Advanced Digital Signal Processing – Wavelets and Mutlirate

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Model No # 01
Lecture No # 30
Building Piecewise Linear Scaling Function, Wavelet

A warm welcome to the thirtieth lecture on the subject of wavelets and multirate digital signal processing. We had began building the piecewise linear multi resolution analysis in the previous lecture that was a difficult task. Let us quickly recapitulate what we talked about last time.

We had seen that the piecewise linear function obtained by convolving the haar scaling function with itself, which we call phi 1 t, was not orthogonal to all its translates, I mean integer translates. The trouble maker translates were translation by 1 and minus 1 and this manifested in the sum of translated spectra not being a constant, but as expected, since it was only 1 and minus 1 which were trouble makers, the sum of translated spectra was kind of constant within two positive bounds, and as a consequence of that one could bring out or extract from that phi 1 t another piecewise linear function which was orthogonal to its own translates, and we were trying to use that function to build a multi resolution analysis, based on piecewise linear scaling functions and wavelets.

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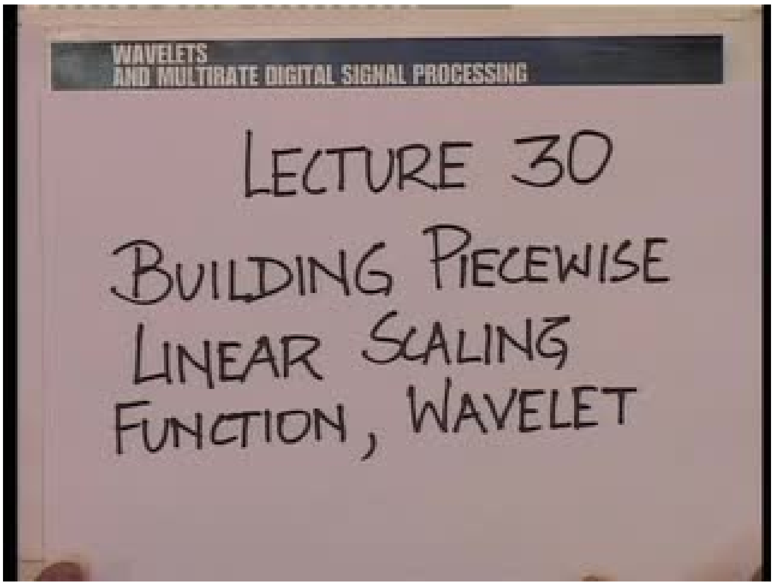

And, therefore today we take further in this lecture the construction of piecewise linear scaling functions and wavelets.

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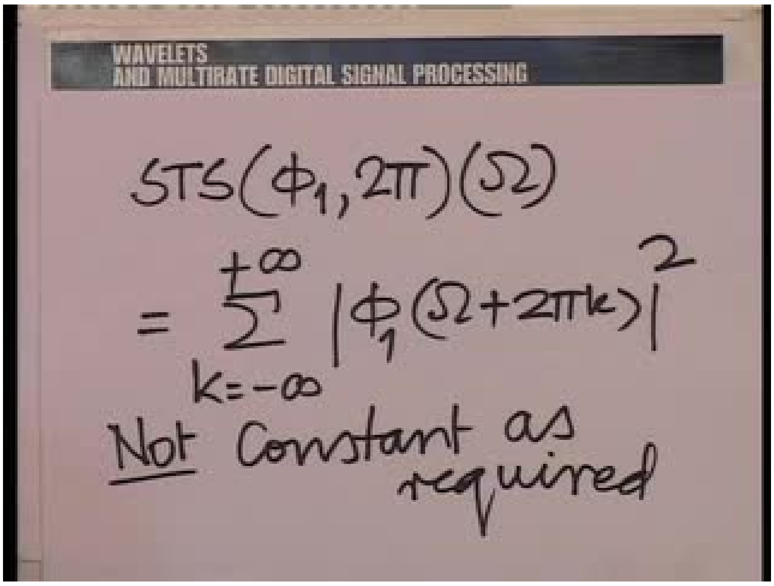

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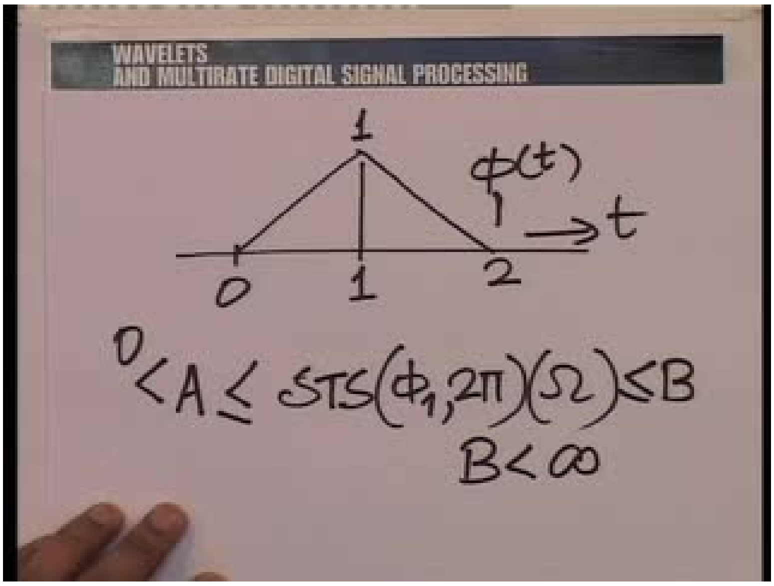

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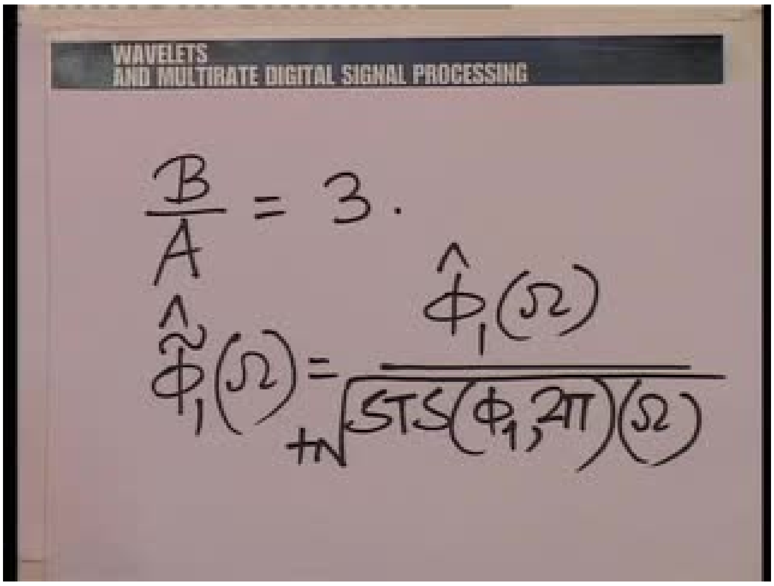

Well we need to put back a few ideas that we did the last time to bring continuity and those are as follows; the first this is we noted that we could construct the sum of translated spectra of phi 1 with translations of 2 phi, evaluated at all omega and this was given by summation k running from minus to plus infinity phi 1 omega plus 2 pi k mod squared, we saw that this sum of translated spectra was not a constant as required, but it lies between 2 positive bounds, so for phi 1 t looking like this.

The sum of translated spectra lies between 2 positive bounds, let us call them A, A is strictly greater than 0 and B, B is strictly less than infinity, and B by A was equal to 3. This is another way of looking at it, you know if you scale a function by a constant, the sum of translated spectra is scale by the square magnitude of that constant, so that does not affect this property and that is why I have just emphasis the ratio B by A, and the fact that both A and B are strictly positive quantity that is all periodically required. Anyway, what we said was that we could construct a phi 1 tilde. In terms of its Fourier transform, and phi 1 tilde cap has a function of omega was phi 1 cap omega, divided by the sum of translated spectra of phi 1 with a translation of 2 pi evaluate at omega and a positive square root take.

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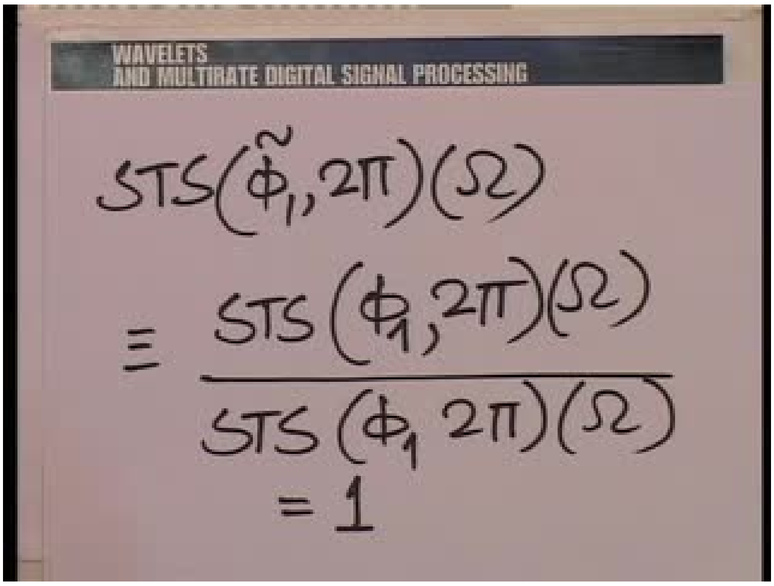

We noted that the sum of translated spectra of phi 1 tilde, in other words this quantity is a constant, in fact it is of the form sum of translated spectra of phi 1 with translations of 2 pi at omega divided by STS phi 1 2 pi at omega, which is in fact equal to 1, and this cancellation of numerator and denominator is acceptable, because a numerator and denominator each neither go to 0 nor infinity.

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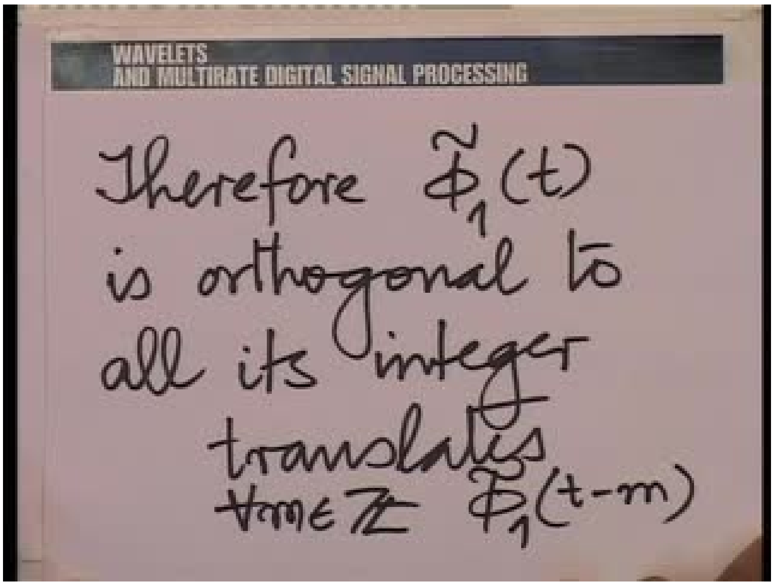

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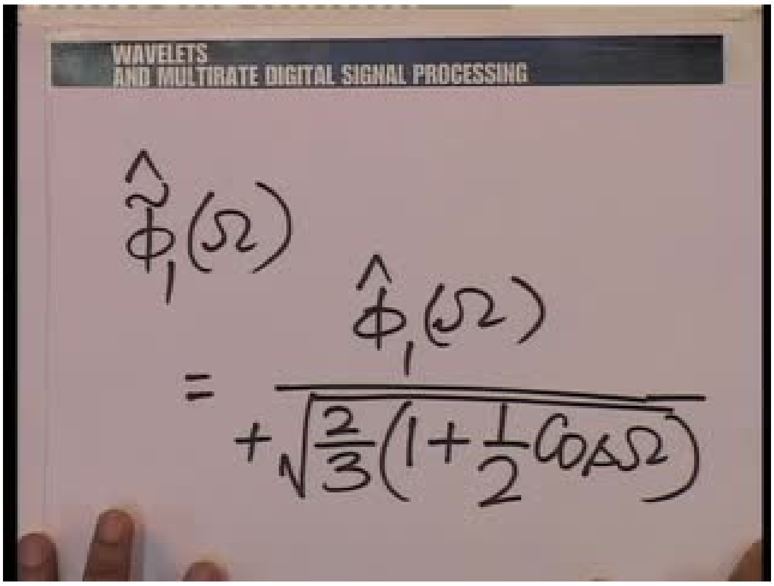

Based on this, we noted that phi 1 tilde is orthogonal to all its integer translates, in integer translates could be written as phi 1 tilde t minus m for all m integer. Now, we were trying to get the nature or a feel for the function phi 1 tilde t, and we have noted that phi 1 tilde cap omega is of the form phi 1 cap omega, divided by the positive square root of 2 by 3 times 1 plus half cos omega.

Now, we are also use the so called binomial expansion of the denominator, so we said that one could expand the denominator in a binomial series, a generalized binomial expansion, or one could also think of it as an expansion in a Taylor series. So, there are 2 ways of looking at it.

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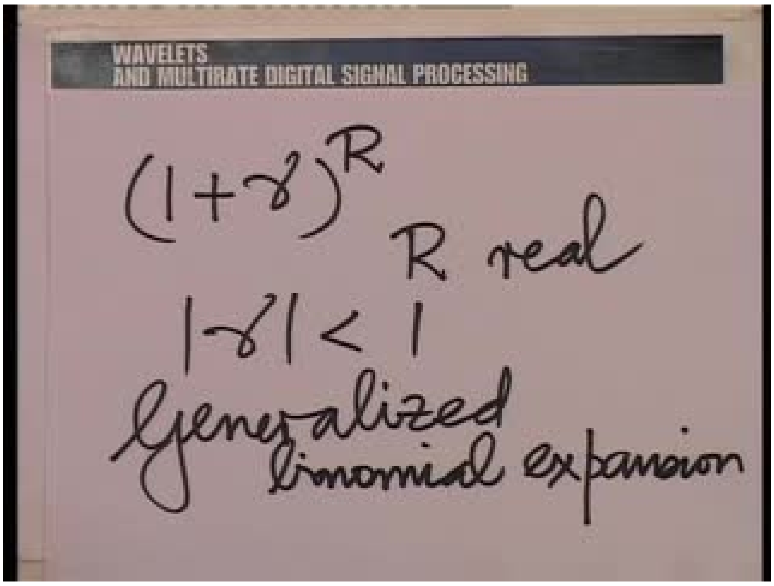

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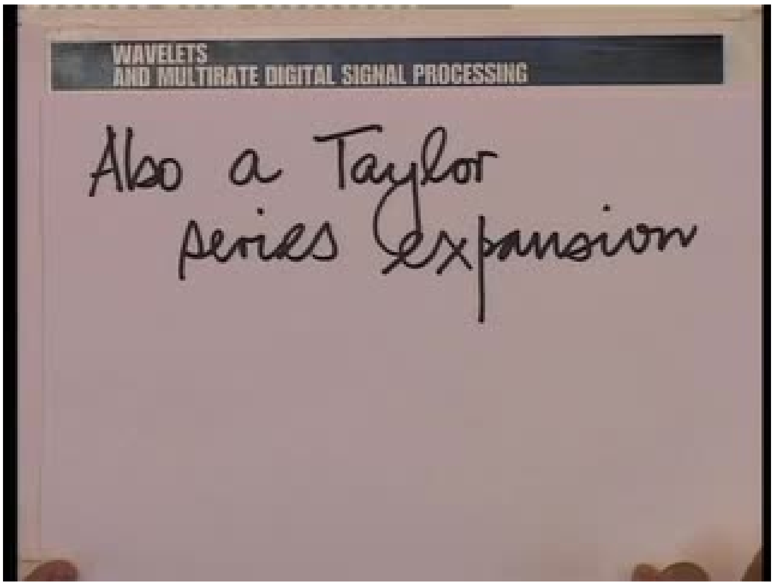

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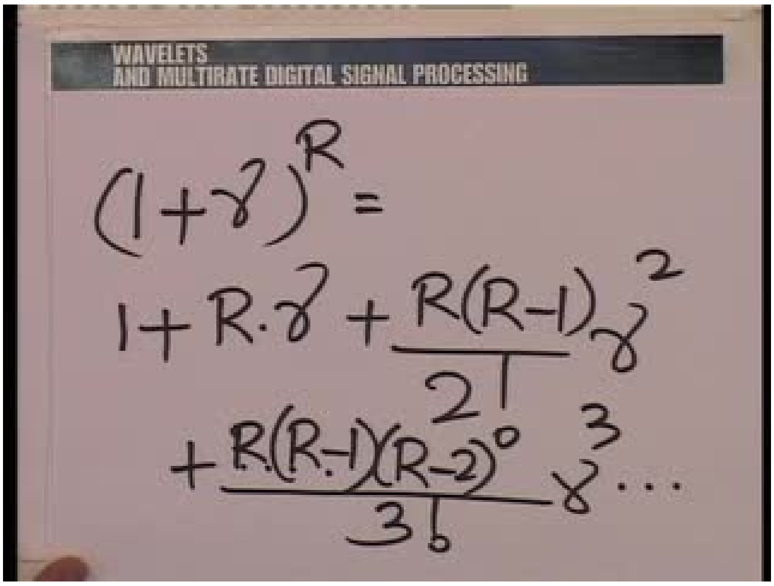

What we have noted was that if you had something like 1 plus gamma to the power R, R any real number, and mod gamma strictly less than 1. One could a generalized binomial expansion or one could also think of the same thing as a taylor series expansion, and essentially this expansion was of the form 1 plus gamma to the power R, is 1 plus R times gamma plus R into R minus 1 by 2 factorial times gamma squared plus and so on. So, of the next term for example would be R into R minus 1, into R minus 2 by 3 factorial gamma cubed and so on and so forth.

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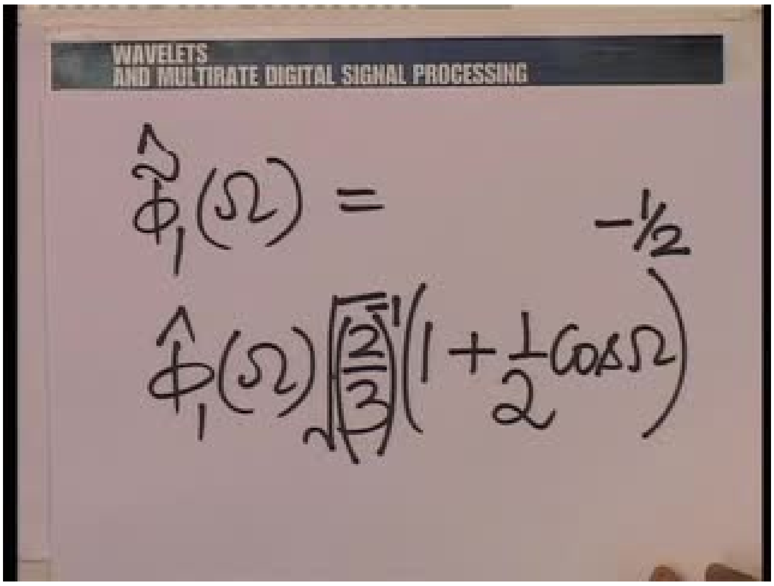

Now, one can view this, as you see to be a taylor series expansion in the variable gamma around the point gamma equal to 0 or one could also view it as I said, as a generalized binomial expansion, in whatever way one views it. What one is saying is that ultimately phi 1 tilde t, rather the Fourier transform of phi 1 tilde t can be written as the Fourier transform of phi 1 multiplied by essentially an expansion of this form.

So, you know this would be divided by, so you would have 2 by 3 inverse here, and 1 plus half cos omega there to the power minus half. Now, one could of course write down the binomial expansion here, I note so much interested in writing down the expansion in all its gory detail, as talking note of the fact that when we so expand, you would get essentially powers of cos omega here, so you would get cos omega, cos squared omega, cos cubed omega, cos the power 4 and so on, and ultimately cos squared omega or cos cubed omega of that matter any positive integer power of cos omega can be regarded again as a positive integer power of e raise the power j omega minus e raise the power minus j omega.

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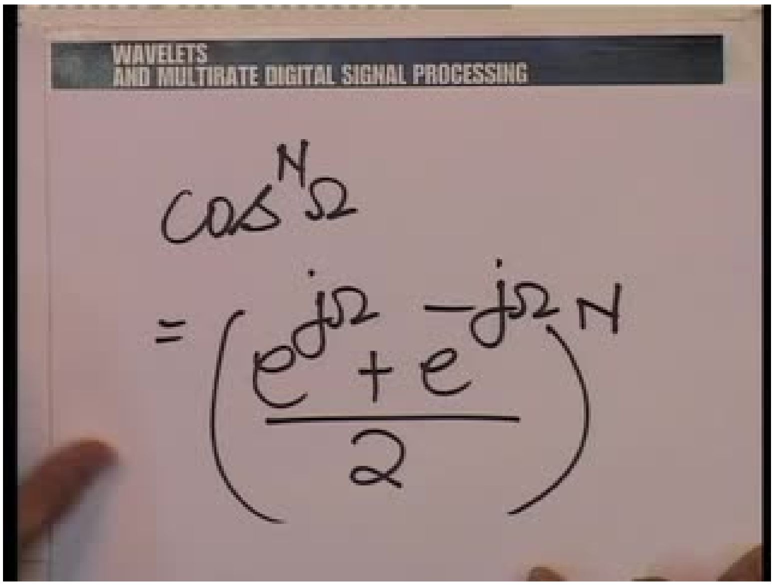

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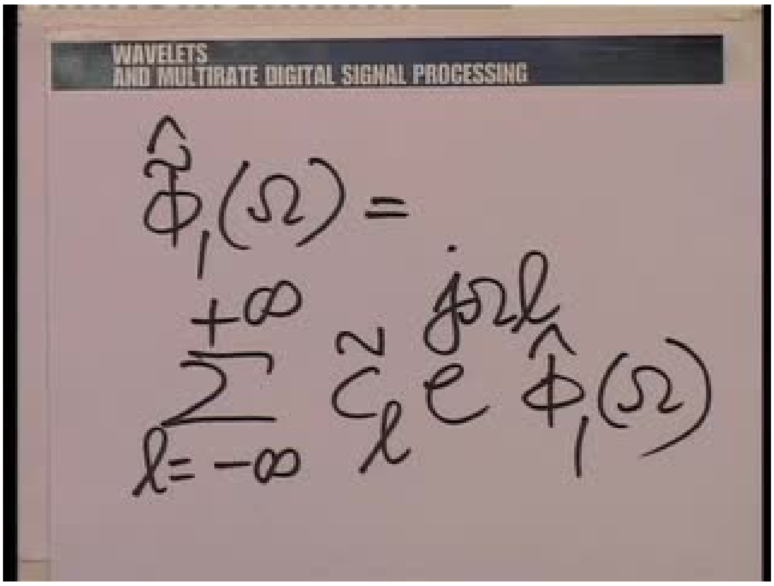

In other words what we said was, cos to the power N omega is essentially e raise the power j omega plus e raise the power minus j omega by 2 to the power N, and therefore what we have here, is that ultimately phi 1 tilde cap omega is equal to summation, let say k running from or lets use a different variable, l running from minus to plus infinity, let us call the constant C l tilde, C l tilde times e raise the power j omega l as we said, time phi 1 cap omega.

Now, this was the critical step, I took the trouble to repeat these steps, because this is the very important idea here, the actual calculations in this are cumbersome, in fact probably it is not even a good idea to do this calculations manually, it might be a good idea to write a small computer program to calculate this coefficient C N or C l tilde, you know incidentally, if you look carefully these coefficient C l tilde each of them will involve writing out a series. So, to get C 0 for example C 0 tilde, you have to write a series, to get C 1 tilde or C minus 1 tilde, you would have to write a series.

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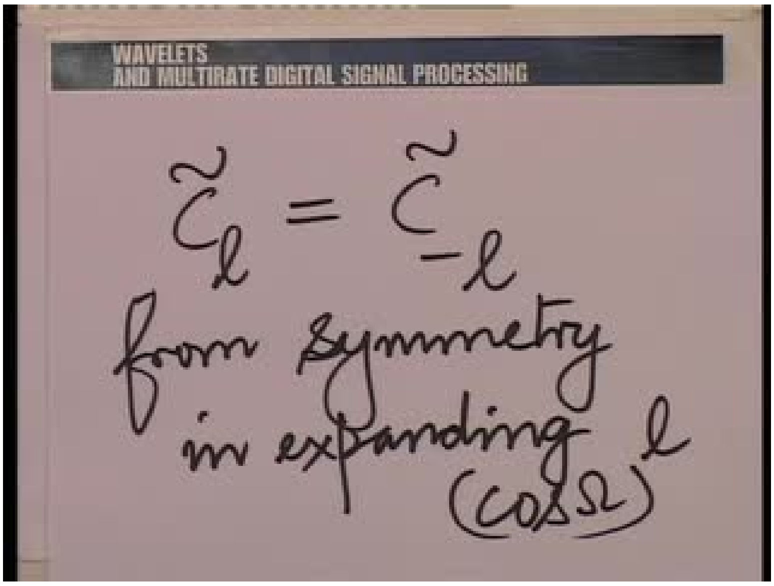

In fact what is also interesting is that, here C l tilde and C minus l tilde are going to be equal from the symmetry in expanding cos omega to the power l, and therefore what we have, is essentially if you take the inverse Fourier transform here, so if we take the inverse Fourier transform of both sides.

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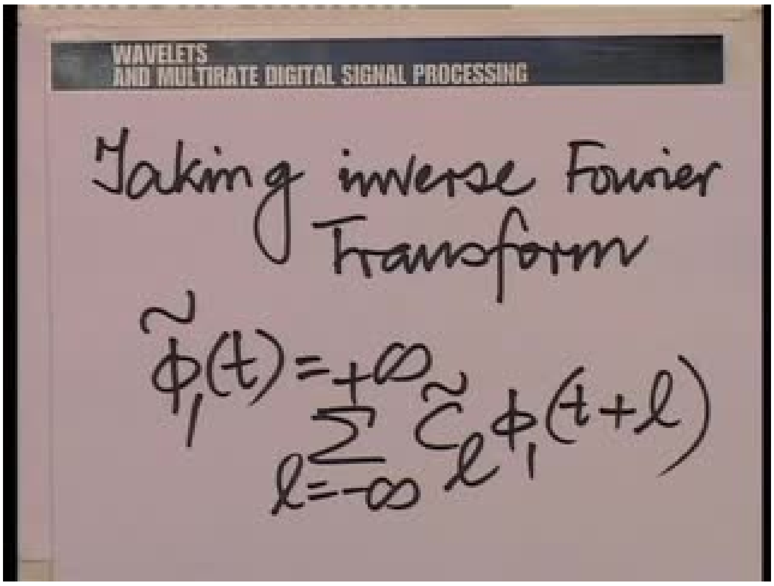

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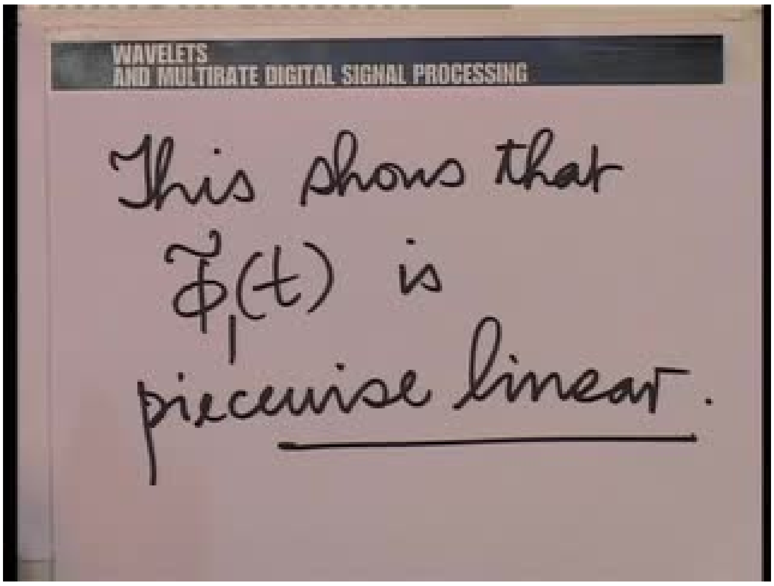

Phi 1 tilde t is summation l running from minus to plus infinity, C l tilde, phi 1 t plus l, and therefore we saw that phi 1 tilde t is a linear combination of phi 1 t and its integer translates. Now, as I said calculating this C l tilde is a cumbersome job, it is not difficult its cumbersome, I leave it to you to write a small computer program to calculate them, that is not so terribly important, what is important is the inference that we draw from here, the inference is that phi 1 tilde is piecewise linear.

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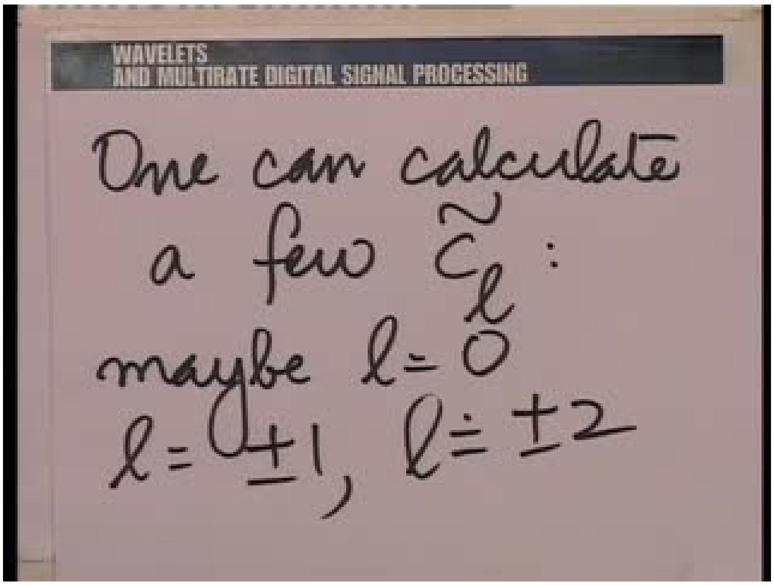

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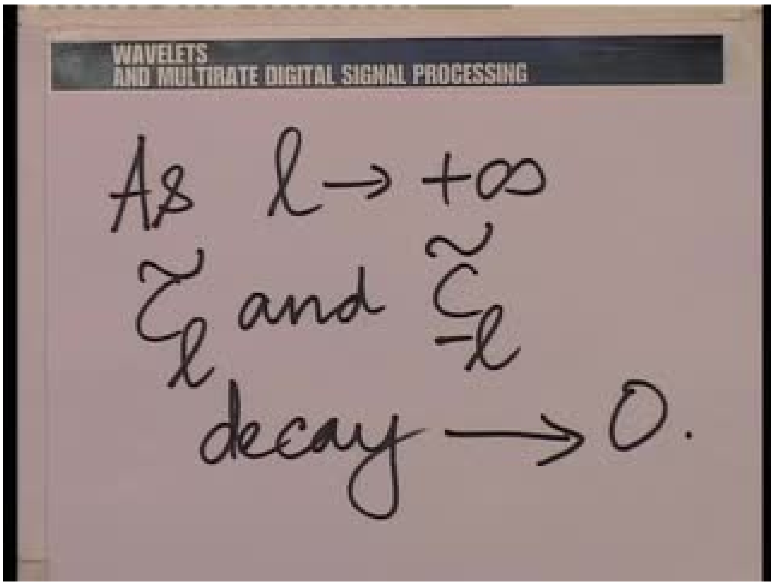

In fact, if you go a step further, and if you just calculate a few of the coefficients, one can by calculating a few of this coefficients, may be for l equal to 0, l equal to plus minus 1, l equal to plus minus 2 and so on, may be up to 3 or 4, and one would see that C l tilde decays, so luckily as l tends to infinity, C l tilde and C minus l tilde decay towards 0. That is what we are saying, is that phi 1 tilde t is going to have an appearance that looks something like this.

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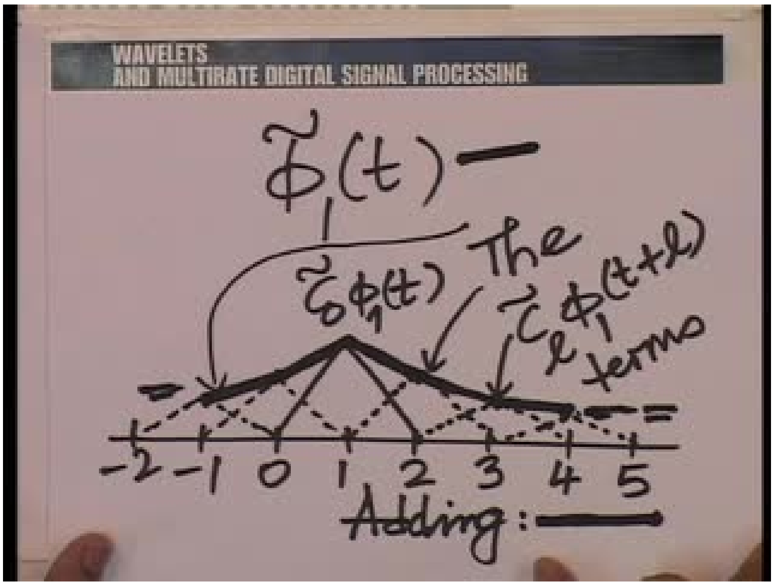

It is essentially going to be well, let’s draw a small compressed diagram here, so between 0 and 2 is going to lie phi 1, between 1 and 3 would be phi 1 t minus 1 and so on, going backwards, so this is phi 1 t here, so you would have something like C 0 phi 1 t, C 1 phi 1 t minus 1 or C 1 phi 1 t plus 1, C 1 and C minus 1 are equal, would be something like this, C 2 of course would be even lower and so on, these would decay fast.

And when you add these, you would get the daub dash line that I am going to draw, so it is very easy to add piecewise linear functions. For example if I wish to add this piecewise linear segment, and this, so you know in the region between say 1 and 2, if I wish to add these 2 segments, just look at the end points, the sums at the end points. So, this linear segment goes to 0 here, and this goes to C 0 tilde here, this linear segment goes to C 1 tilde here, and this one has gone to 0. So, at this point it is going to be C 0 tilde, at this point it is going to be C 1 tilde, and therefore this is the sum here. So, middle what I will do is instead of using a dot dash, I will use a thick line like this; this is the sum, this thick line, and when you go to 3 it is going to be again another thick line, go to 4, you know they are all pieces of straight lines here and so on, continued. So, phi 1 tilde t is going to look like this thick line that I have drawn here, and obviously phi 1 tilde t is piecewise linear, it is a sum of piecewise linear functions.

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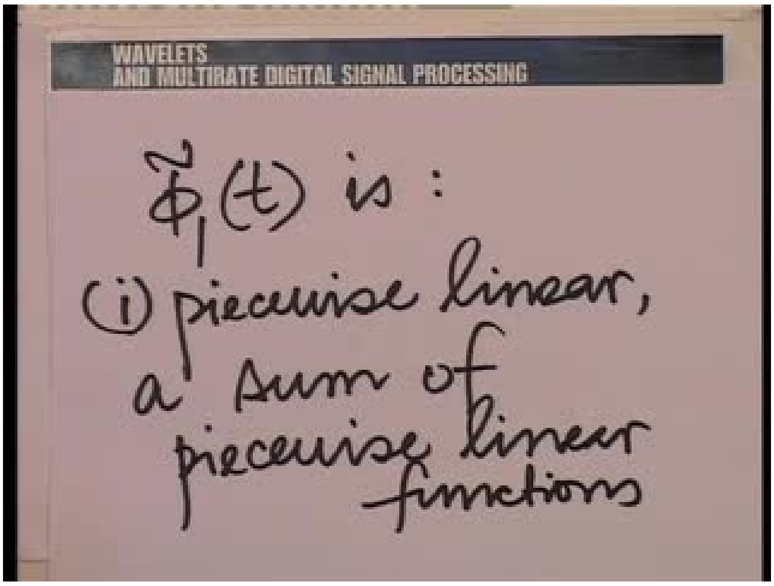

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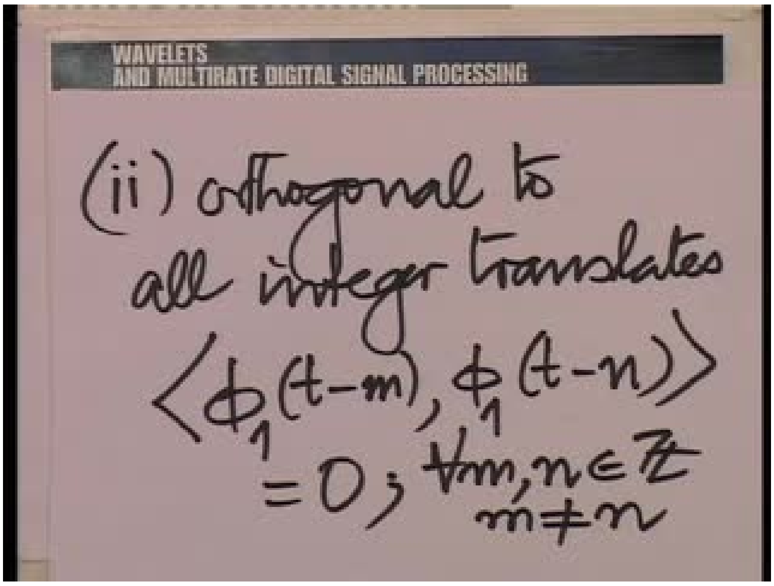

So, it is piecewise linear, a sum of piecewise linear functions. Secondly, phi 1 t as we know, well it is also going to all its integer translates. Another, third property and that is the most important property when we wish to build a multi resolution analysis, is that phi 1 obeys a dilation equation leading to a dilation equation on phi 1 tilde, you see it is the dilation equation that ultimately builds the MRA. So, that is phi 1 tilde obey a dilation equation.

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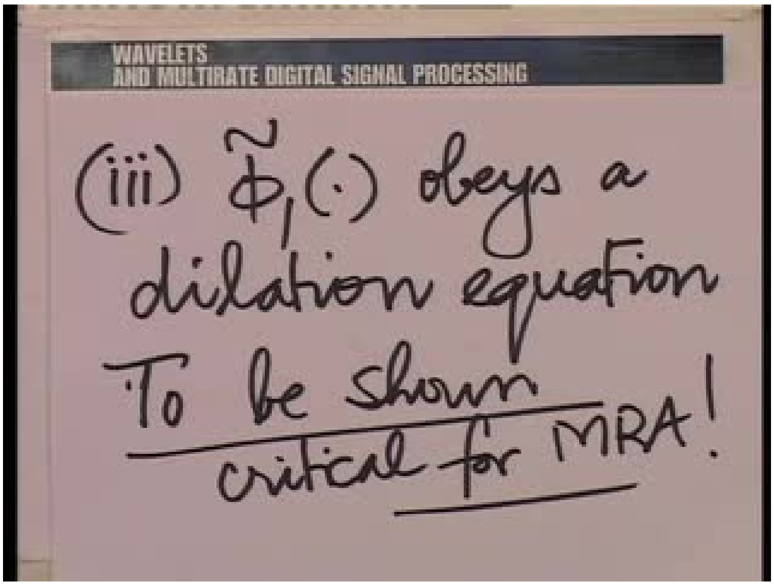

We need to show this, because it is the most critical, this is critical for MRA, it is critical, because it is this dyadic dilation equation that ensures that phi 1 tilde t when dilated by a factor of 2 and then translated by all the integers, constructs a basis for the next sub space, so you can talk about a sub space v 0 which is span by phi 1 t and its integer translates and then you have a sub space which is span by phi 1 t, phi 1 tilde t contracted by a factor of 2 and integer translated. So, we need to prove this dyadic dilation equation obeyed by phi 1 tilde, and we will do it again using the frequency domain.

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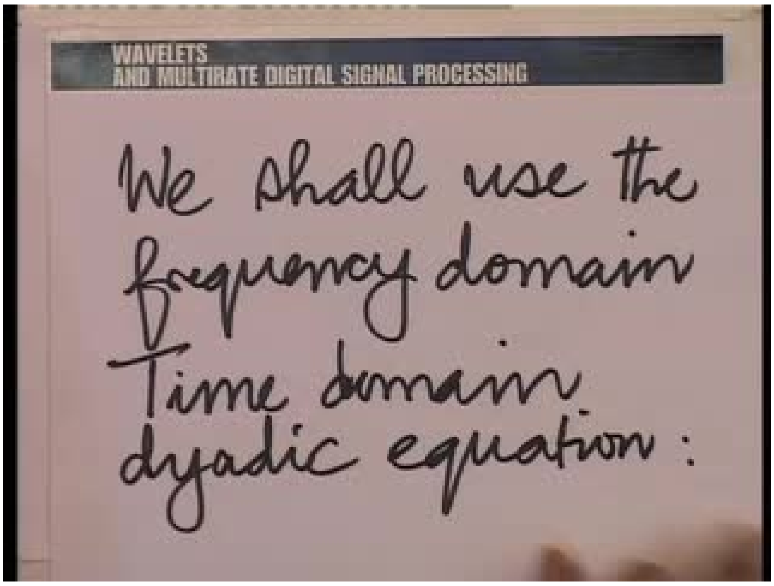

Now, let us go back to the basic dyadic dilation equation that we have, on the scaling function phi t, so you see the time domain equation is as follows.

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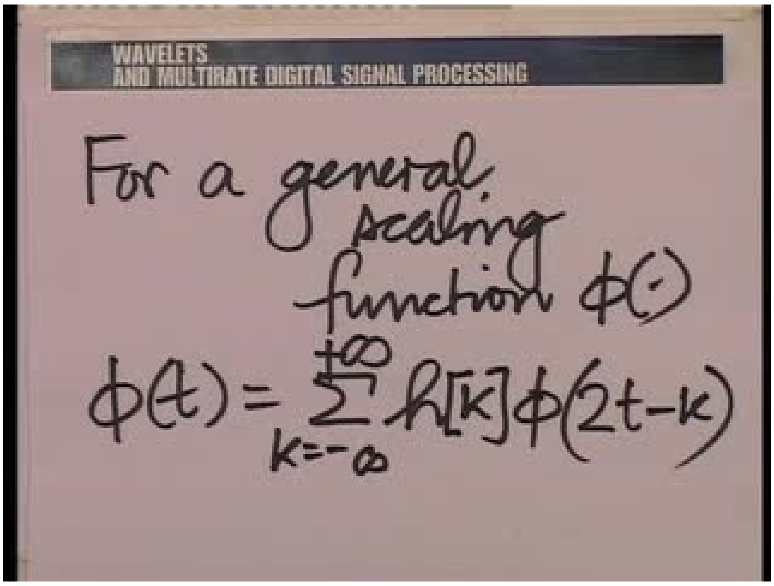

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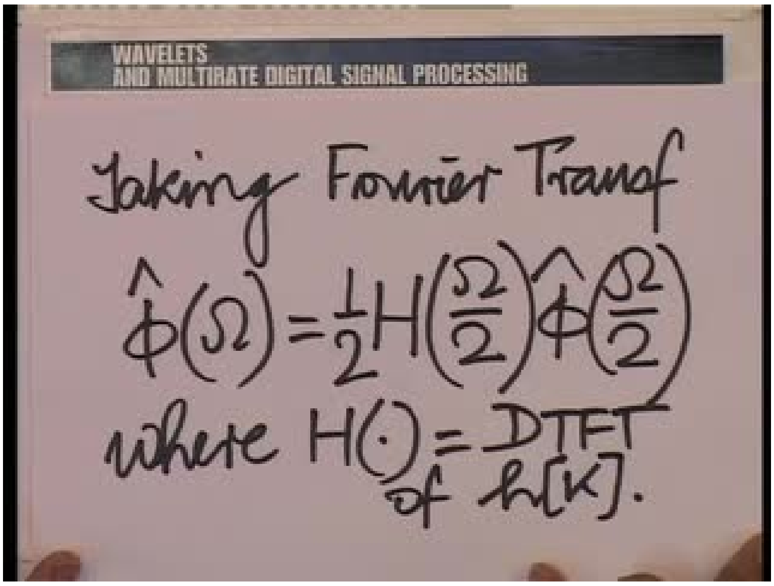

For any scaling function, for a general scaling function phi t, phi t is summation K overall the integers in general, h K phi of 2 t minus K. So, if you look at the Fourier domain, you would have phi cap omega is sum, let us call it capital H of omega by 2 times phi cap omega by 2.

In fact, very strictly speaking you also have a factor of half here, where H is essentially the discrete time Fourier transform of the sequence h K. So, you know the critical idea here in establishing that there is a dyadic dilation equation, is to show that phi cap omega divided by phi cap omega by 2 is essentially a discrete time Fourier transform with the frequency variable essentially scaled, omega replace by omega by 2.

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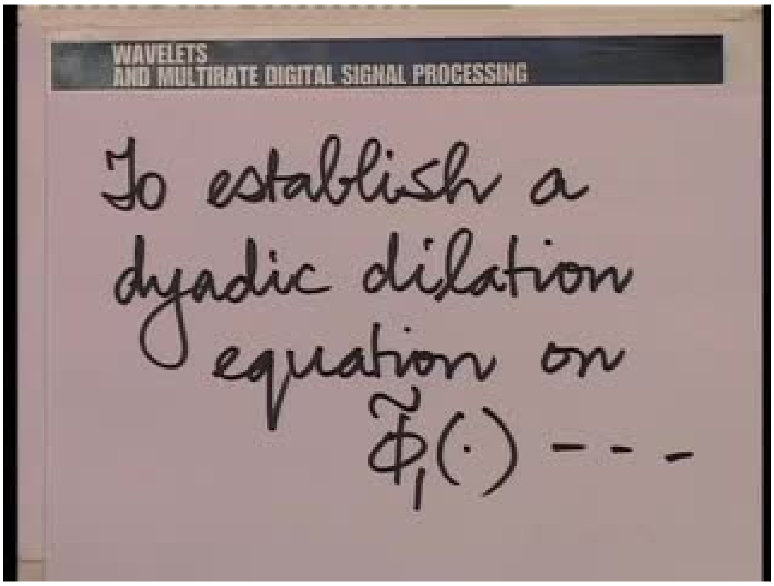

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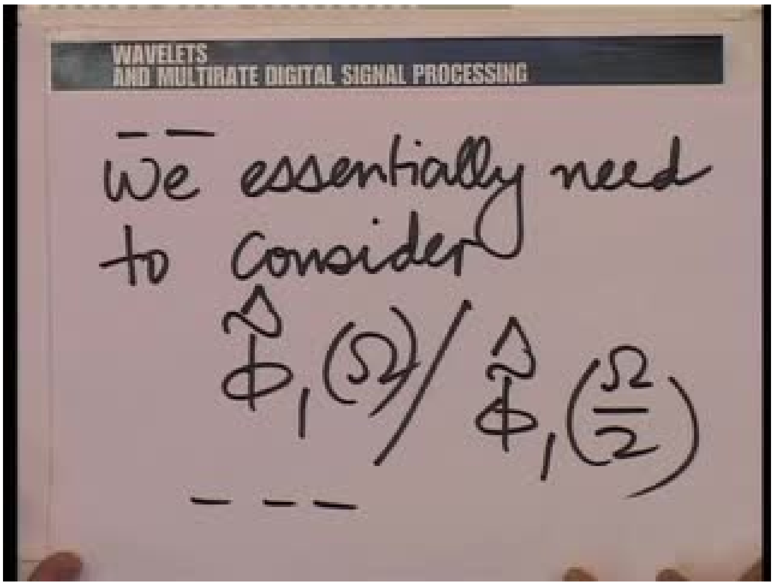

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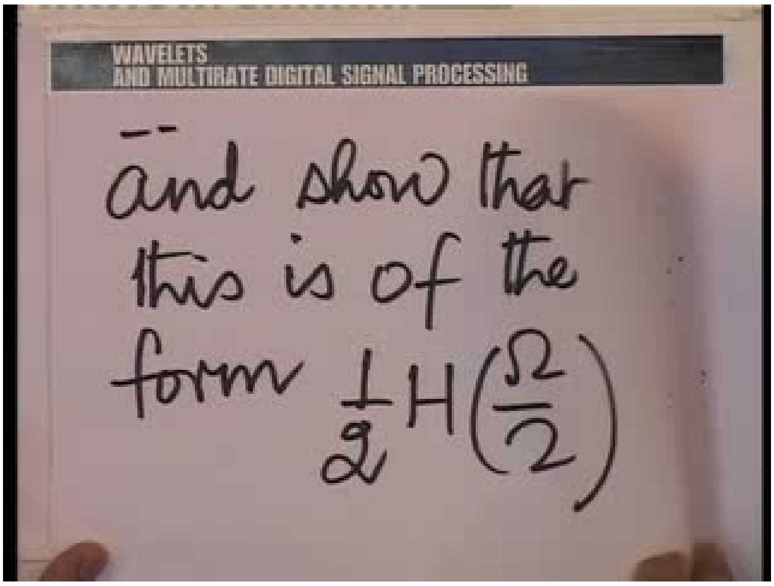

So, what we are saying is, to establish a dyadic dilation equation on phi 1 tilde, what we essentially need to do, we essentially need to consider phi 1 tilde cap omega divided phi 1 tilde cap omega by 2, and show that this is of the form half H omega by 2.

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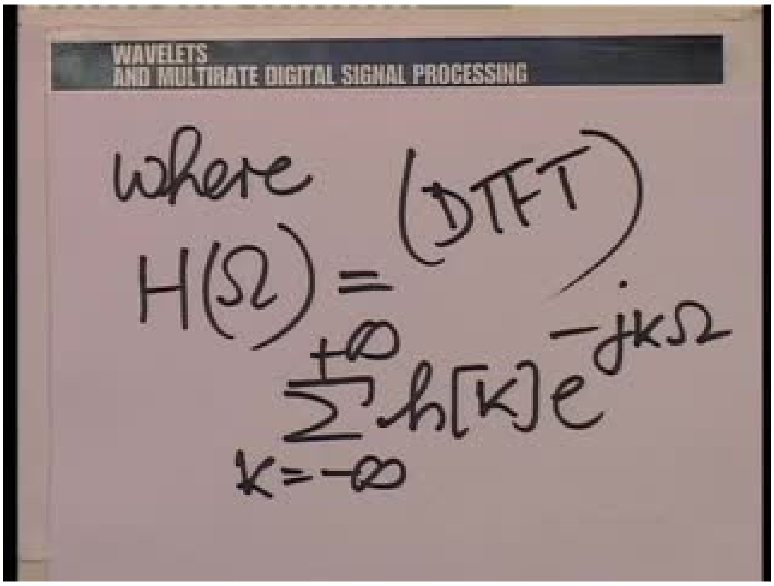

Where, H omega is summation k running from minus to plus infinity, h K e raise the power minus j k omega, essentially a discrete time Fourier transform a DTFT. Now, when would something be a DTFT, when would some function in capital omega be a DTFT, well there are 2 things are characterize a DTFT, one is periodicity in omega, so a function must be a periodic with a period of 2 phi in capital omega. Secondly, it must be inverse discrete time Fourier transformable.

So, in fact if the function is bounded, all over the interval between 0 and 2 phi are on any contiguous interval of 2 phi, then it is of course inverse Fourier, inverse discrete time Fourier transformable, because the inverse DTFT integral would converge, so all that we need to establish is that the following is true.

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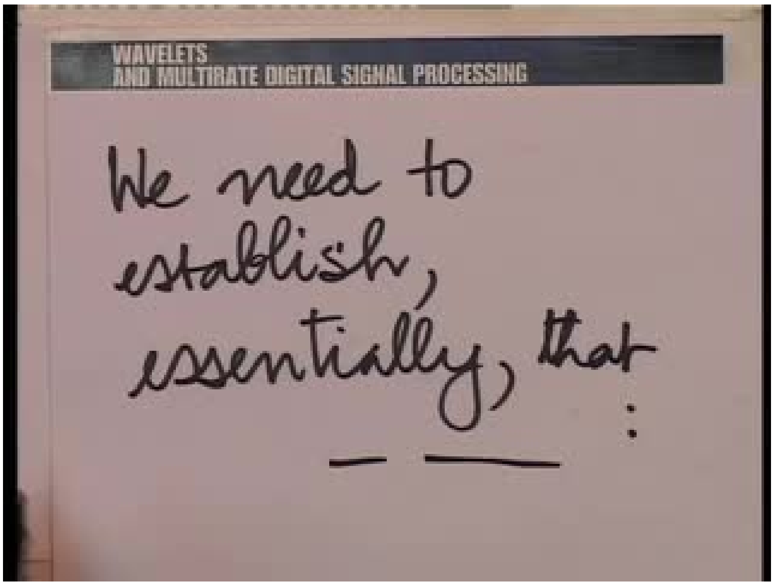

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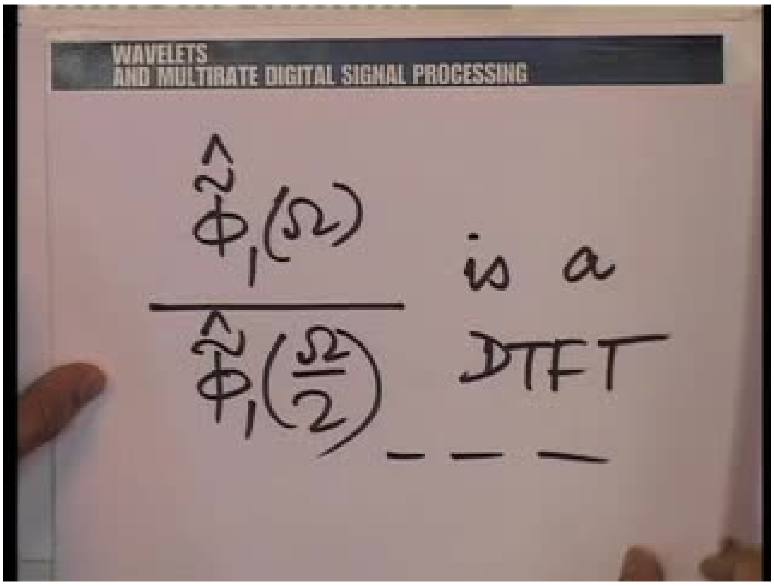

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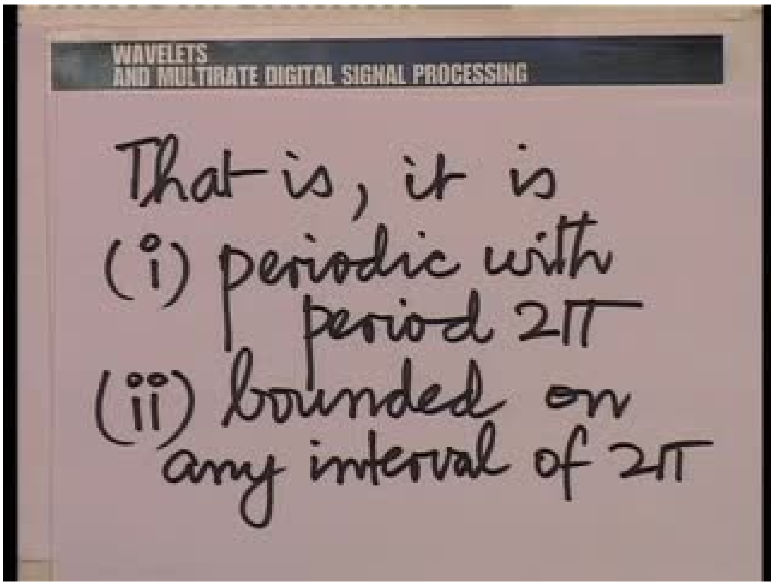

So, we need to establish essentially that phi 1 tilde cap omega, divided by phi 1 tilde cap omega by 2 is a DTFT, and that amongst to saying, that is it is periodic with period 2 pi, it is bounded on any interval of 2 pi, you know the reason why we wanted to be bounded on any interval of 2 pi is as I said inverse DTFT calculation.

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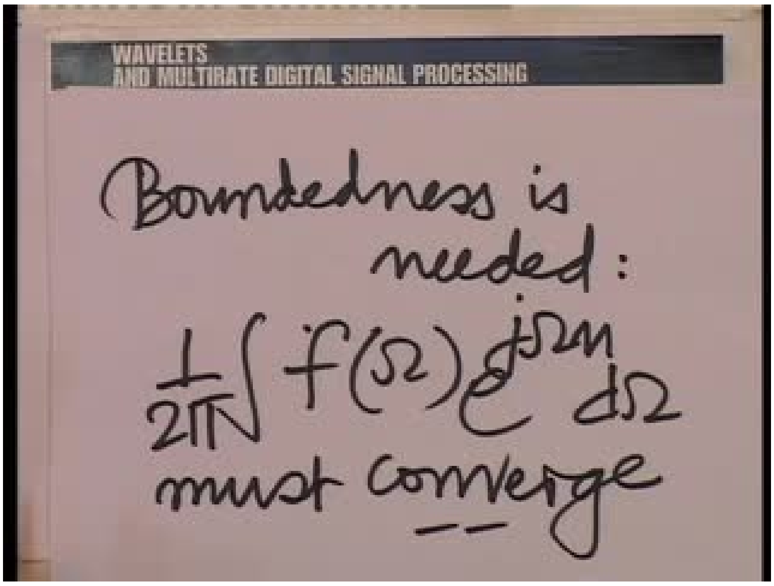

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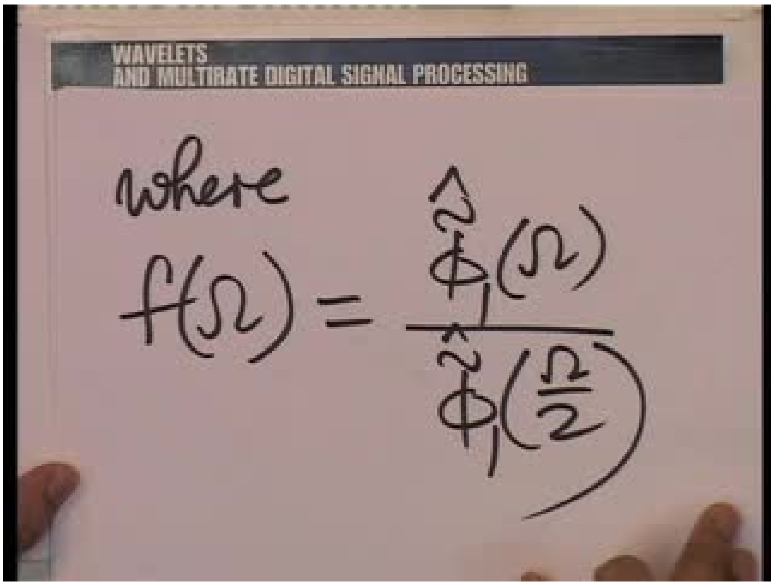

So, the boundedness is needed, because we wish this integral to converge, whatever that function is, that function in terms of omega; e raise the power j omega n d omega by 1 by 2 pi must converge, where f omega is phi 1 tilde cap omega by phi 1 tilde cap omega by 2, quite a mouthful, but not so difficult really.

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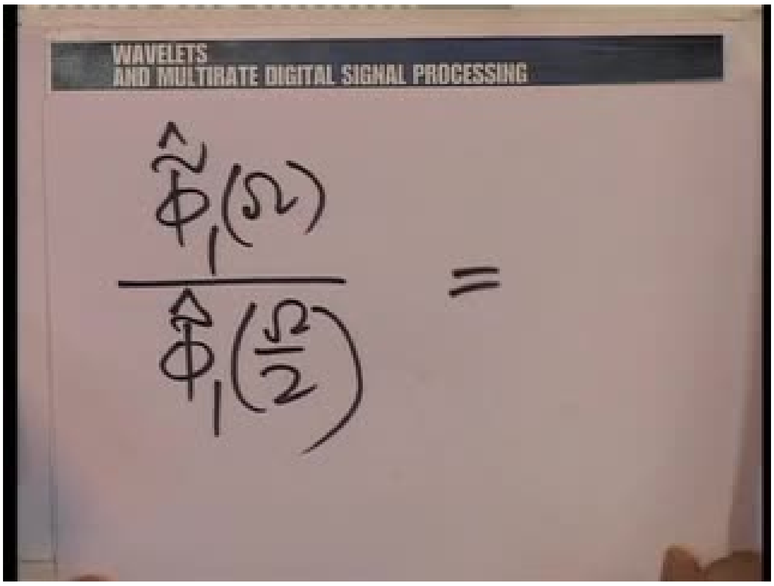

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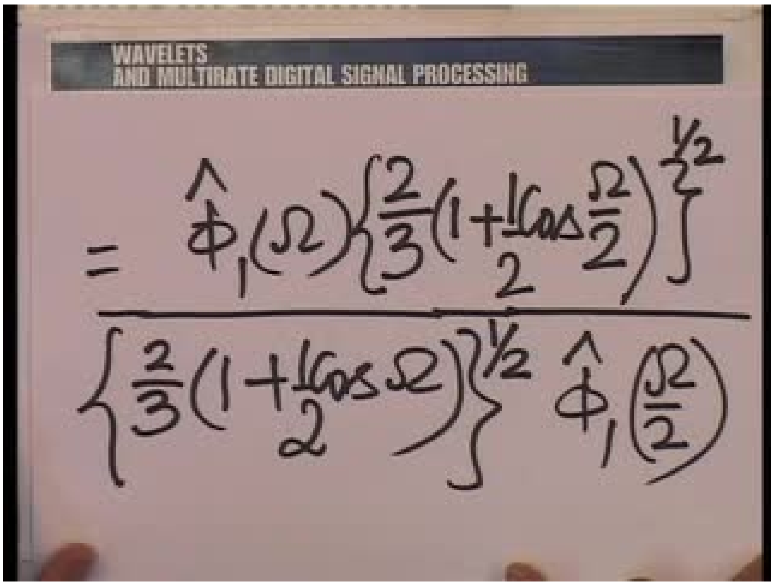

What we essentially need to do here, is to write down the expression for this and immediately note the qualities that we are trying to access. In fact it is very easy to write down phi 1 tilde cap omega, divided by phi 1 tilde cap omega by 2. Let's, write that down by noting, that phi 1 tilde cap omega is actually phi 1 cap omega, divided by something like 2 by 3 into 1 plus cos omega multiplied by half, and the whole raise the power of 1 by 2 here, and the numerator you would have the same thing, but with omega replace by omega by 2 and the denominator here we have phi 1 cap omega by 2, we are trying to write phi 1 tilde cap omega by 2, that is phi 1 cap omega by 2 divided by this, put in the numerator, therefore an omega replace by omega by 2.

Now, we are in the very comfortable situation, look at this expression carefully, for the moment forget about these terms, and focus your attention only on these new terms that have come up, this term is obviously periodic with period 2 pi, in fact if you look at this, this is the same term, but stretched by a factor of 2. So, this is periodic with a period 2 pi, this is going to be periodic with a period 4 pi. In fact well, let's look at it little more carefully, so you know focus on:

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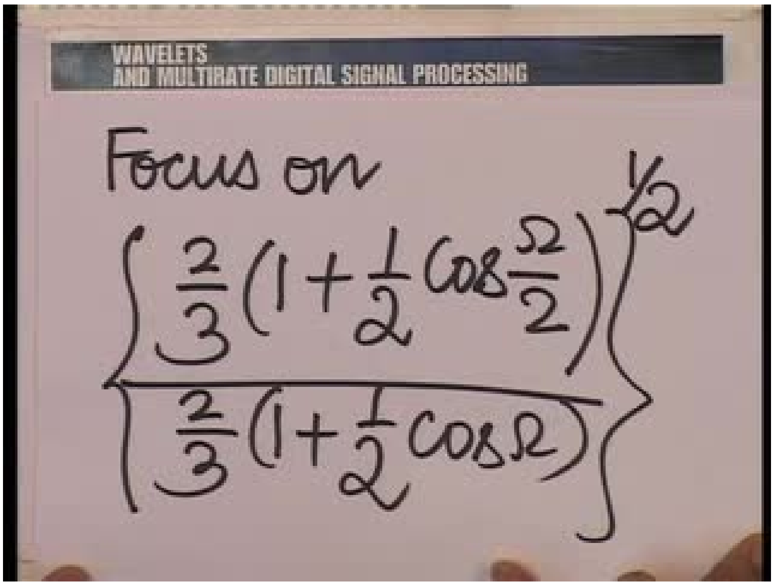

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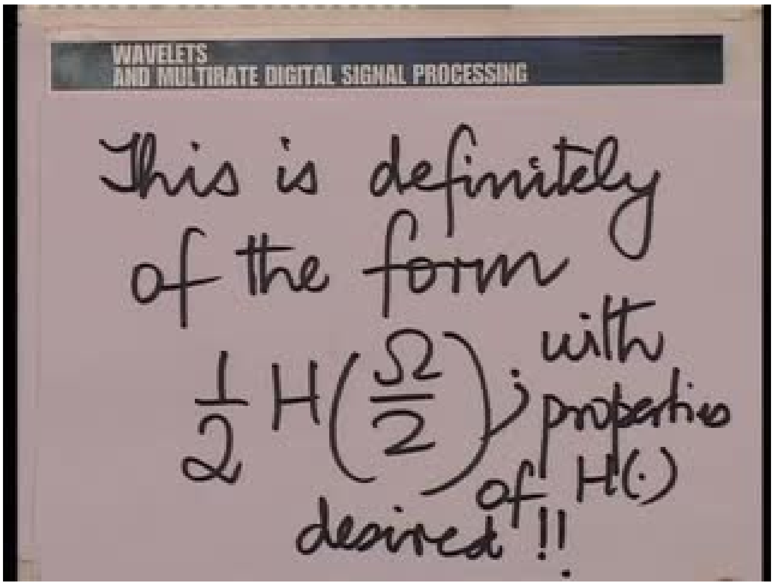

Now, if you look at this is a function of omega then it is obviously periodic, so this is definitely of the form half H omega by 2 with the properties of H desired. Let me justify this.

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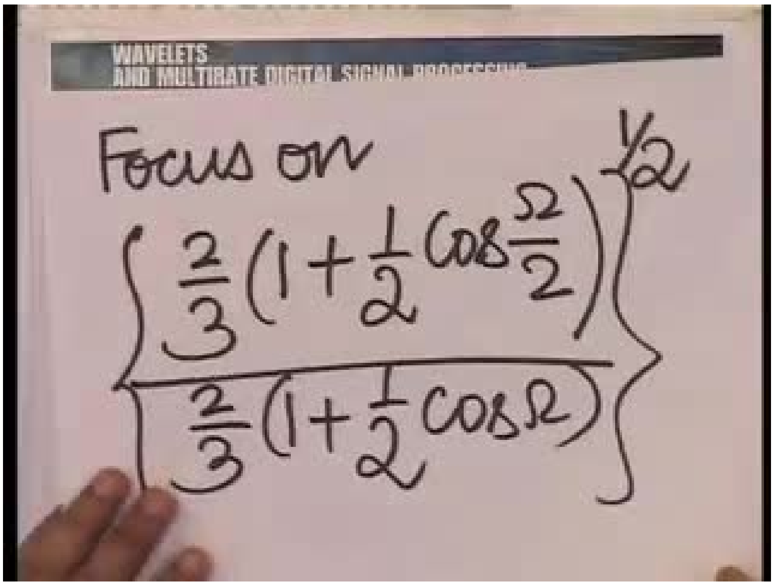

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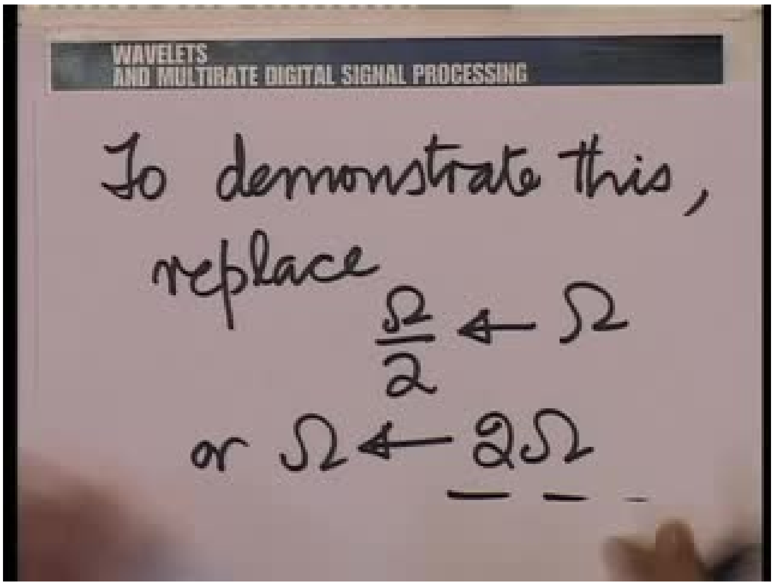

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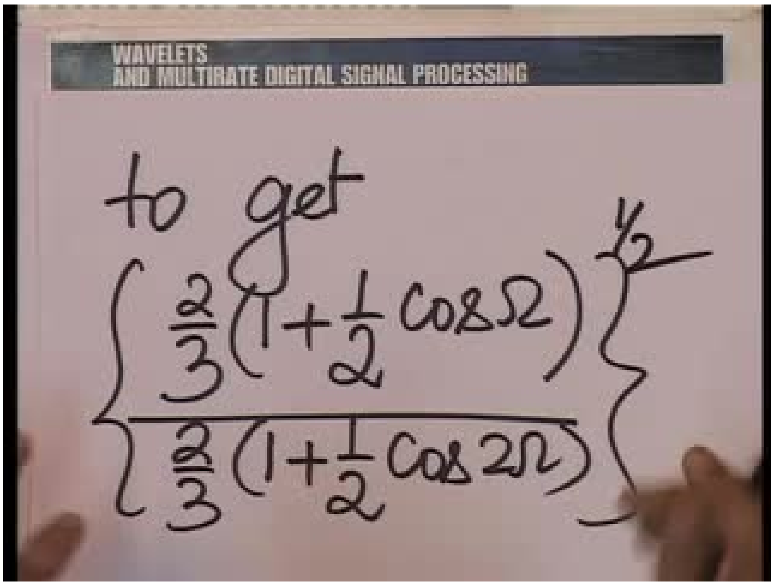

What is the properties desired, forget this factor half that is just a constant, the properties desired are that, if you replace omega by 2 omega here, that means in place of omega by 2 if you had just omega here, and place of omega therefore you had 2 omega here. So, you know this whole thing has to be of the form H omega by 2, so another way of looking at it is replace omega by 2 omega, omega by 2 by omega or omega by 2 omega to get.

Essentially, 2 by 3 1 plus half cos omega here in the numerator, and 2 by 3 1 plus half cos 2 omega in the denominator, the all of that the power half, let me just put it back before you to show you.

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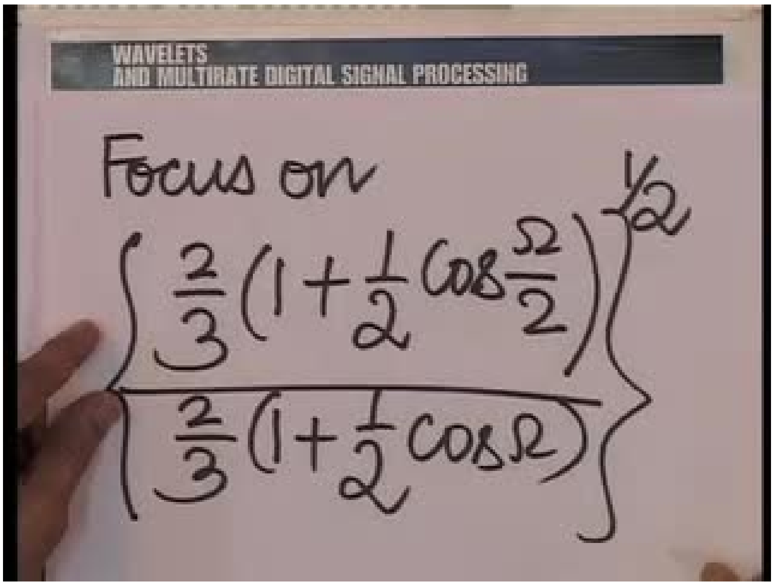

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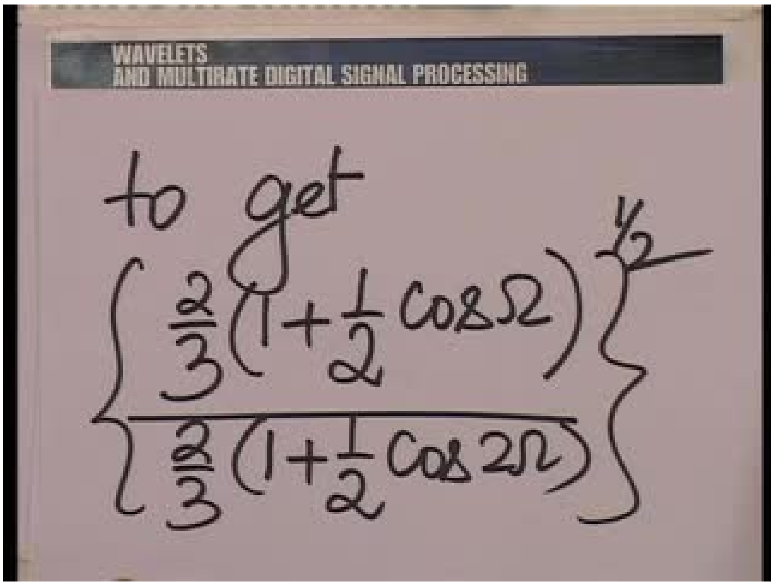

So, what I have done is, I have replaced omega by 2 by omega here, and omega by 2 omega there. Now, look at this expression carefully, this is periodic with period 2 pi, all this is periodic with period 2 pi, cos omega is periodic with period 2 pi, and this is not periodic with period pi, because you have double the frequency here.

Now, if this is periodic with period pi, it is also periodic with period 2 pi, so both the numerator and the denominator are periodic with the period of 2 pi, of course the denominator is periodic also with the period of pi, but being periodic with the period of pi also means being periodic with the period 2 pi. So, this entire expression is periodic with a period 2 pi, so the periodicity is established. Now, for the boundedness, to establish boundedness we need to establish, you know when you have a ratio of 2 quantities and you wish to establish the boundedness of that quantity, you essentially need to show that the numerator and the denominator are only between 2 positive bounds, if you can show that, then you can also show the boundedness of the fraction.

Let me go back to this fraction to illustrate, so you know that the numerator is between 2 positive bounds say A and B, the denominator is also between 2 positive bounds again say A and B. So, let us write that down explicitly.

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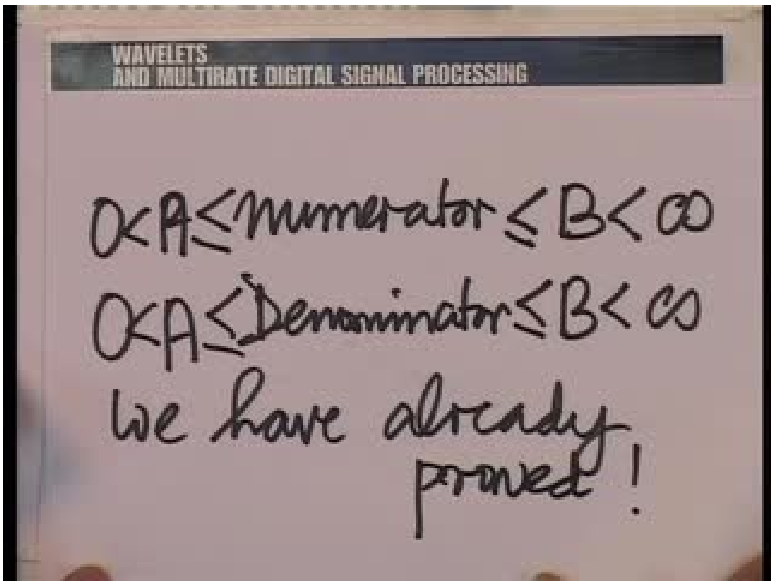

The numerator here is between 2 positive finite bounds, and so is the denominator. In fact the bounds are the same that is very easy to show, we have already proved this, we know these bounds, in fact B by A is 3 we know that.

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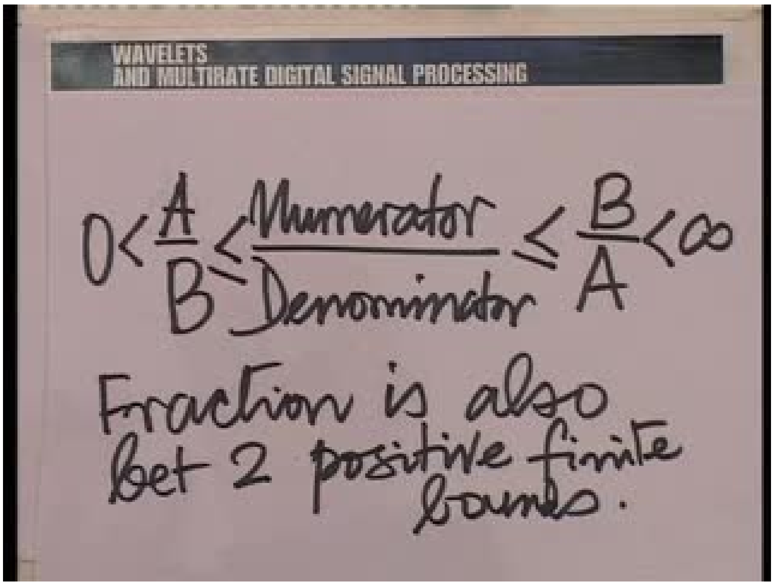

Therefore, the numerator by denominator can at most been when the numerator is a maximum and the denominator a minimum, and B by A is of course less than infinity, and it can at least be when the numerator is a minimum and the denominator a maximum, and this is also greater than 0, the fraction is also between 2 positive finite bounds, and therefore the boundedness has also been established.

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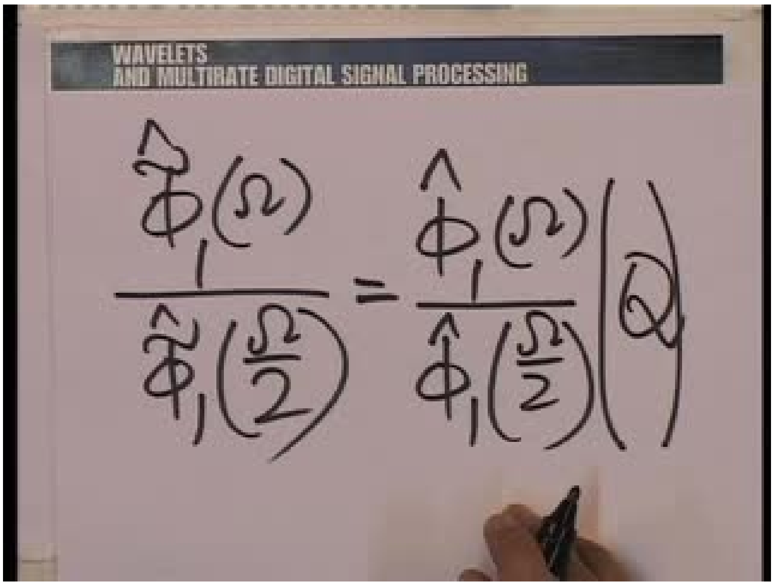

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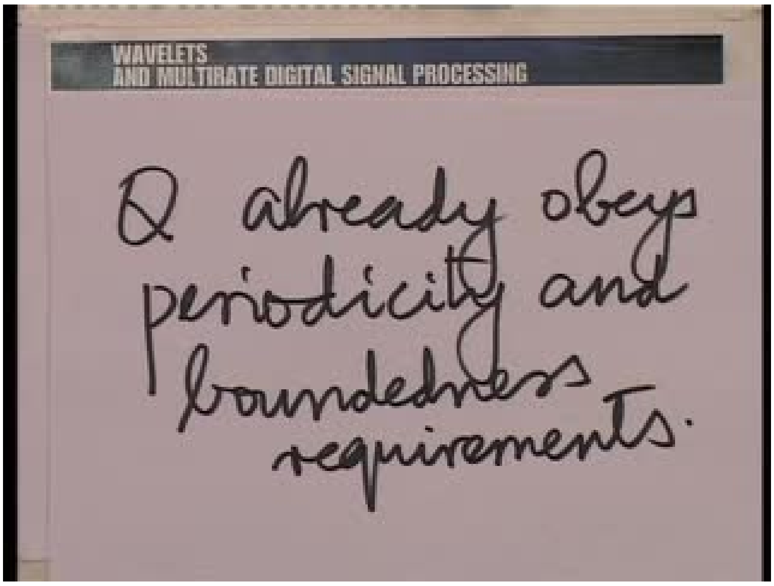

So, if we go back to that ratio, if you go back phi 1 tilde cap omega divided by phi 1 tilde cap omega by 2, is of the form phi 1 cap omega by phi 1 cap omega by 2, multiplied by some Q, where Q already obeys the periodicity and boundedness requirements.

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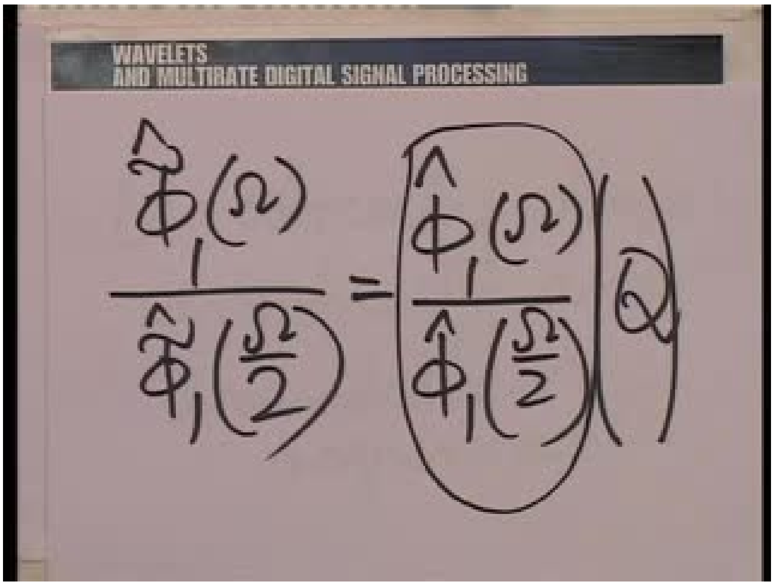

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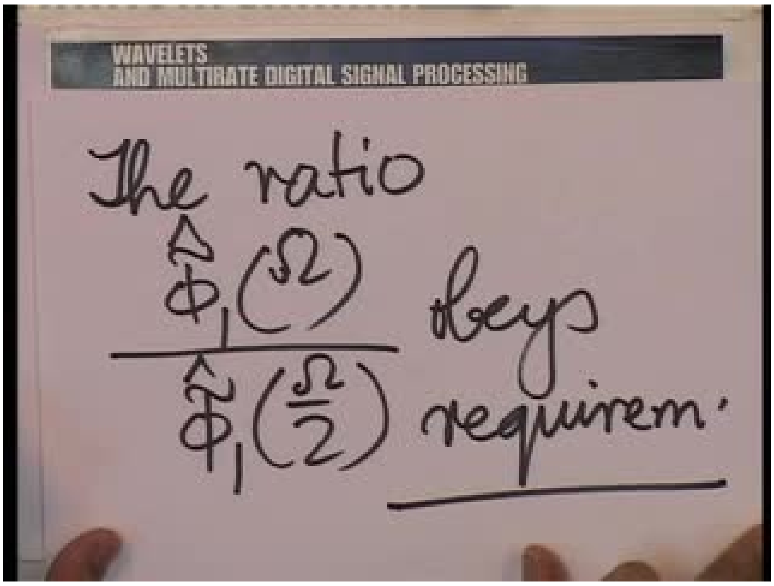

So, if I put this expression back here, this already has been shown to obey the periodicity and boundedness requirements, as far as this concerned we already know that it does, phi 1 cap omega divided by phi 1 cap omega by 2 is known to us, we know that phi 1 obeys a dyadic dilation equation, in fact we also know the coefficients of the sequence h K there, they are half one and half. So, if you look at this composite term, this one obeys phi 1 cap omega by phi one cap omega by 2 obeys the requirements.

This also obeys the requirements, this is periodic with period 2 pi, this is periodic with period 2 pi, so of course the product is periodic with period 2 pi, this is bounded, this is bounded, so the product is bounded. So, all in all, the ratio phi 1 tilde cap omega divided by phi 1 tilde cap omega by 2 obeys the requirements.

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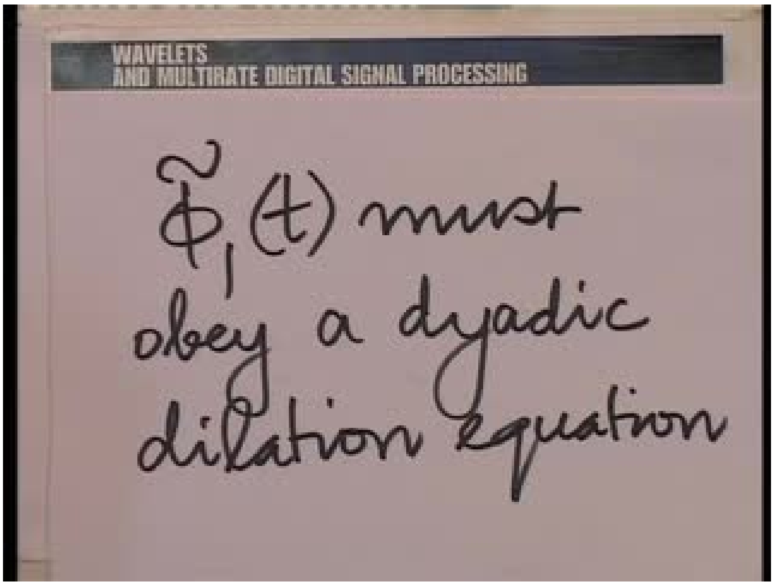

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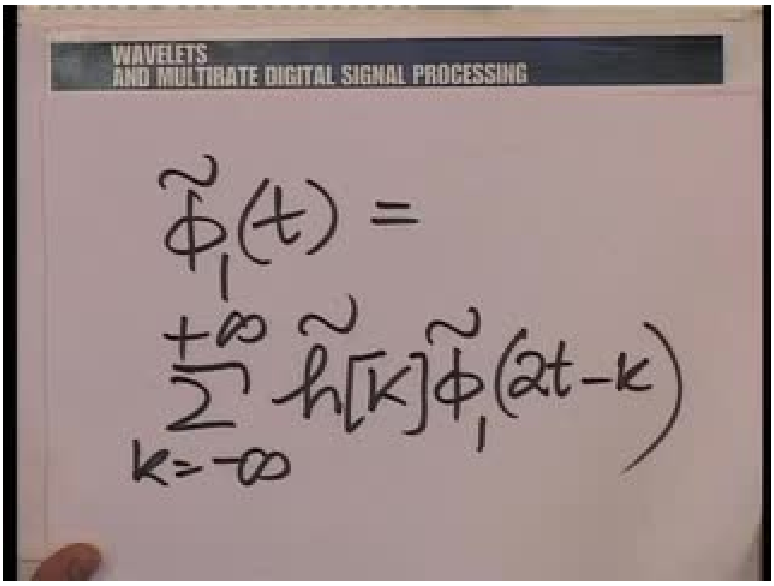

And therefore, it is very clear that phi 1 tilde must obey a dilation equation, the dyadic dilation equation is of the form phi 1 tilde t, is summation k going from minus to plus infinity h K tilde, phi 1 tilde 2 t minus k, and h K tilde is essentially obtained from the following considerations.

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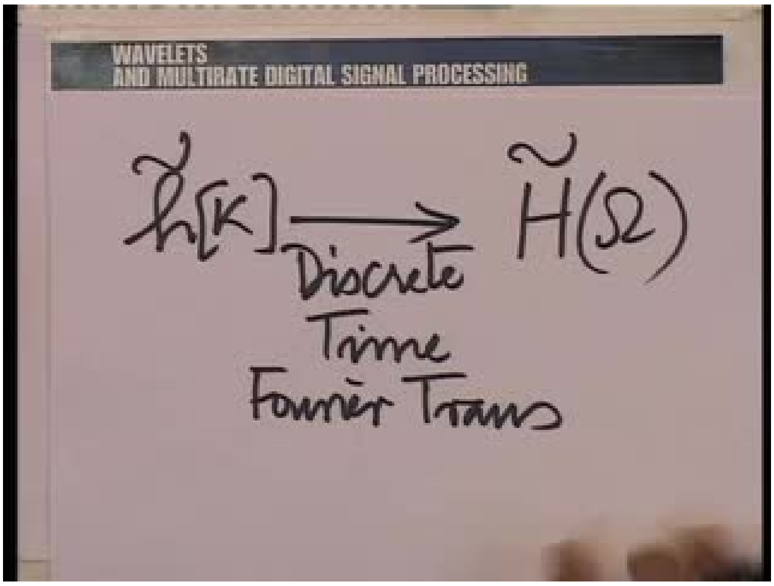

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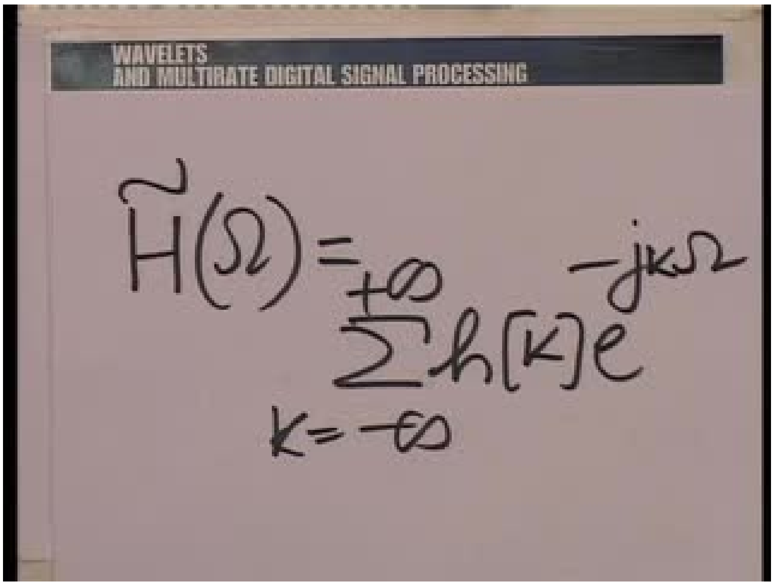

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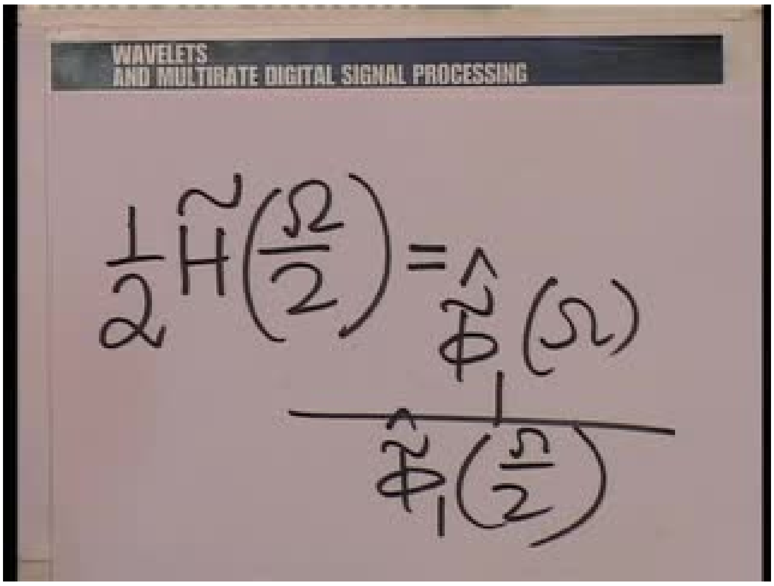

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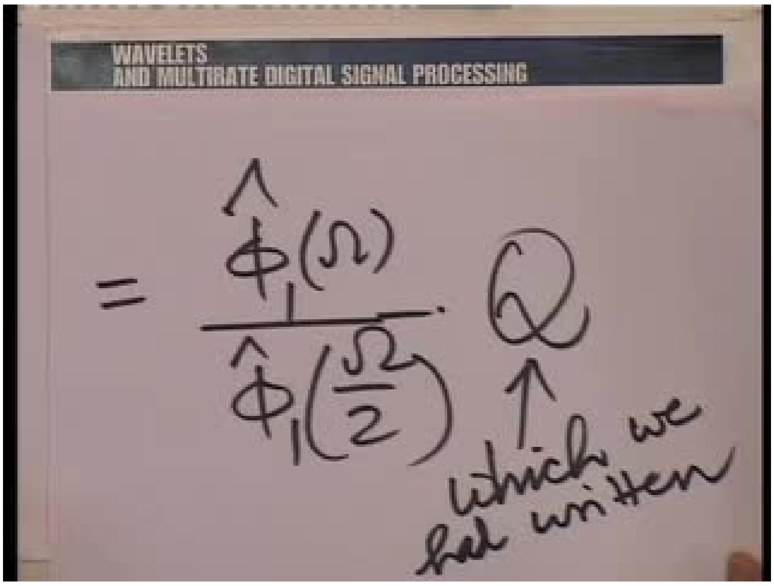

Well if h tilde K has the discrete time Fourier transform capital H tilde omega. In other words, capital H tilde omega is summation k going from minus to plus infinity, h K e raise the power minus j k omega, if this is true, then half H tilde omega by 2 is essentially phi 1 tilde cap omega, divided by phi 1 tilde cap omega by 2, and this is essentially phi 1 cap omega by phi 1 cap omega by 2 times, that quantity Q which we had calculated, and let us write the following the complete thing in all its gory detail.

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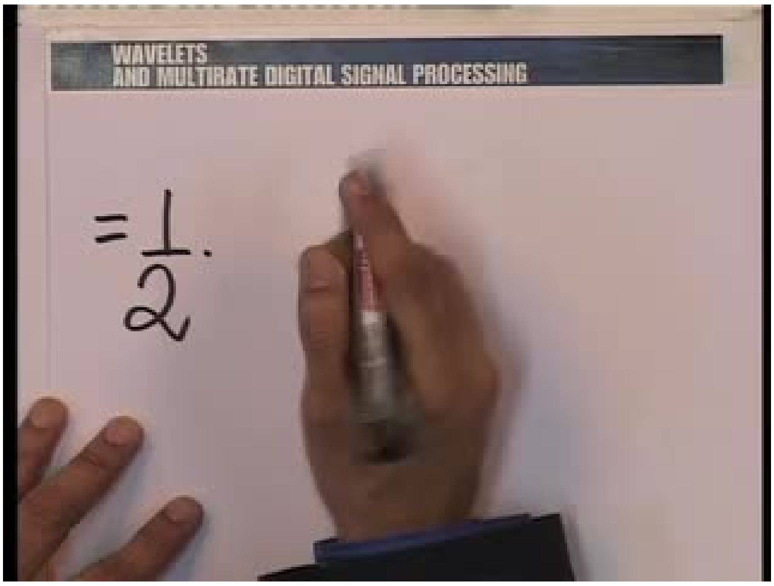

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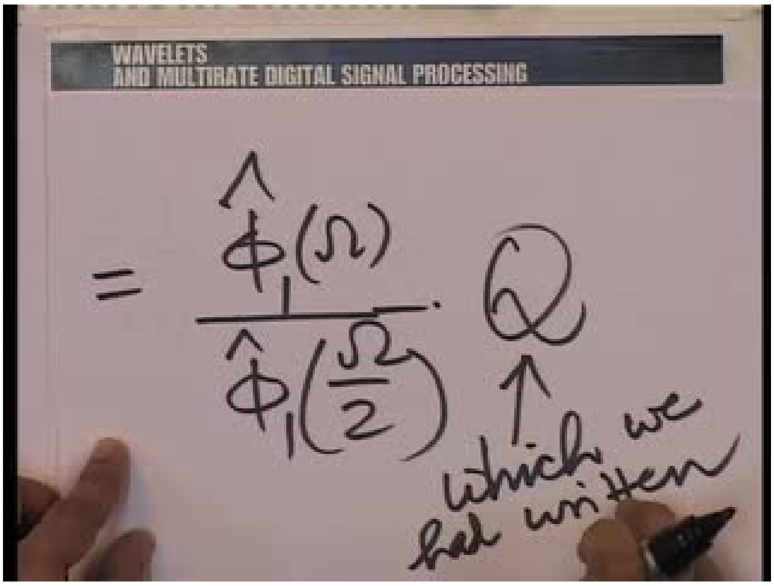

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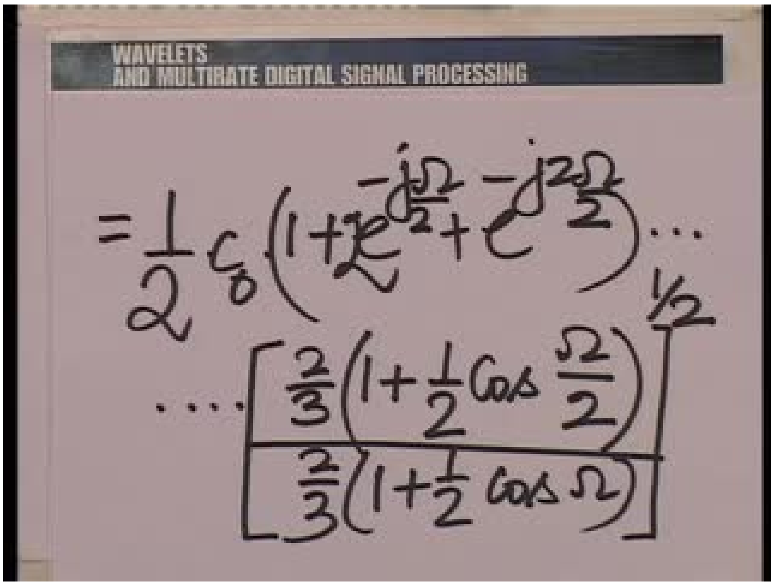

For this would essentially be half, well you know you know what this ratio is, it is essentially the discrete time Fourier transform of the sequence 1 2 1 or half one half, so it is of the form some constant, let say some c 0 times, 1 plus e raise the power minus j omega, plus e raise the power minus j 2 omega squared, well this does not have to be squared this is, but omega has to be replaced by omega by 2 here.

Because, you do replace the variable omega by omega by 2 as you know, this one multiplied by Q, Q is of the form, well if you forget about the two third, so if you want you can keep the two thirds where they are, two thirds, 1 plus half, cos omega by 2; this whole thing is of the form half H omega by 2. So, this whole thing gives you the sequence h tilde.

Now, with this working, because this is definitely a difficult discussion as we see, it is non-trivial, what we have done is to establish the existence of this h tilde K as you could see, actually calculating h tilde is a difficult job, not at all convenient, all that we have shown is the existence of this h tilde, we have also shown how to construct h tilde. If you do manifest the courage to carry out this inverse discrete time Fourier transformation, you could arrive at h tilde K, it is a difficult job, but not impossible. Moreover, once you have h tilde K, you have the impulse response of the low pass filter in the orthogonal MRA.

Once, you have a dyadic dilation equation, the coefficients of the dyadic dilation equation give you the low pass filter response. Once you have the analysis low pass filter, you could construct all the other filters; the synthesis low pass filter, the analysis high pass filter, and the synthesis high pass filter agree, but possible, and many things need to be noted here. The frequency response of this low pass filter is not only infinite in length these h tilde K's are going to be infinite in length. Let me put back this expression before you, if you look at the inverse discrete time Fourier transform this, you could calculated by making a binomial expansion of the numerator and denominator here, you could do that, but as you can see this expansion will last forever, so it is going to have infinite number of coefficients, it is going to have all the wises which make it unrealizable.

It is going to be infinitely non causal, it is going to be irrational and these are going to make this unrealizable as a filter, I mean one can write down the impulse response, but the filter may or may not be realizable. So, to get an orthogonal piecewise linear multi resolution analysis, one requires unrealizable filters. Now, we could get it so much more easily by using the biorthogonal multi resolution analysis, which we did in the 5 3 case, having spent these 2 lectures on constructing the orthogonal MRA with piecewise linear phi, we can now see the advantage of biorthogonal MRA. There, you get at least the piecewise linear phi on one side, the side where you have a length three low pass filter.

Now, just to complete the lecture 1 little detail. Once, you know the impulse response of the low pass filter on the analysis side, you also know how to construct the wavelet, because the analysis high pass filter coefficients, impulse response coefficients would construct the wavelet from the scaling function, you know that psi t can be expanded in terms of the dilates of phi t, phi 2 t minus n with the coefficients in the linear combination given by the impulse response coefficients of the high pass filter. So, once you know the low pass filter, you know the high pass filter, impulse response, those impulse response coefficients or samples give you the linear expansion of psi t. So, constructing psi t is again existentially possible though extremely cumbersome.

So, this is a clear illustration of how we could bring in the variant of introducing infinite length impulse response filters, get piecewise linear multi resolution analysis which are orthogonal, but a better option is then to go for biorthogonal MRA which we did earlier, now we understand why. With that then we come to the end of this lecture and next lecture, we shall take up one more variant of the multi resolution analysis, the wave packet transform.

Thank you