Advanced Digital Signal Processing – Wavelets and Multirate
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Lecture - 03
The Haar Multiresolution Analysis

A warm welcome to the third lecture on the subject of wavelets and multi rate digital signal processing. Let us spend a minute on what we talked about in the second lecture, we had introduced the idea of a wavelet in the second lecture and we had done so by using the Haar wavelet. Essentially, where piecewise constant approximations are refined in steps by factors of 2 at a time. In today's lecture, we intend to build further on the idea of the Haar wavelet by introducing what is called a multiresolution analysis or an M R A as it is often referred to in brief.

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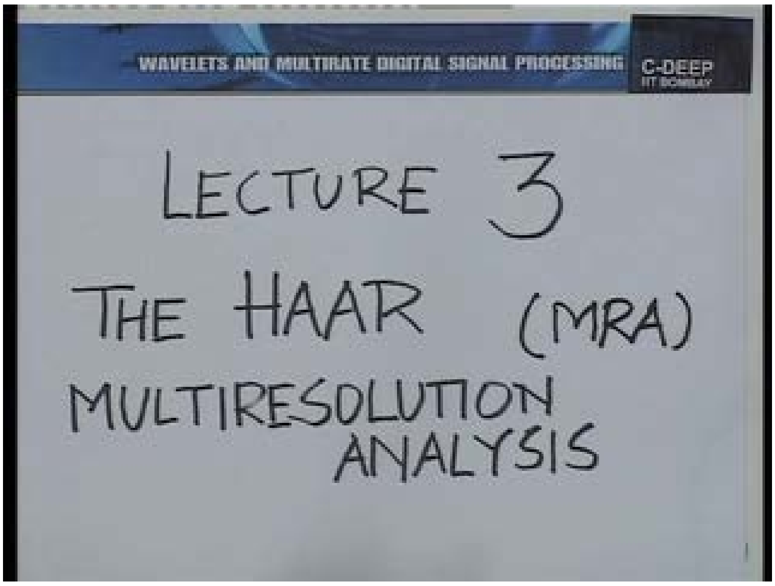

So, Let me title today's lecture, we shall title today's lecture as the Haar Multiresolution analysis and in fact let me also put down here the abbreviation for Multiresolution analysis M R A. You see, the whole idea of Multiresolution analysis has been briefly introduced in the context of piecewise constant approximation. So, recall that we said that the whole idea of the wavelet is to capture incremental information. Piecewise constant approximation inherently brings in the idea of representation at a certain resolution. We took the idea of representing an image at different resolutions in fact we use the term resolution when we represent the images on a computer, 512 cross 512 is a resolution lower than 1024 cross 1024.

And one way to understand the notion of wavelets or to understand the notion of incremental information is to ask if I take the same picture the same 2 dimensional scene or same 2 dimensional object so, to speak and represented first at a resolution of 512 cross 512 and then at a resolution of 1024 cross 1024. What is it that I am additionally putting in to get that greater resolution of 1024 cross 1024 which is not there in 512 cross 512, the Haar wavelets captures this. So, in some sense you may want to think of the Haar wavelet has been able to capture the additional information in the higher resolution and therefore, if you think of an object with many shells, this is the very common analogy.

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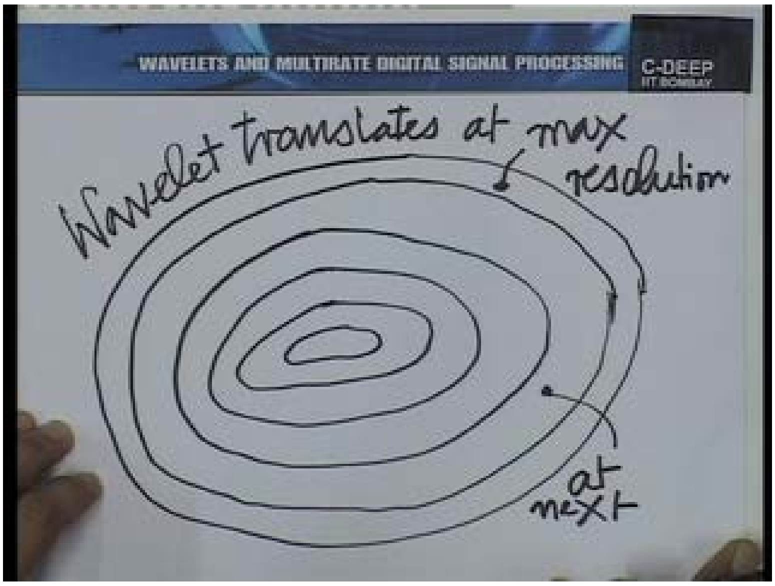

You know, if you think of the maximum information may be as a cabbage or as an onion informally and if you visualize the shells of this cabbage or this onion like this then the job of the wavelet is to take out a particular shell. So, the wavelet at the highest resolution, wavelet translates at highest resolution at max resolution would essentially take out this, at next resolution it would take out this shell and so on. So, when you reduce the resolution what you are doing is to peel of shell by shell. In fact I think this idea is so important that we should write it down.

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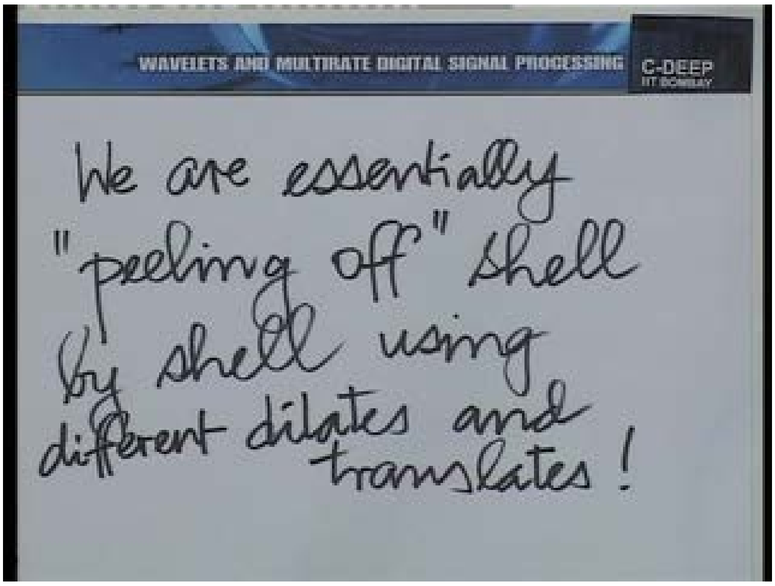

We are essentially, peeling off shell by shell using different dilates and translates of the Haar wavelet. And there again a little more detail, different dilates correspond to different resolutions and different translates essentially take you along a given resolution that is the relation between peeling of shells and dilates and translates. Now, all this is an informal way of expressing this, we need to formalize it and that is exactly what we intent to do in the lecture today. Again, we would now like to talk in terms of linear spaces. So, without any loss of generality, let us begin with a unit length for piecewise constant approximation.

I say without loss of generality because after all what you consider as unit length is entirely your choice. You can call 1 meter unit length, you can call 1 centimeter unit length or if you are talking about time you could talk about 1 second as unit length or unit piece and so on. So, unit on the independent variable is our choice and in that sense without any loss of generality let us start, make the focal point piecewise constant approximation at a resolution with unit interval.

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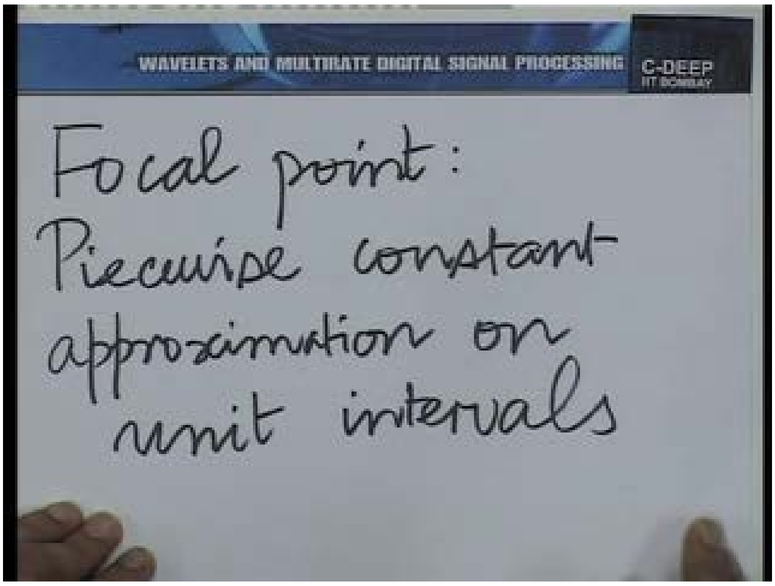

Let us write down formally piecewise constant so you know so called fulcrum or focal point is piecewise constant approximation on unit intervals.

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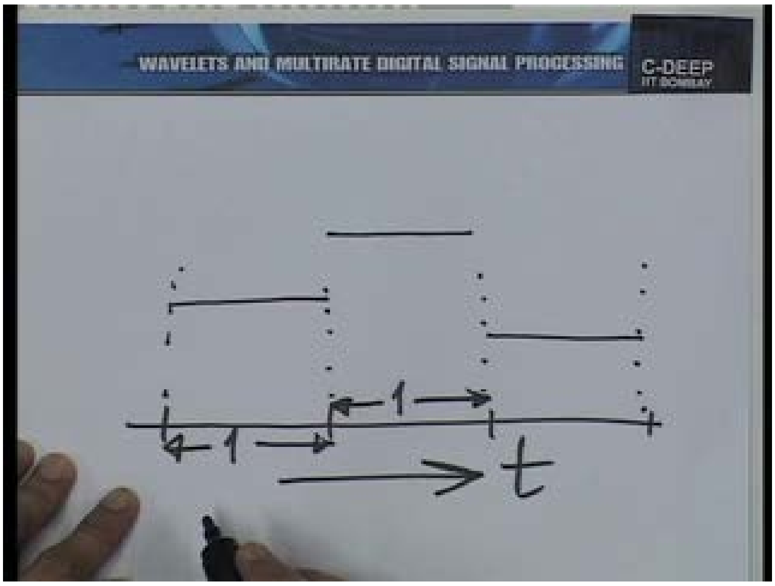

And let us sketch this to explain it better. So, what we are saying is you have this independent variable again without any loss of generality let that independent variable be t, you have unit intervals on this. And on each of this unit intervals, you write down a piecewise constant function, essentially, corresponding to the average of the original function on that interval. So, this is the average of the function on this interval, this one on this interval and this one on this interval. Now, how can we express this function mathematically with a single function and its translates. So, essentially we want a function, let us call it phi (t) now.

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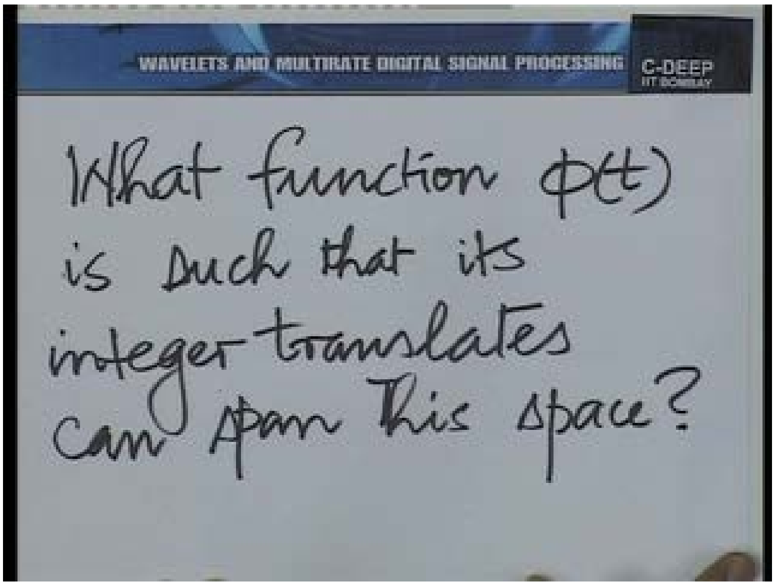

So, what function phi (t) is such that its integer translates can span this space, what space?

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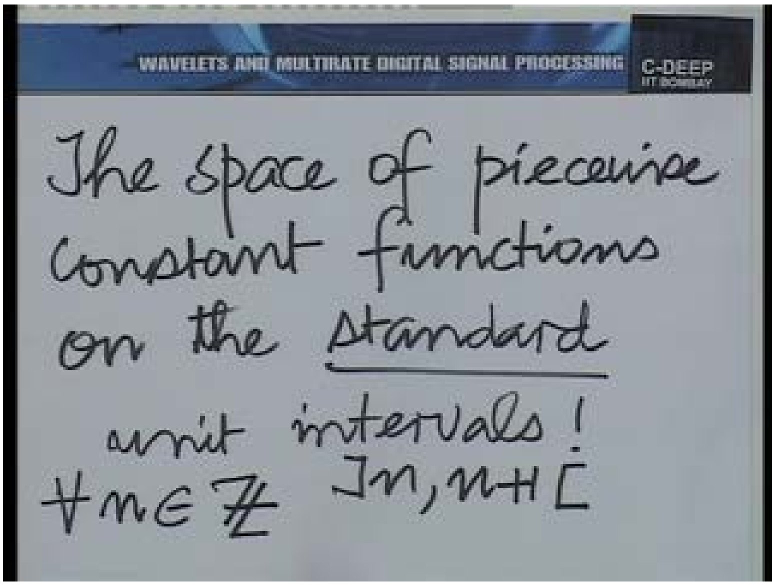

First, the space of piecewise constant functions on the standard unit intervals. What are these standard unit intervals? The standard unit intervals are the open intervals n, n plus

1 for all n over the set of integers. Now, I wish to slowly start using notation which is convenient. So, this notation script z would in general in future refer to the set of integers. I think we should make a note of this.

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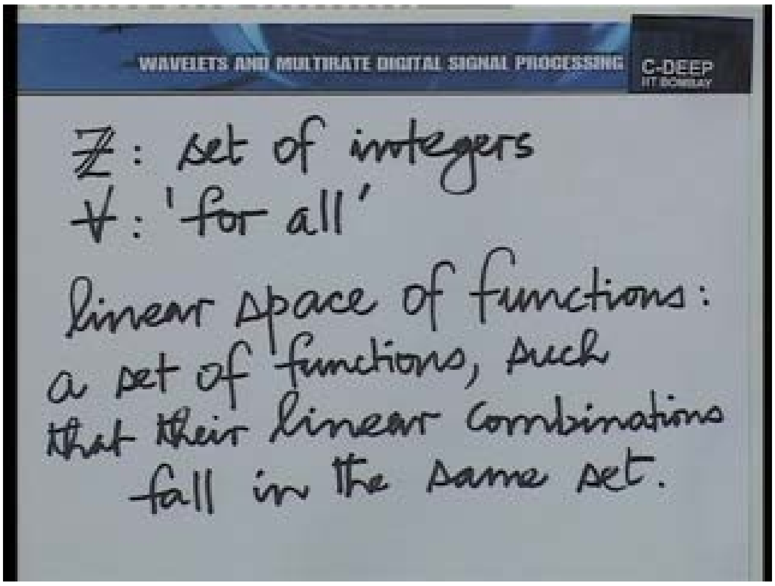

Script Z is the set of integers and this refers to for all.

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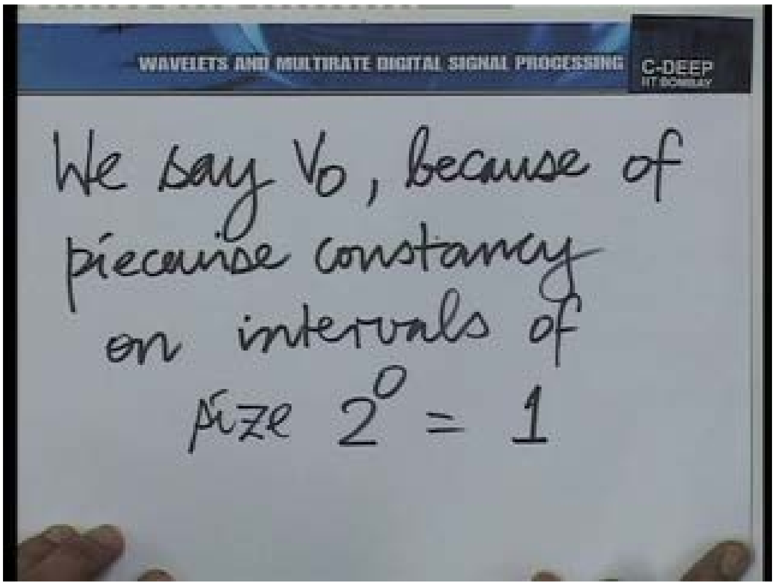

So, what are we saying here let us go back we are saying we have this space, now again I must recapitulate the meaning of space, a linear space of functions is a collection of functions, any linear combination of which comes back into the same space. So, if I add

2 functions it goes back into the same space, if I multiply a function by a constant it goes back into the same space, if I multiply 2 different functions in that space by different constants and add up these resultants it would still be in the same space.

In general, we would say any linear combination; we say a set of functions forms the space a linear space if it is closed under linear combinations. So, we say a linear space of functions is a set of functions such that linear combinations fall in the same set.

I am making it a point to write down certain definitions and derivations in this course, and there is an objective behind that, I believe that a course like this is the best learnt by working with the instructor. So, although one could just listen and try, and remember that does not give the best flavor in a course like this. It does require in depth reflection and thinking and therefore, I do believe that the student of the course would do well to actually, note down certain things and work with the instructor for it is then that the full feel of derivation and the full feel of concepts would dawn upon the students.

Anyway, with that little observation and instruction let us go back to what we are doing here. So, you see a linear space of functions is one in which any linear combination of functions in that set fall back into the same set. Now, here there is a little bit of clarification require. You see, in general if you consider the space of functions that we talked about a minute ago, namely the space of piecewise constant functions on the standard unit intervals. Which are the standard unit intervals, the intervals of the form open interval of the form n, n plus 1 for all n over the set of integers then there is infinity of such functions and naturally when you talk about linear combinations you could have finite linear combinations and you could have infinite linear combinations.

Now, for this point in time when we talk about linear combinations we are essentially referring to finite linear combinations. So, that is just a little clarification for the moment well the idea could be extended to infinite linear combinations too. But I do not want to go into those niceties at this point, in time they would carry us away from our primary objective. So therefore, these sets of functions that we talked about for a minute ago is indeed a linear space that is why I have called it as space here.

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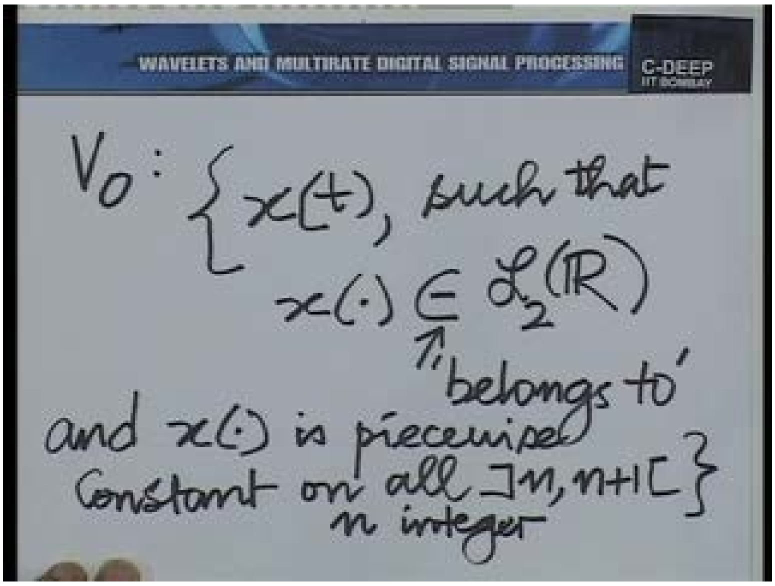

And we give that space a name we will call that space V 0. So, V 0 is the set. Now, I am going to write mathematical notation. V 0 is the set of all x (t), such that 2 things happen x (t). Now, you know x is a function, when I write like this, what I mean is I am suppressing the explicit value of the independent variable. But, I recognize that there is an independent variable here but I am treating the whole thing as an object. It is a function I am treating the whole function as an object and this object belongs to L 2 R recall that L 2 R is the space of all functions which are square integrable and this stands for belongs to such that x belongs to L 2 R and x is piecewise constant on all intervals of the form n, n plus 1 n integer. Now, once we have talked about V 0 in fact the reason for giving the sub stead 0 here is that we are talking about 2 to the power 0 as a size of the interval that is important enough I think to make a noting.

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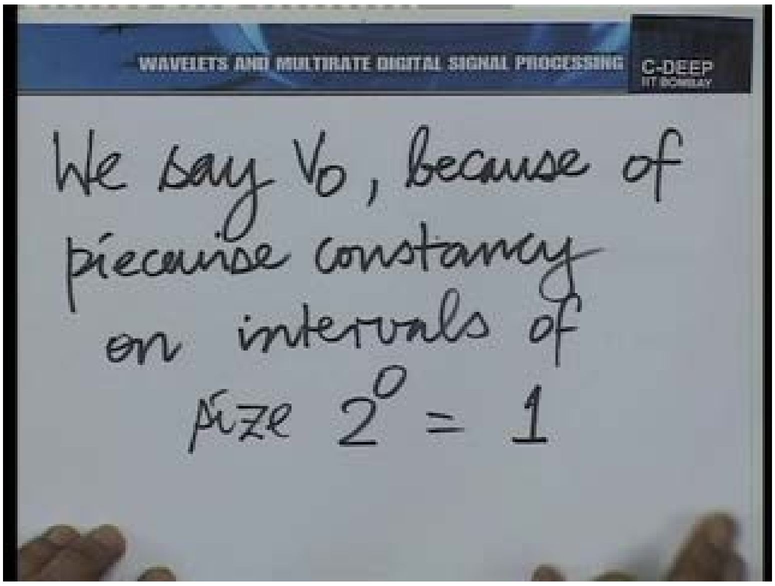

So, we say V 0 because of piecewise constancy on intervals of size 2 raise the 0, which is 1. And similarly, therefore, in fact you know you could call it 2 raise the power of 0 or you could call it 2 raise the power of minus 0. We will prefer to use 2 raise the power of minus 0 because it will be consistent in future.

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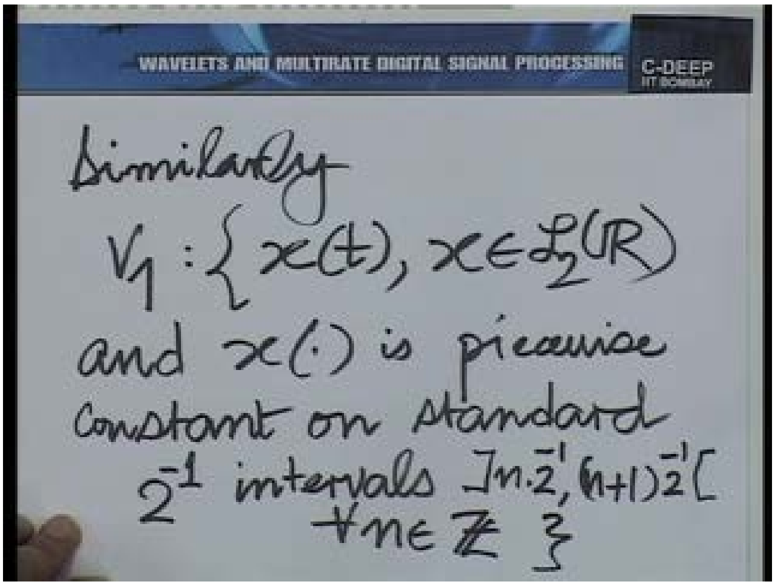

So, we could similarly, have V 1 is the set of all x. Let us define it, the set of all x x belongs L 2 R and x is piecewise constant on standard 2 raise the power minus 1 intervals that is intervals of the form n into 2 raise the minus 1, n plus 1 into 2 raise the minus 1 for all n integer.

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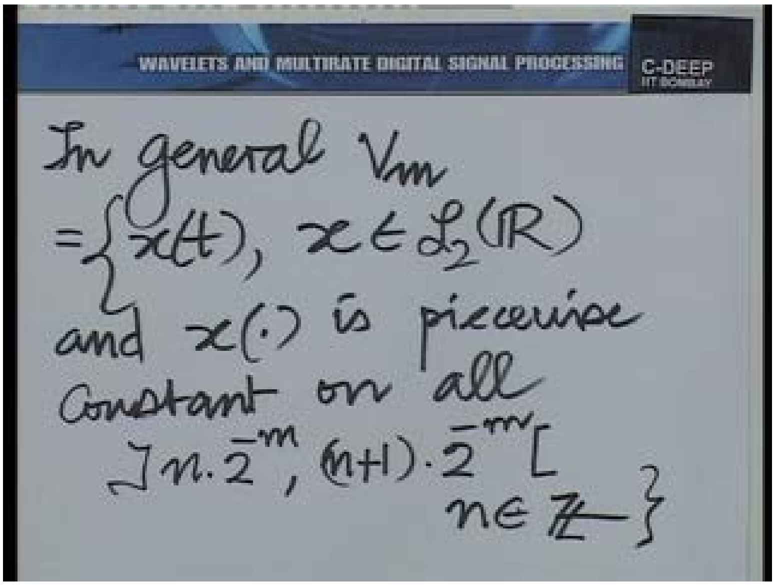

And therefore, we have V m, the set of all the x t for completeness, we should write down the definition properly and x is piecewise constant on all open intervals of the form simple enough. To fix our ideas let us sketch a couple of examples.

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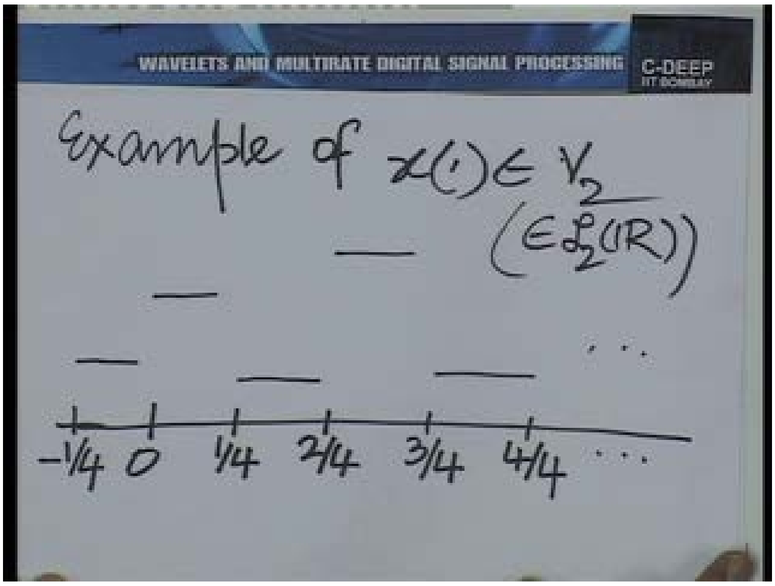

So, let us take an example of x belong into V 2, it would look something like this. You would have intervals of one-fourth here and in fact to be complete we should also include intervals before 0 and so on. We have piecewise constancy on this and so on there, and please remember x is also in L 2 R. So, when you say it is in V 2 is automatically of course, in L 2 R and that means that if I take the sum squared of all these constants at sum squared is going to be finite that is an important observation. The constants that we assign here must be such that when we sum the square of all of them, magnitude square of all of them that is sum must converge. This observation is important that I think we should make a note of it.

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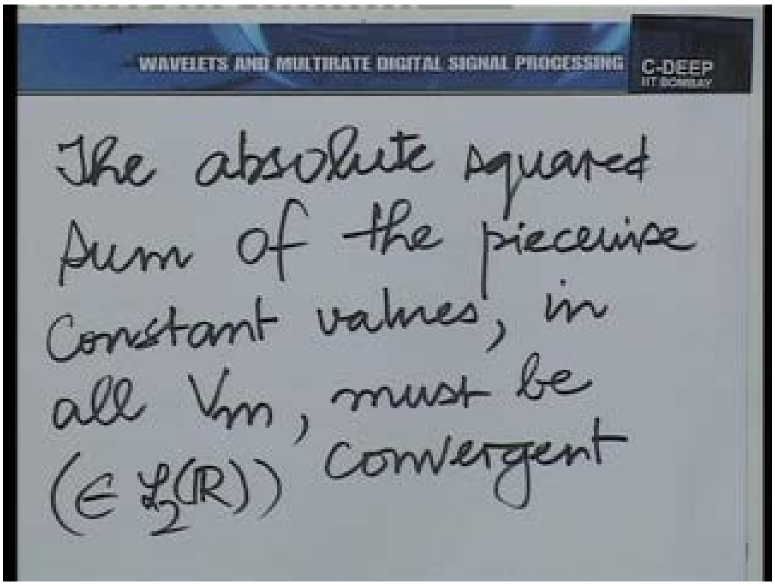

So, we are saying the sum squared, the absolute squared sum of the piecewise constant values in all these v m must be convergent and this follows from belonging to L 2 R.

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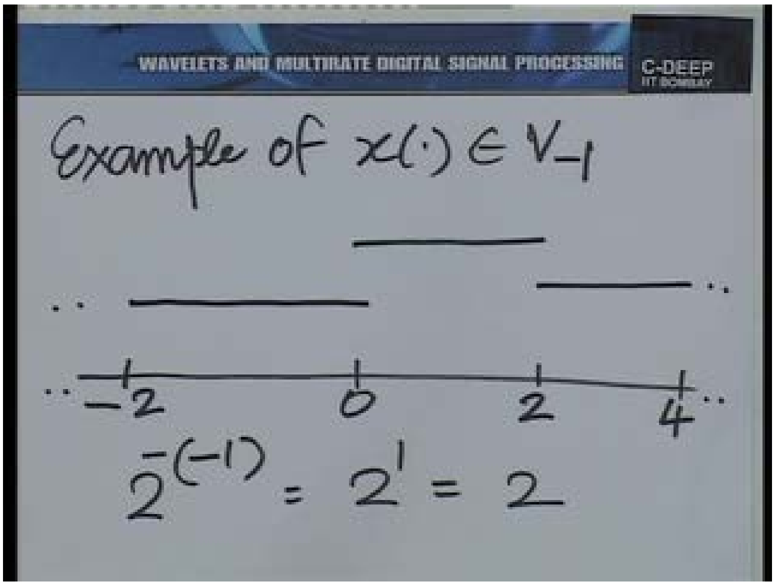

Let us also take an example of a function belonging to v minus 1. So, minus 1 means intervals of size minus of 2, 2 raise the power minus of minus 1, intervals of size 2 and so on there and so on there and we have piecewise constant there and so on here and so on there. So, now we get our ideas fixed, what we mean by the spaces V m. At the moment we put down these spaces with this example so clearly we see a containment relationship, there is the relation between these spaces they are not arbitrary they are not just totally disjoint and unrelated. In fact you can notice that if a function belongs to V 0 for example, which means that it is piecewise constant on the standard unit intervals. It is also going to be piecewise constant on the standard half intervals. And for that matter if a function belongs to V 1 which means that it is piecewise constant on the standard half intervals, it is automatically going to be piecewise constant on the standard one-fourth intervals.

To exemplify this, let me go back to this example of x belonging to V minus 1 that I have here notice that this function is piecewise constant on the standard intervals of size 2. So, obviously if you take this standard intervals of size 1 for example, 0 to 1, 1 to 2, 2 to 3, 3 to 4, minus 2 to minus 1, minus 1 to 0 and so on. The function is still piecewise constant. Therefore, a function that belongs to V minus 1 automatically belongs to V 0, a function that belongs to V 0 automatically belongs to V 1.

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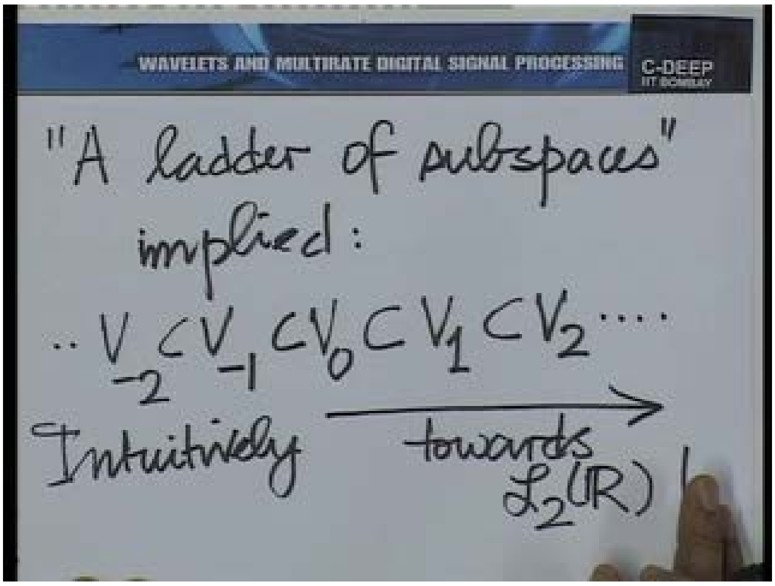

And therefore, there is a ladder of subspaces that is implied here. What is that ladder? The space V 0 is contained in V 1, the space V 1 is contained in V 2 and so on, this way and of course, the space V minus 1 is contained in V 0 the space V minus 2 in V minus 1 and so on. And we expect intuitively as we move in this direction we should be going towards L 2 R. Of course, it is an important question what happens when we go in this direction that is interesting we will spend a minute now and reflect on that. So, you see what happens when we go leftwards.

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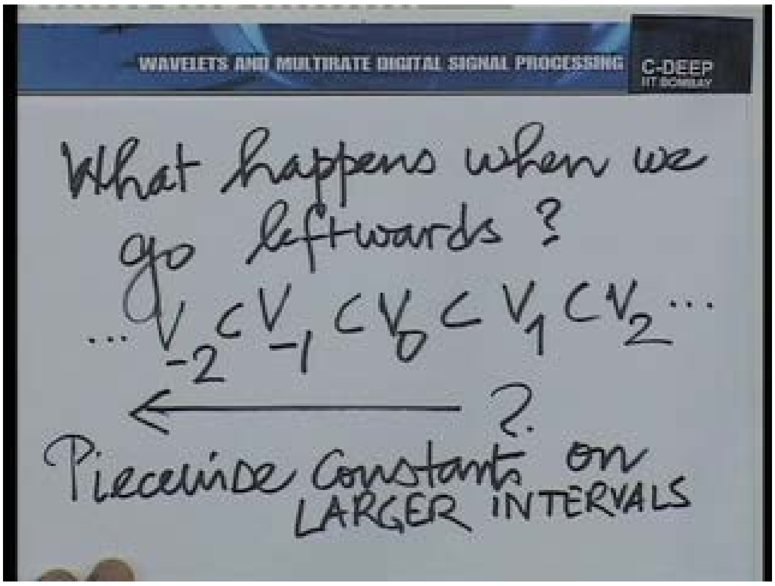

what I mean by that is first you have V 0 contained in V 1 contained in V 2 and then V minus 1 contained I mean contained in V 0, yes and so on here and so on there. What happens when we go this way? What do we think should happen, what are we doing we are taking piecewise constant functions on larger and larger intervals. Let us write that down the piecewise constant function on larger intervals.

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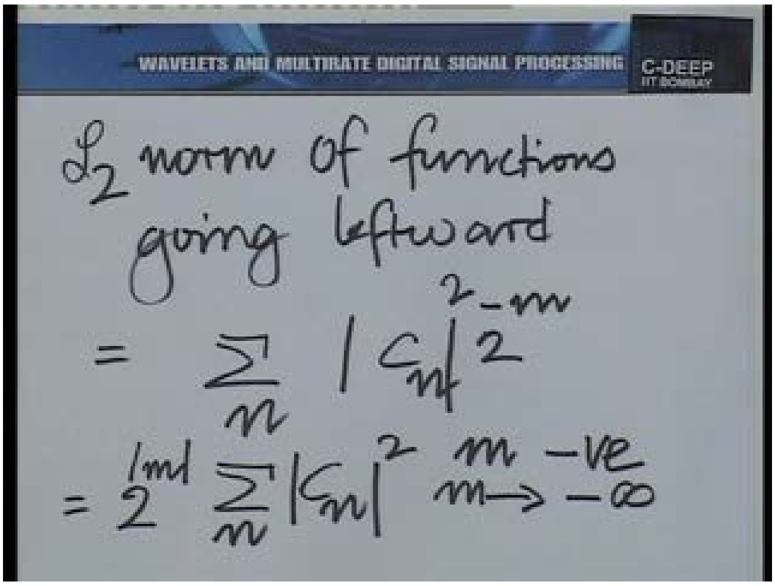

Now, you see what is the L 2 norm of functions as you go leftwards, what kind of a form will it have, it is going to have a form like this summation on n. Now, you see remember the L 2 norm is the integral of the absolute squared of the function and please remember the function is piecewise constant. So, you have one constant let us call it C n on the n th interval and the interval is of size 2 raise the power minus m.

So, this is essentially, you know you are talking about integrating mod C n squared. It is a constant over an interval of 2 raise the power of minus m and please remember m is negative and m goes towards minus infinity as you go leftwards. So, that is the same thing as 2 raise, now you see 2 raise the power minus m is 2 raise the power of mod m in the context of negative m and summation on n mod c n squared. Now, you see the scuttle point z if this needs to be finite irrespective of how large m is, we have no control on this except that this part must be finite. But, then when we say finite if it is non zero and if we allow m to go without bound this is going to diverge. The only way in which this can converge no matter how large I mean large in the sense large in magnitude, how large in magnitude m is no matter, how large in magnitude m is if this is to converge then this must be 0, a very important conclusion.

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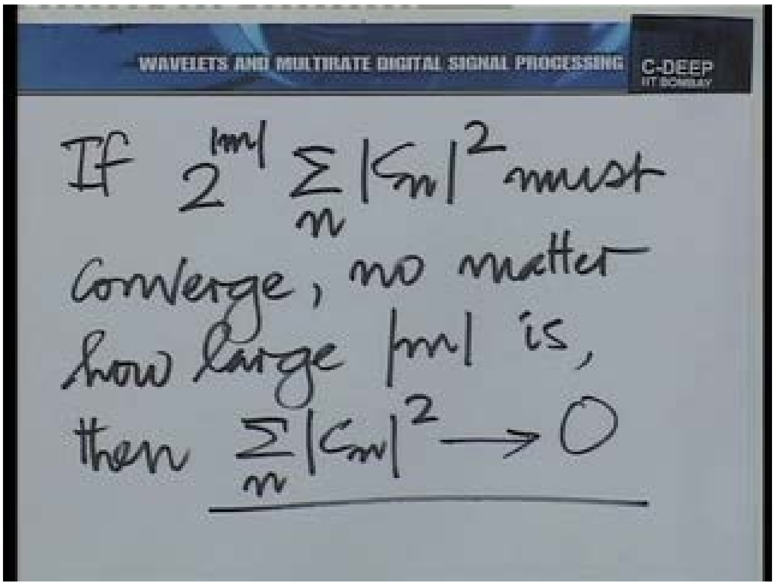

So, we are saying that if 2 raise the power mod m summation n mod c n squared must converge no matter how large or how negative. Then we must have summation over n c n squared tending to 0. So, essentially, what we are saying is as we move leftwards we are going towards the 0 function, a point that takes a minute to understand but it is not so difficult as you can see. So, now we have very clearly an idea of our destination as we move up this ladder towards plus infinity and as we move down the ladder towards minus infinity, and we can formalize that.

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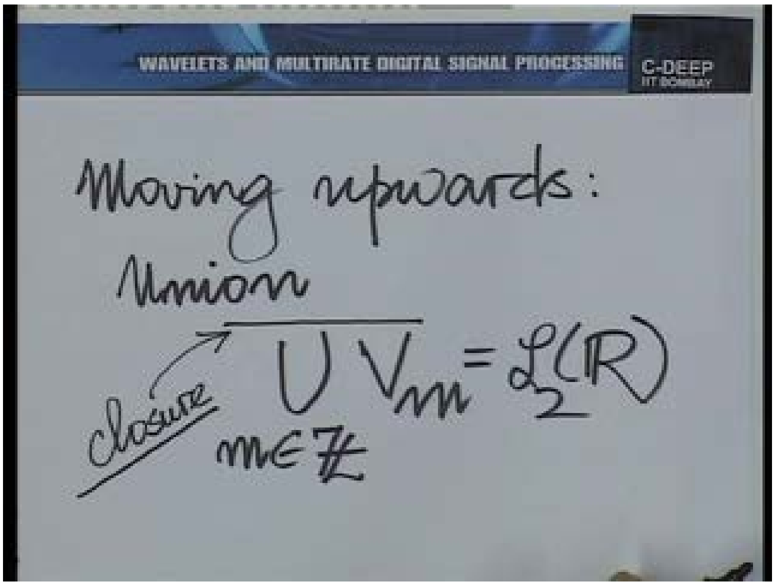

What we are saying is moving upwards now you know one has to use proper notation, we would have been tempted to say something like limit as m tends to infinity or plus infinity or something like that but, you see it is not really correct to talk about limit of sets. So, we need to use that notation that is appropriate in the context of sets namely union. So, when we take a union of 2 sets and if one set is contained in the other we are automatically taking the larger set. Moving upwards is attained by using union. In other words we are saying the union of V m, m over all the integers should almost be L 2 R now that is where the little catch is I mean we would have been happy to write is equal to L 2 R. But, you know we need to make a little detail here we need to put something called a closure. I will explain what I mean by a closure.

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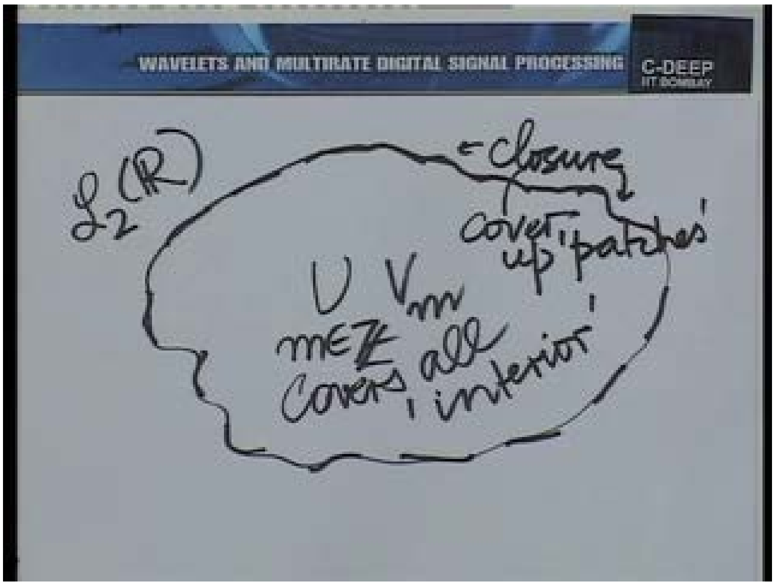

Suppose, you were to visualize L 2 R to be like an object with a boundary, these were L 2 R just notional and this is the boundary of L 2 R, so to speak. So, it is a space you know. Now what we are saying is as we go in union that is union m over all integers of V m it would cover all the inside covers, all the interior but then it might leave out some peripheral things on the boundary. So, it may also cover some part of the boundary.

Now of course, do not ask me at this stage what we meant by boundary and interior. Say that you know we are talking about situations in a boundary. You know informally when you say boundary you are talking about functions where moving in a certain direction does not remain L 2 R moving in the other one does. So, you know it is see this boundary and the interior at the movement needs to be understood only informally but what we are saying is as far as this union goes it can take you almost all over L 2 R. It covers all the interior, it may also cover quite a sizeable part of the boundary but it might leave some patches of the boundary untouched and therefore, when we do closure we are covering up those patches. What we just did was covering up those patches. So, closure means cover up boundary patches.

Now, this is a small detail and we need not spend too much of time in reflecting about this idea of closure and so on. But to be mathematically accurate we do need to note that it is after closure that the union overall m integer of the m becomes L 2 R. Otherwise it is almost L 2 R which means that when you take this union that is when you make piecewise constant approximation on smaller in smaller and smaller interface. You can go as close you desire to function in L 2 R. So, you can reduce the l 2 norm of the function to 0. So, if we look at it what we mean by that is implied by boundary.

You know, you can go as closures you like to certain function you can make the l 2 norm 0 but still it would not quite reach there. So, you know you could just visualize that you might just be a teeny weeny bit in side that boundary but not quite on the boundary. And how teeny weeny as small as you like there that is where the union takes you that is this subtle idea of closure. Anyway as I said, we do not need to spend too much of time in talking about this closure but we should be aware of this idea because when we read literature on wavelets of that matter when we really wish to put down the axioms of Multiresolution analysis properly we must be aware that this closure is required so much so anywhere.

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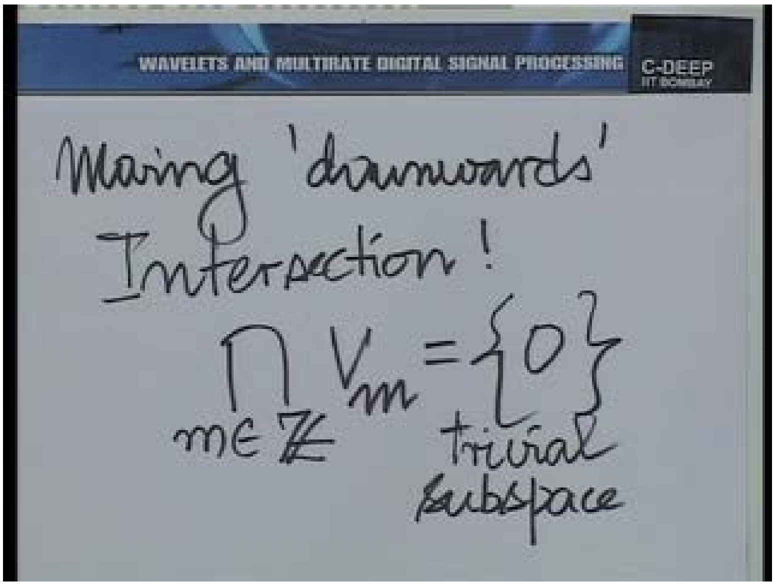

Now, let us take the second of our of our inference here moving downwards. So, how would be move downwards just as union takes you upwards intersection takes you downwards. If you take an intersection on all m belonging to Z of V m there I do not need to worry about closure and anything of that kind I can simply put down this is a trivial subspace essentially the subspace of L 2 R with only the 0 function included which is called trivial subspace.

Now, again we must make an observation here to clarify the trivial subspace is not the same as the null subspace. The trivial subspace has only the trivial 0 element in it, the null subspace does not have any element. So, that is a subtle distinction and we must bear in mind that we are talking about the trivial subspace. Now, you know yesterday I told you that there is this beautiful idea about just one function psi t its dilates in translate going all over to capture incremental information. Now, we need to state that formally too but in order to move in that direction we first need to bring in as I said another function which will span V 0. So, we need to bring in these ideas of spanning.

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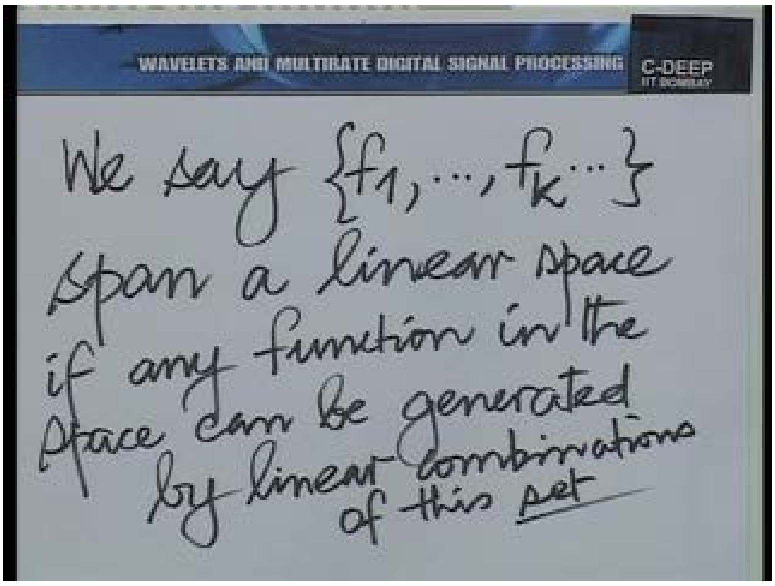

We say a set of functions we say a set of functions let us say f 1 to let us say f k and so on. Span a linear space if any function in that linear space can be generated by the linear combinations of this set. Again there this subtle distinction between finite linear combinations and infinite linear combinations. I do not wish to do well on those distinctions at the moment. But what we are saying what we mean by span, when we talk about the span of a set of functions we are talking about all linear combinations of those function. Then therefore, the set or the space of function in fact the space of functions generated by all linear combinations of that set. So, now we ask a question which should make our life easy.

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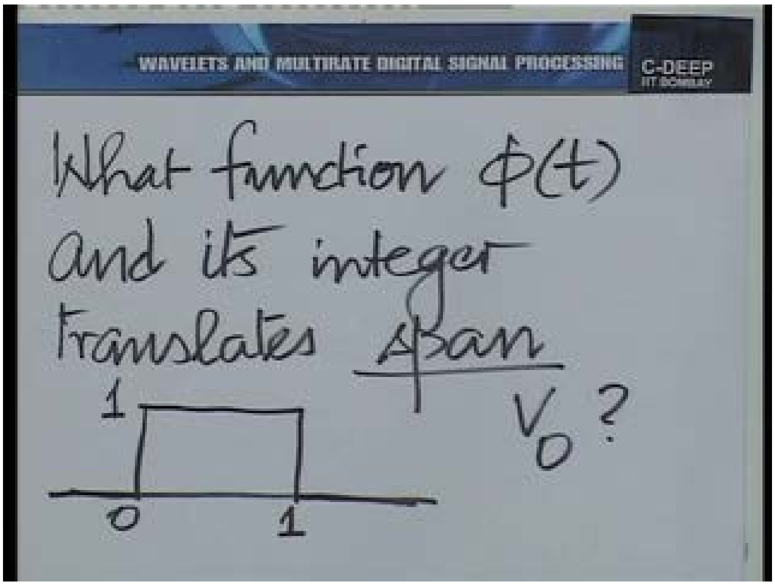

What function, we ask this question before but now we answer it, what function suppose we call it phi (t) and its integer translates. Span V 0 and the answer is very easy. In fact, if you were to visualize a function which is one in the interval from 0 to 1 and 0 else low and behold you have the answer.

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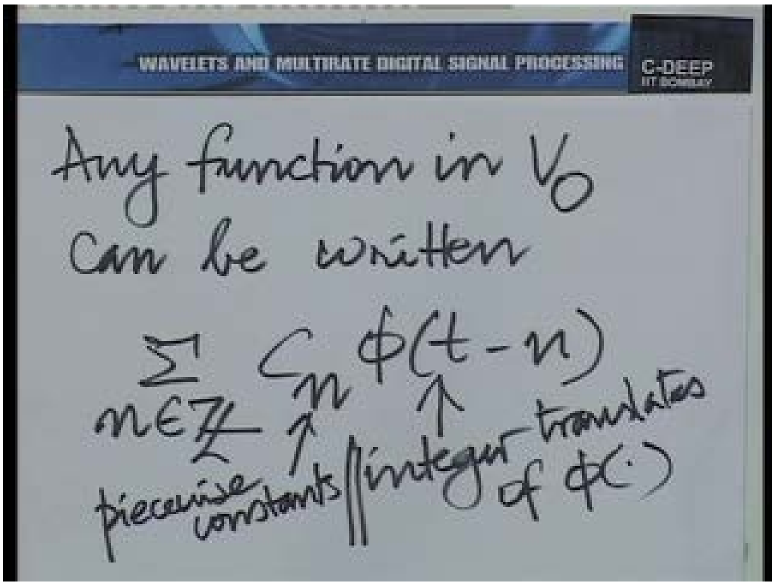

So, what we are saying is any function in V 0 can be written like this, summation n over all the integers C n phi t minus n. So, essentially integer translates of phi and these are the piecewise constants here is to fix our ideas let us take an example here.

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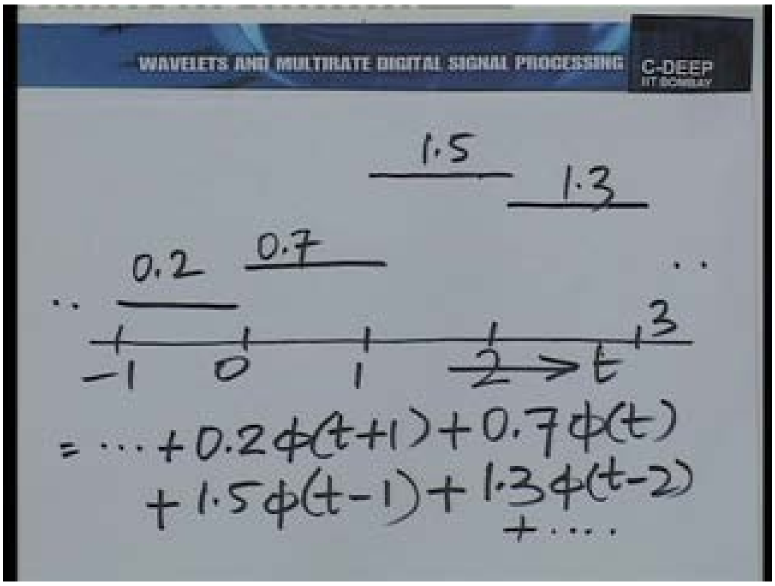

So, what we are saying is for example, suppose I have this example of a function in V 0. So, I have 0 1 2 3 and so on. The value here let us say 0.7, the value here is 1.5, the value here between 2 and 3 is 1.3. The value between minus 1 and 0, let us say is 0.2 and so on. Then and this could continue then this function can be written as well dot, dot, dot plus 0.2 times phi (t) plus 1 plus 0.7 times phi (t) plus 1.5 times phi (t) minus 1 plus 1.3 times phi (t) minus 2 and plus dot, dot, dot and so on. Simple enough, not at all difficult to understand. So, we have this single function phi t whose integer translates span V 0. Now the subtle point is that if you where to go to any space v m, the same thing would carry forth.

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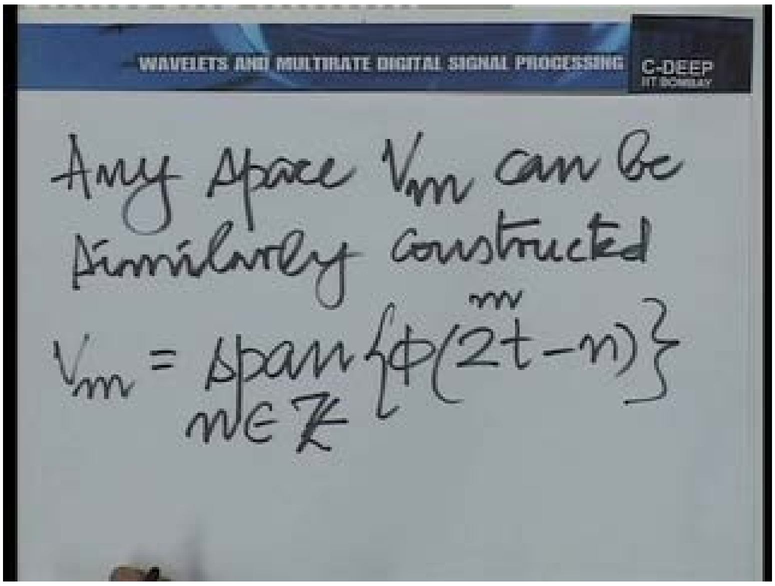

It is very easy to see that any space v m can be similarly, constructed. In fact we can be more precise, we can write down v m is essentially the span over all m belonging to Z of phi 2 raise the power of m t minus n. So, just as we looked at the wavelets yesterday and said it is the single function which can allow details to be capture, we now have this function phi (t) which captures representation at a resolution.

It is a very powerful idea, if we think about it. If you want to capture information at a resolution at a certain level of details that is all the information up to that resolution you have the function phi. If you wish to capture the additional information in going from one subspace to the next you have the function psi the wavelet. Now, we need to give this function phi t a name we shall call it a scaling function.

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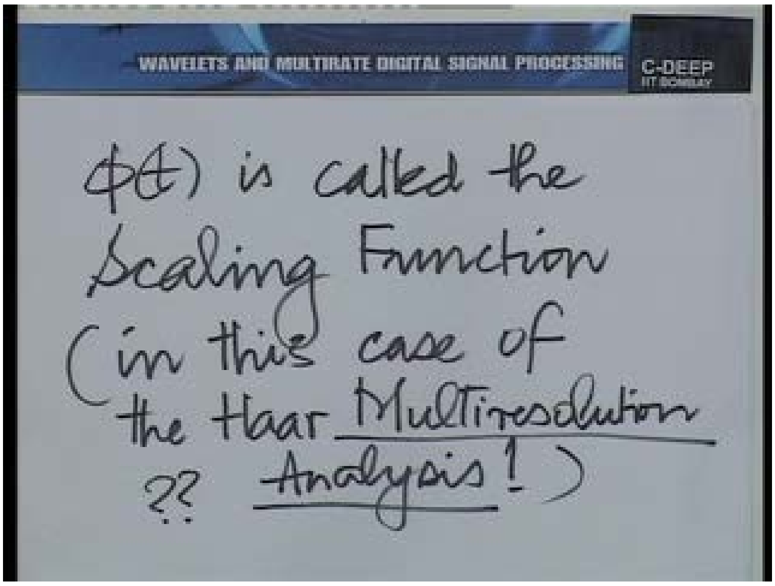

And of course, here the phi t that had drawn in this context is the scaling function of the Haar wavelet or the Haar Multiresolution analysis. Now, what is this Multiresolution analysis? I have suddenly brought in this work. So, what is this?

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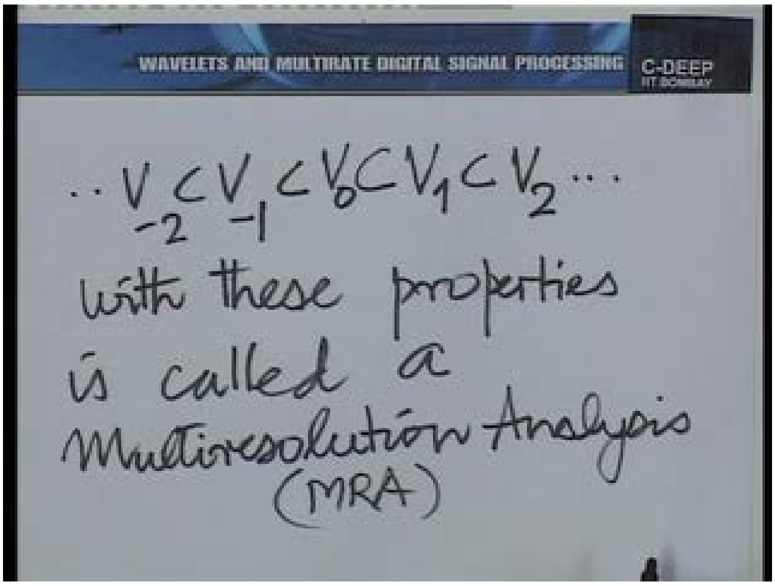

Well, this ladder of subspaces that we are talking about here with these properties is called a Multiresolution analysis or an M R A for brief of course, in this case the Haar Multiresolution analysis. Now, what properties we need to put them down formally once again. We have introduced two of them and one more subtly but we now need to put down axioms very clearly. So, let us put down what are called the axioms of a Multiresolution analysis.

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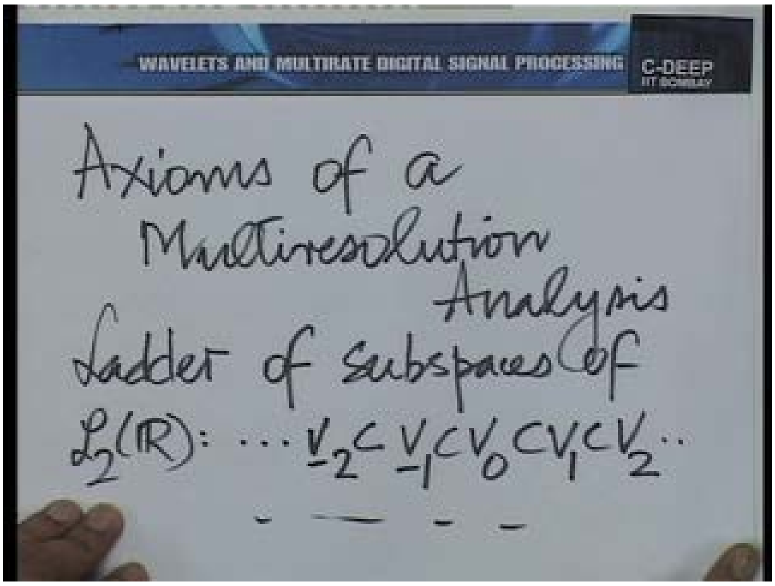

The first axiom is of course, there is a ladder of subspaces of L 2 R and we know what that ladder looks like.

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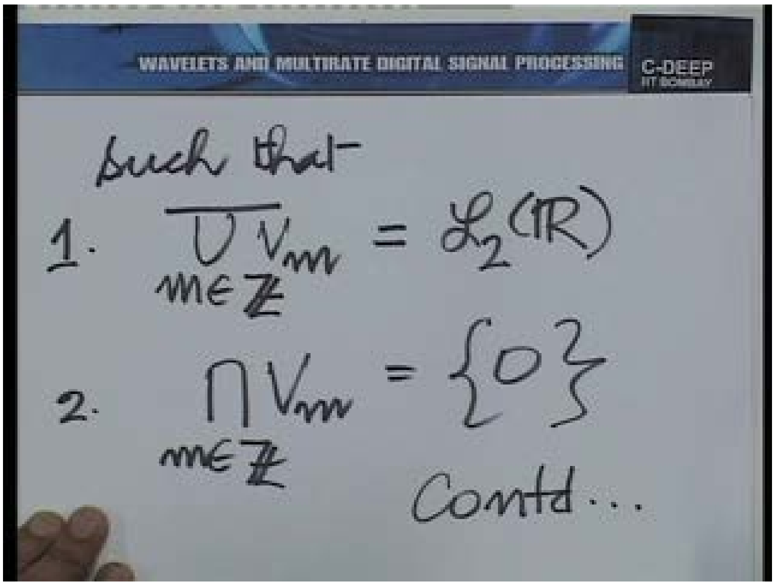

Such that axiom number 1 union over all integers closed V m is equal to L 2 R. Intersection over all integers V m is the trivial subspace with only the 0 element is a not all.

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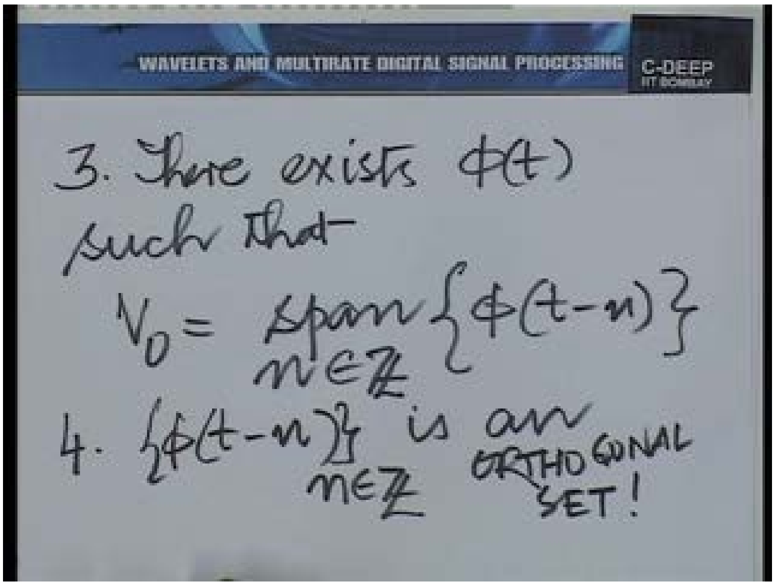

Further there exists a phi (t) such that V 0 is the span over all integer n of phi (t minus n). 0.4. In fact, you know it is not just span there is something more. This phi (t minus n) over all n is an orthogonal set this is a deeper issue here, now we will explain in more detail the notion of orthogonallity in the next lecture. But, for the moment let us pick the content to put this down as an axiom.

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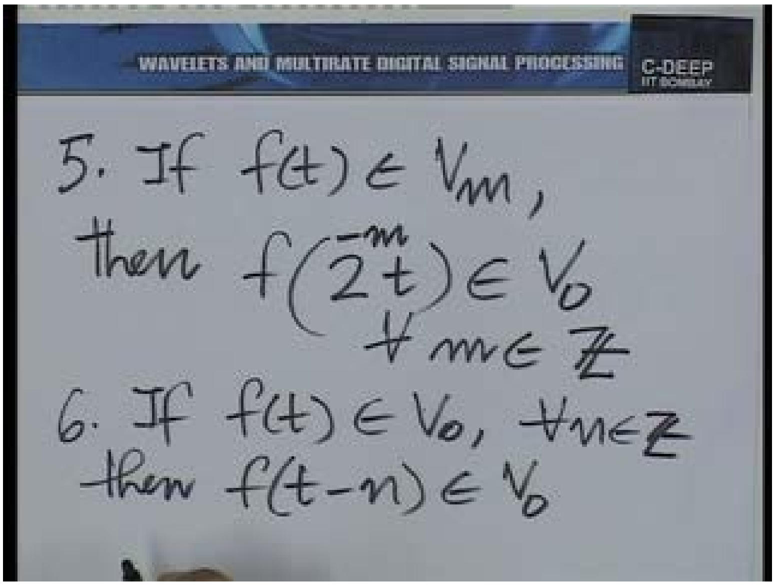

Next, if f (t) belongs to V m then f (2 raise the power of m t) a (2 raise the power of minus m t) belongs to V 0. So, for example, if f t belongs to V 1 then f t by 2 or f 2 raise the power minus 1 t belongs to V 0 for all m belonging to Z and if f t belongs to V m to V 0 then f (t) minus n also belongs to V 0 for all integer n. So, these are the axioms of a Multiresolution analysis that means this is what constitutes a Multiresolution analysis and here we have taken the Haar Multiresolution analysis to build the idea up. But, the whole abstraction is that we can have several different files and then to end these lecture the corresponding size. So, where does the psi come in it, comes in what is called the theorem of Multiresolution analysis.

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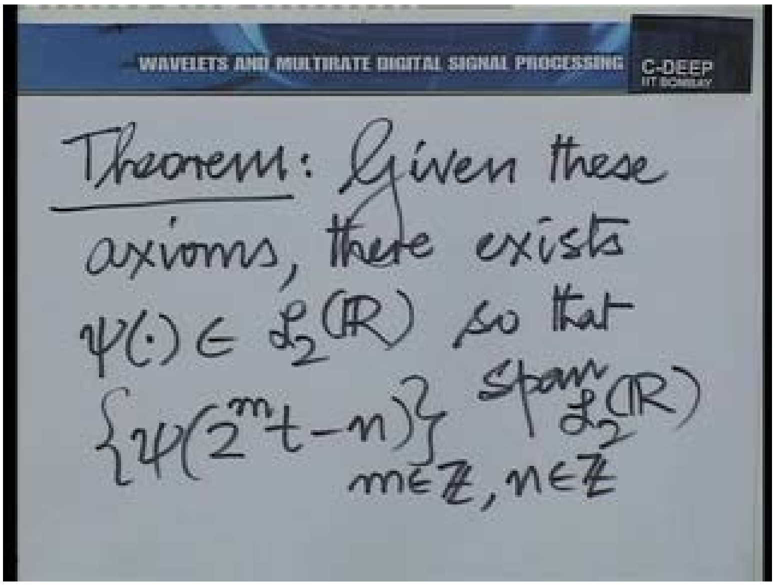

Given these axioms there exists a psi belonging to L 2 R so that psi 2 raise the power of m t minus n for all integer m and all integer n span L 2 R this is a very significant idea. In other words this is exactly what we said yesterday take dyadic dilates and translates of the functions psi, and you can cover all functions go arbitrally close to any function in L 2 R as you decide. We have built this idea from the Haar example but in the next lecture, we shall try and build a little more abstraction into what we have done, and proceed further from there. Thank you.