noble identities, let us apply the noble identities now to the context of the haar multi resolution analysis. Our objectives was to go towards the wave packets transform in the context of the haar MRI. So, let us take 2 iterations. You know actually the wave packet transformation makes sense when you go down at least by 2 steps in the ladder of the multi resolution analysis. So, let us go down to 2 steps; so let us take the haar MRI.

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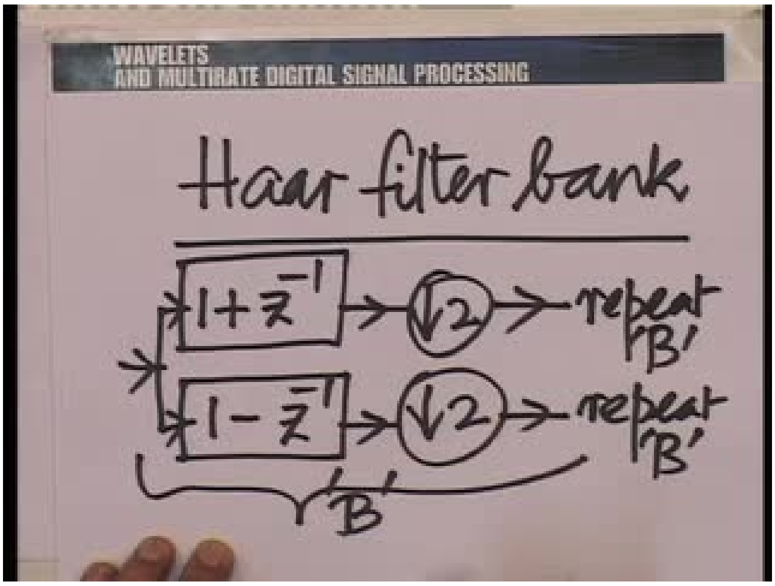

We have 1 plus Z inverse, forget about the factors of half inverse and so on. Let us just focus on the basic filter system function 1 plus Z inverse and 1 minus Z inverse. This is the analysis filter bank for the haar. Now, hold this whole thing in the bank B and what we do in the wave packet transform in haar would be to repeat B here and repeat B there.

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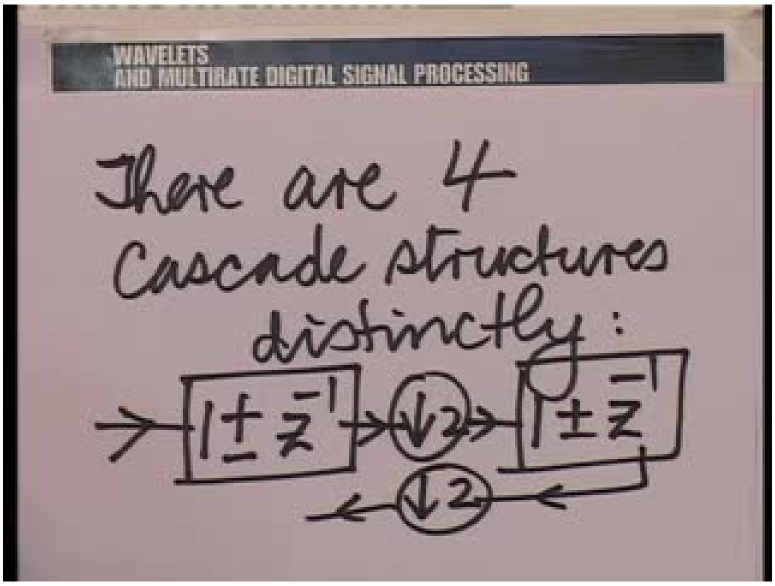

In other words, we would have 4 structures of cascade. You know, visualize it; you have this and then you have this whole structure being repeated here. So, you have 2 structures of cascade emerging from this this cascade with this and this cascade with this when you repeat this whole structure here (Refer Slide Time: 41:20); you have this cascaded with this and this cascaded with this. So, there are 4 structures and let us draw those 4 structures of cascade; let us take just 1 of them first to make matter simple.

So let us make a remark; there are 4 cascade structures distinctly. In fact, we can capture them in one expression; if we could write 1 plus or minus Z inverse followed by down sample by 2 followed by 1 plus or minus z inverse, followed by down sampled by 2 again.

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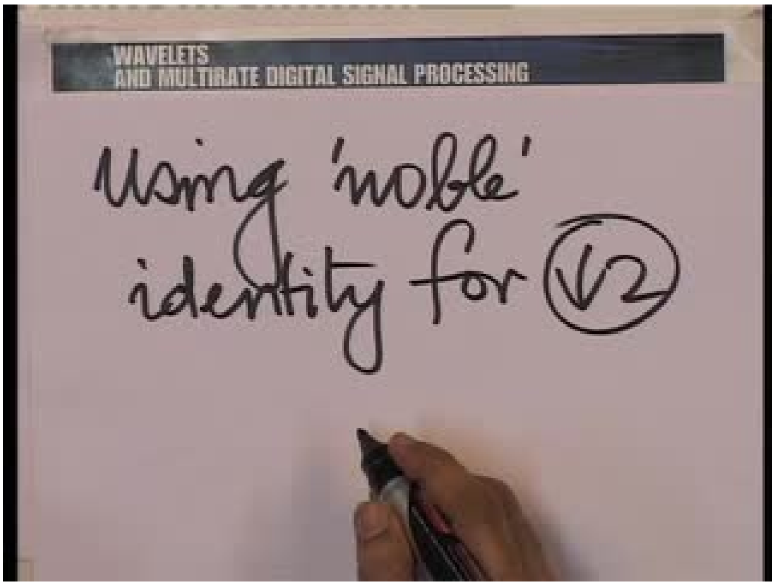

Now, it is very easy to use the noble identities to replace this. So, as I said here there is 1 example where I would like to interchange this pair here. So, I would like to bring this filter in place of the down sampler and the down sampler in place of the filter. I can put the down samplers together using the noble identity.

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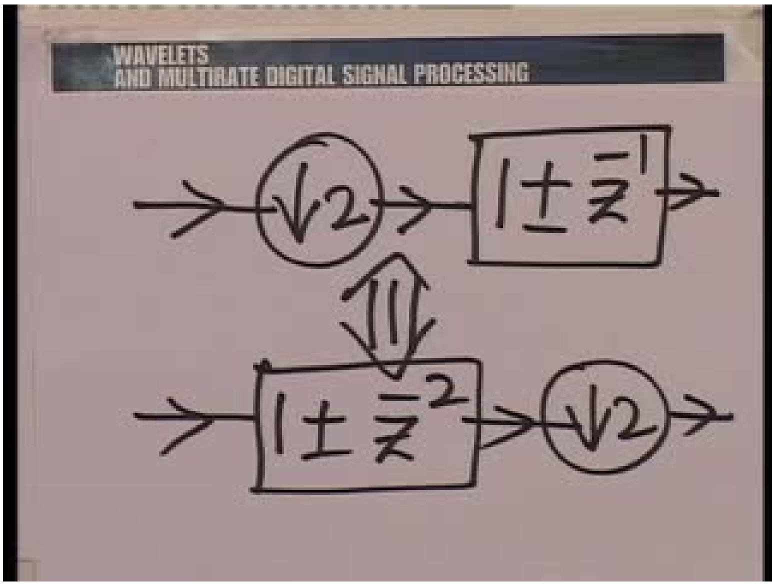

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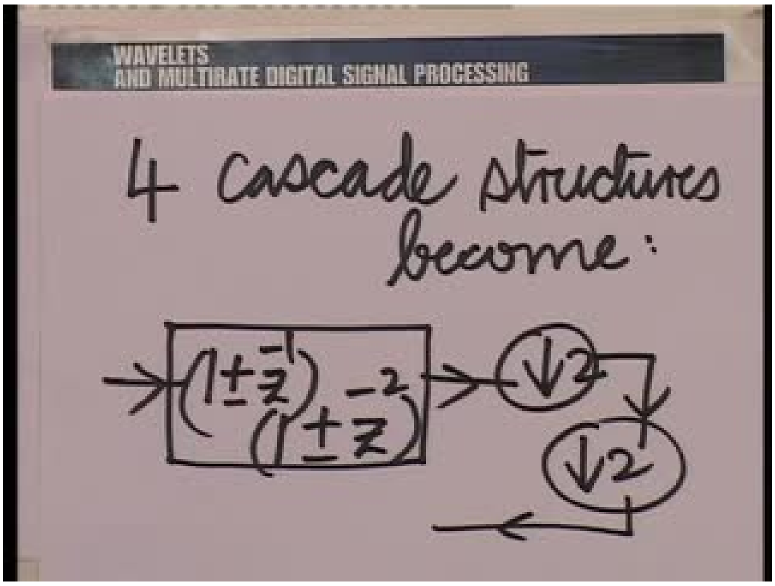

For down sampling, I would have down sample followed by 1 plus or minus Z inverse is equivalent to 1 plus or minus Z inverse, followed by down sampling by 2 Z to the power minus 2. Remember, here instead of Z to the power minus 1 there and therefore the 4 distinct cascade structure become as follows. Essentially, 1 plus or minus Z inverse times 1 plus or minus Z to the power min us 2 followed by down sampled by 2 and down sampled by 2 again and down sampled by 2 again. If you down sampled by 2 twice, it is equivalent to the down sampling by 4. That is easy to say; when you down sample by 2 the first time, you are taking every second sample and putting it in half of the location;

when you down sample by 2 the second time, you are again taking every second sample and putting it back in half the location place. So, in effect, you are taking every fourth sample and putting it in one fourth the place of location.

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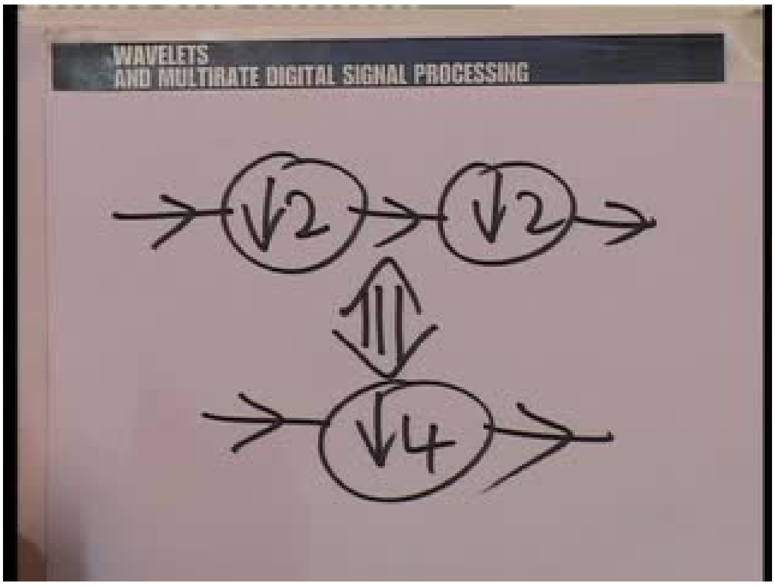

So, what we are saying in effective is down sample by 2 followed by down sample by 2 is equivalent to down sample by 4 and therefore, we have 4 filters that come of out of the this process.

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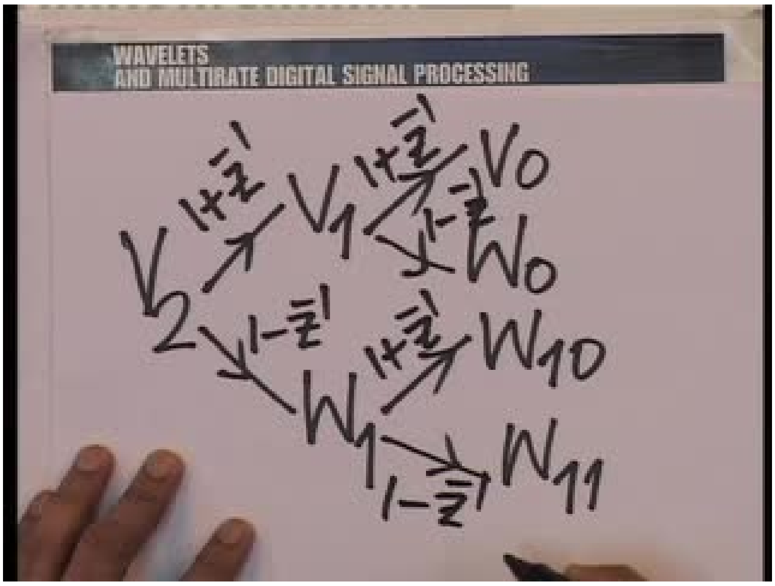

In other words, what we are saying is, when we consider this movement from V 2, you know what we are doing. If you remember, in the unless analysis process is to decompose V 2 into V 1 and W 1 and then followed by a decomposition of V 1 into V 0 and W 0.

Now, W 1 is also being decomposed here into what we might call W 1 0 and W 1 1 and that is the beauty of the wave packet transform; this is special to the wave packet transform. So, in fact, you know notionally we can show the filters also here. When we operate the filter 1 plus Z inverse we are going from V 2 to V 1 and when we operate the 1 minus Z inverse V go from V 2 to W 1.

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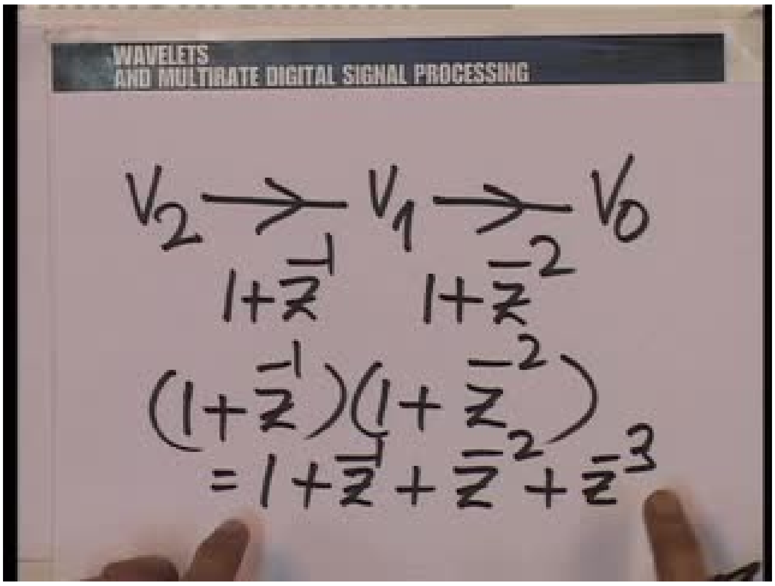

Here again, it is and 1 plus Z inverse, which takes as us to V 0 and 1 minus Z inverse as that takes as us to W 0. Here again, its 1 plus Z inverse 1 that takes us to W 1 0 and 1 minus Z inverse, which takes as us to W 1 1. Therefore, let us take this these 4 filters; let us write down these 4 filters systematically. So, 1 plus Z inverse - V 2 to V 1 to V 0 will come from 1 plus Z inverse there and 1 plus Z to the power minus 2 there. So, we have 1 plus Z inverse into 1 plus Z to the power minus 2 which is 1 plus Z inverse plus Z to the power minus 2 plus Z to the power minus 3.

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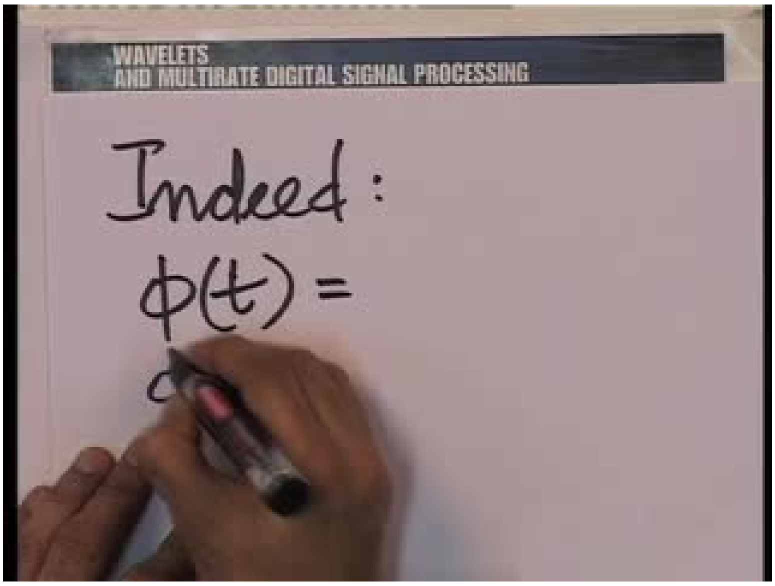

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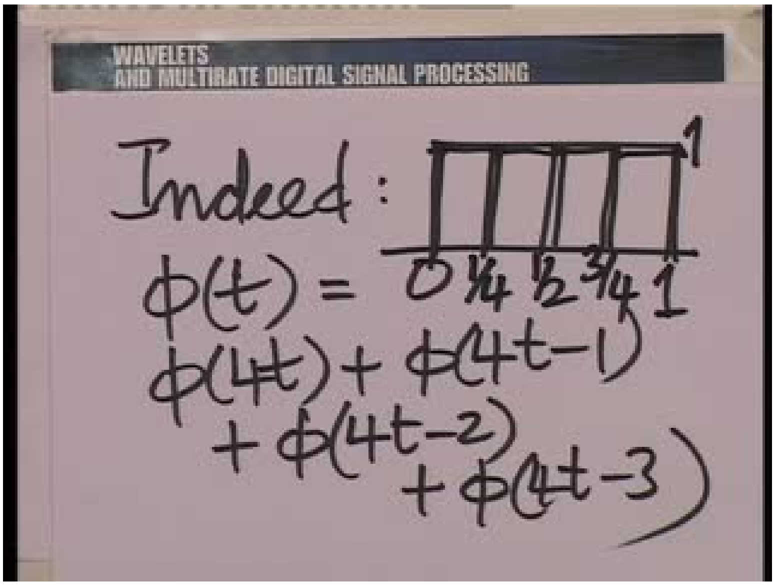

In fact, this sequence tells us how to expand an element of the basis here in terms of the basis here and as you can see this corresponds to the sequence 1 1 1 1 at the points 0 1 2 and 3. So, we are saying the sequence 1 1 1 1 expands phi t a typical basis element in V 0 in terms of phi 4t minus k phi 4t n and it translates you know, when you go from four V 0 to V 2 when you are going to dilate by a factor of 4; as what is meant by a down sampling by 4 and indeed, it is quite clear indeed.

phi t is phi 4t plus phi 4t minus 1 plus phi 4t minus 2 plus phi 4t minus 3 and that is very easy to see. That is, essentially saying that this is, this plus this plus this plus this; here you have 0 to 1 1, there one fourth half three fourth there and 1. Now, once we have done this for 1 of these filters and cascades very easy to do them for the others and that in fact gives us basis for each of these spaces W 1 W 1 0 W 0 1 and W 0.

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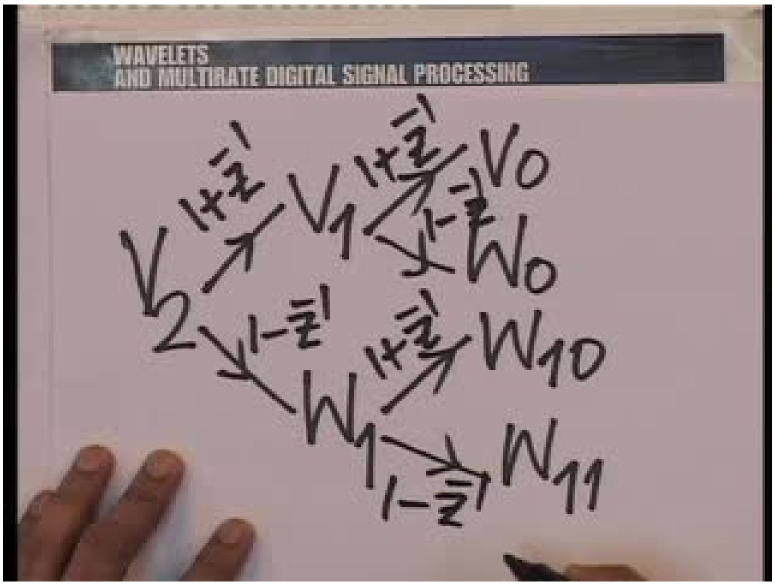

Correspondingly, if you go back to the structures here you realize that to come from V 2 to W 0 you should do 1 plus Z inverse and then followed by a 1 minus Z to the power minus 2 and to go from V 2 to W 1 0 you should do a 1 minus Z inverse followed by a 1 plus Z raised to the power minus 2. And, to go from V 2 to W 1 1 you must do a 1 minus Z inverse followed by a 1 minus Z raised to the power minus 2; so let us do these and complete the job.

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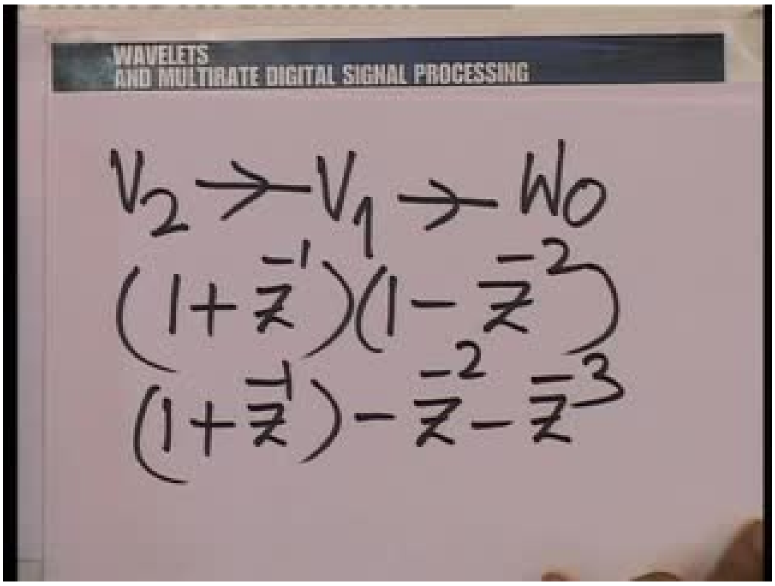

So, what we are saying is V 2 to V 1 to W 0 should be obtained.

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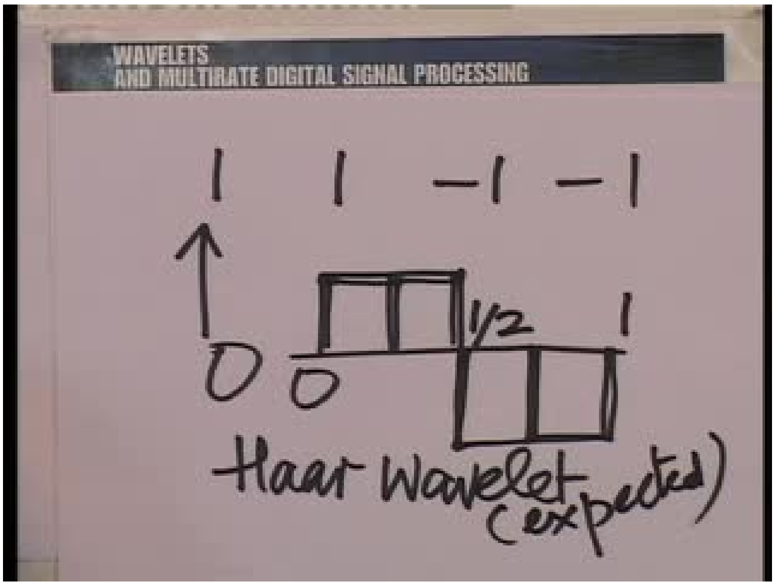

By 1 plus Z inverse followed by a 1 minus Z raised to the power minus 2. Essentially, a 1 plus Z inverse minus Z raise to the power minus 2 minus Z raise to the power minus 3 and back; that gives raise to the sequence 1 1 minus 1 minus 1 and the function 1 1 minus 1 minus 1 0 half 1, the haar wavelet as expected.

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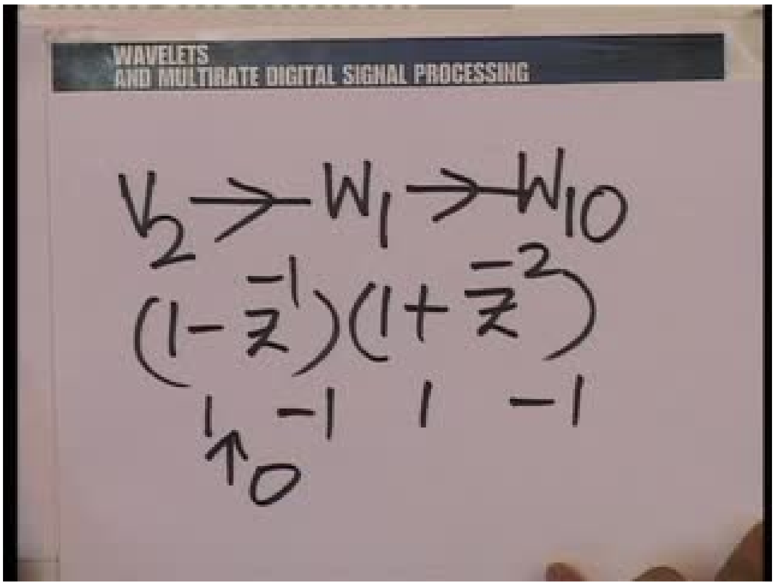

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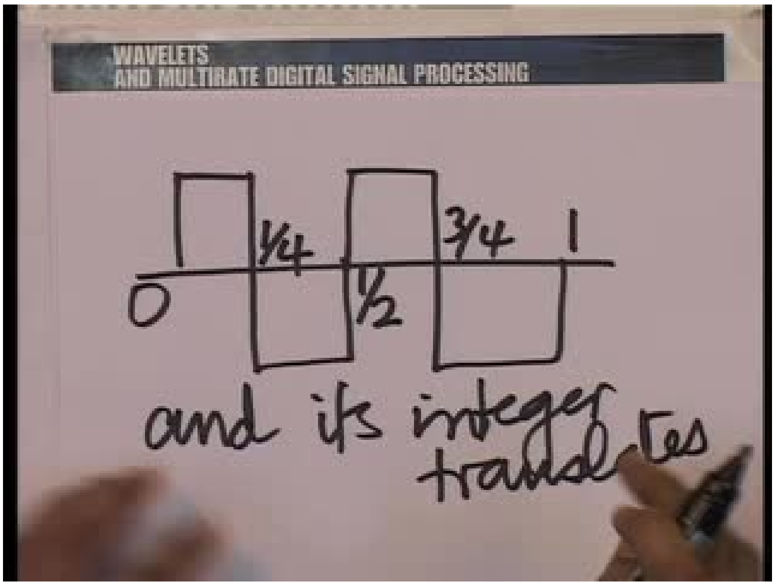

Similarly, we could complete the others. So, V 2 to W 1 to W 1 0 is essentially 1 minus Z inverse and 1 plus Z raise to the power minus 2; essentially, the sequence 1 minus 1 and again 1 minus 1 and correspondingly the function 1 minus 1 1 minus 1.

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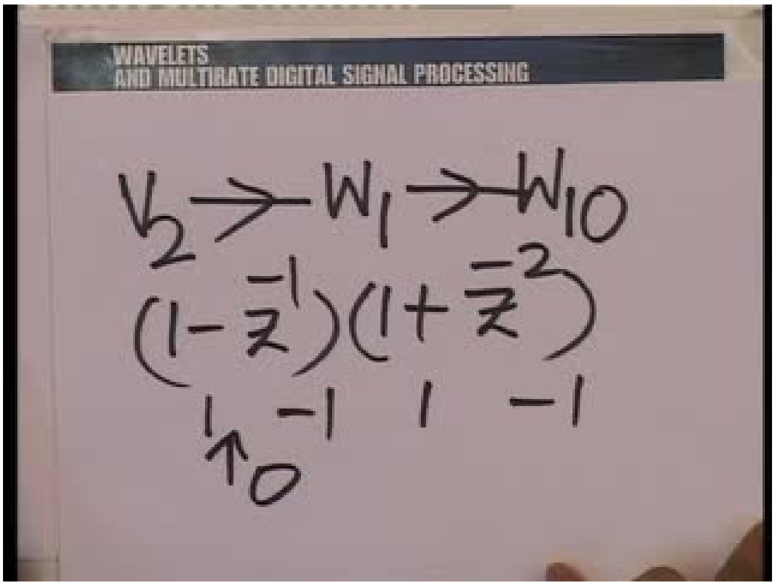

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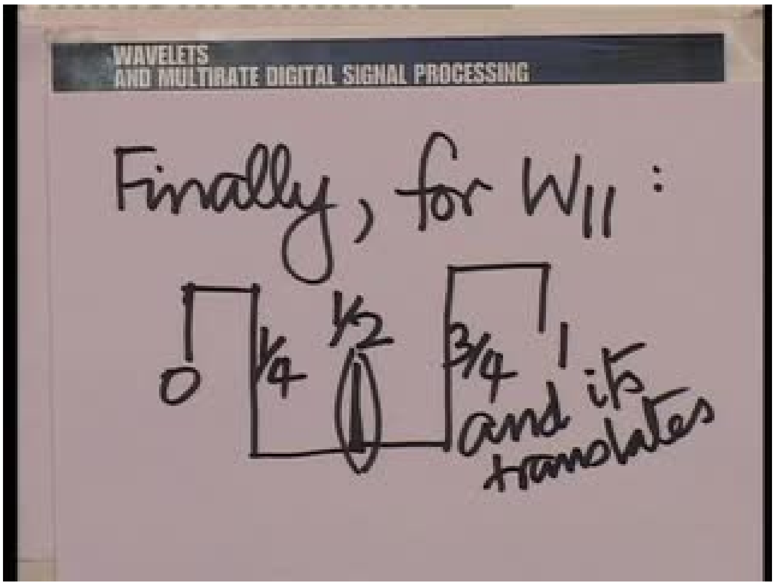

So, this function and its integer translates span W 1 0 and finally, I leave it to you to show for W 1 1; we would have 1 minus 1 minus 1 and 1. So, I do not really need this; this is just to exemplified 0 half 0 one fourth here, half there, three fourth here and 1 this function and its translates.

Therefore, we have now built the wave packet transform in the context of the haar multi resolution analysis. What we have done is to establish the 1 function whose integer translates span and each of this these sub spaces V 0 W 0 W 1 0 and W 1 1.

So, in fact, the property of a single function and its integer translates spanning spaces was also for the spaces immerging from the wave packet analysis with this. Then we illustrated one instance of the wave packet analysis and we shall go on to other uses of the noble identities in the next lecture. Thank you.