Advanced Digital Signal Processing-Wavelets and Multirate
Prof. V. M. Gadre
Department of Electrical Engineering
Indian Institute of Technology, Bombay

Module No.#01
Lecture No. #29
Orthogonal Multiresolution Analysis with Splines

A warm welcome, to the twenty-ninth lecture on the subject of Wavelets and Multirate Digital Signal Processing. In this lecture, we shall continue to discuss one more variant of the idea of multiresolution analysis. In the previous lecture, we had built an orthogonal or perfectly construction with one filter, by extending it to bi-orthogonal or perfectly constructions, starting the design from two filters and we had taken the specific example of the 5 3 filter bank in j peg 2000.

We saw that, when we extend it, the multiresolution analysis to two filters instead of one filter; in other words, where we built the idea of, of course, what we did was to build the filter bank, not really the multiresolution analysis underlying it.

So, we have essentially built a perfectly construction filter bank, where the filters were of unequal lengths, but, we saw that, we could get the advantage of linear phase. We could get the advantage of symmetry in the impulse response. And further, we could extend what we did in the Haar case to a piecewise linear function.

So, if we take the 5 3 filter bank and if we look at the 3 length low pass filter, the filter with impulse response the square of 1 plus z inverse essentially, it would essentially give us the triangular function as the scaling function, and the disadvantage with the triangular function was that, it was not orthogonal to all its translates. It was orthogonal, once you translate it by two units or more, but, it was not orthogonal to the 1 translate, 1 and minus 1 translate. And, this was the reason, why we needed to venture to other shores or venture to other lands, as they said, by looking at variants of the filter banks that we had discussed up to that point, and bringing in the idea of a bi-orthogonal filter bank.

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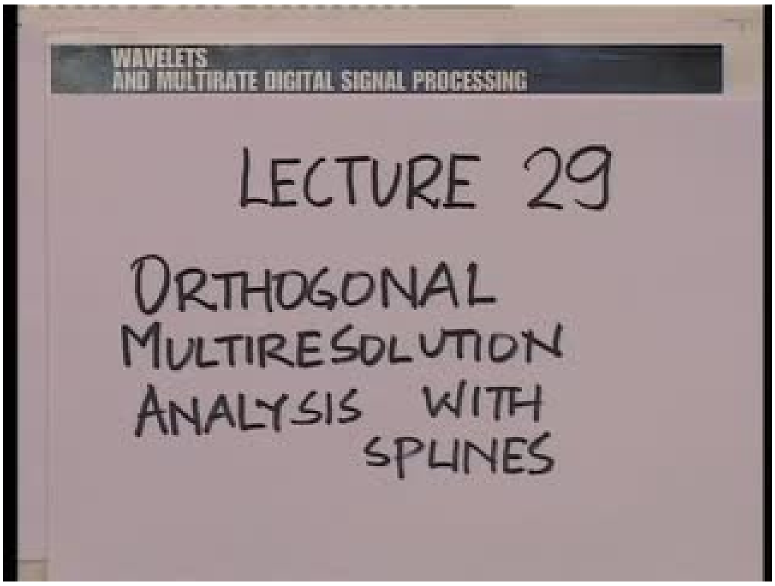

Now, today we will take again, the same 1 plus z inverse the whole squared, the length 3 low pass filter that we talked about in the 5 3 filter bank, but, we will deal with it in a slightly different way. And that would bring us to the idea of orthogonal multiresolution analysis with splines, where we need to make a compromise in the nature of the scaling and wavelet functions that we construct, and also in the nature of the filter banks that we would build, that underlie this multiresolution analysis. In fact, what is going to happen as a consequence of our demanding and orthogonal multiresolution analysis today, is that, we shall have to go from finite length to infinite length filters and with some more difficult things as we shall see. Anyway, with that little discussion to put things in perspective, let us look again at the 3 length low pass filter in the 5 3 filter bank.

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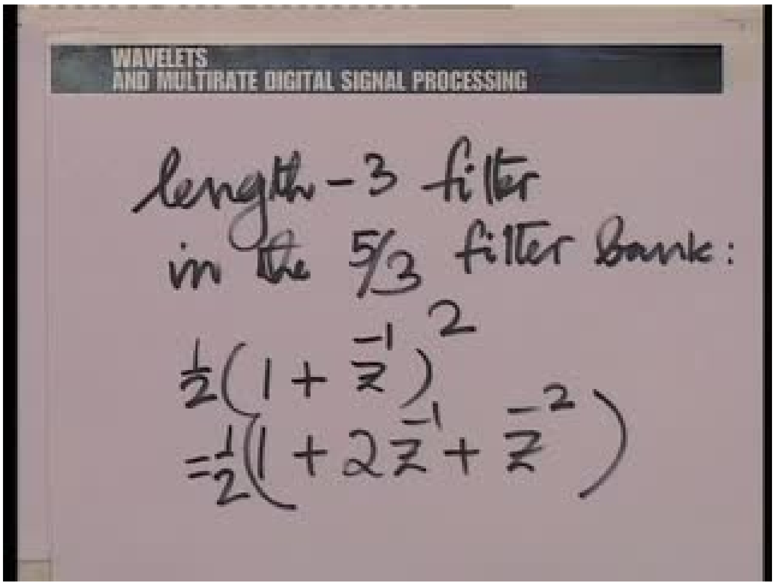

So, if you look at the length 3 filter in the 5 3 filter bank, it essentially has 1 plus z inverse the whole squared as the system function, which is 1 plus 2 z inverse plus z raised to the power minus 2, all be it with the factor of half if you like, the half is not terribly important, but, let us keep it there for the moment.

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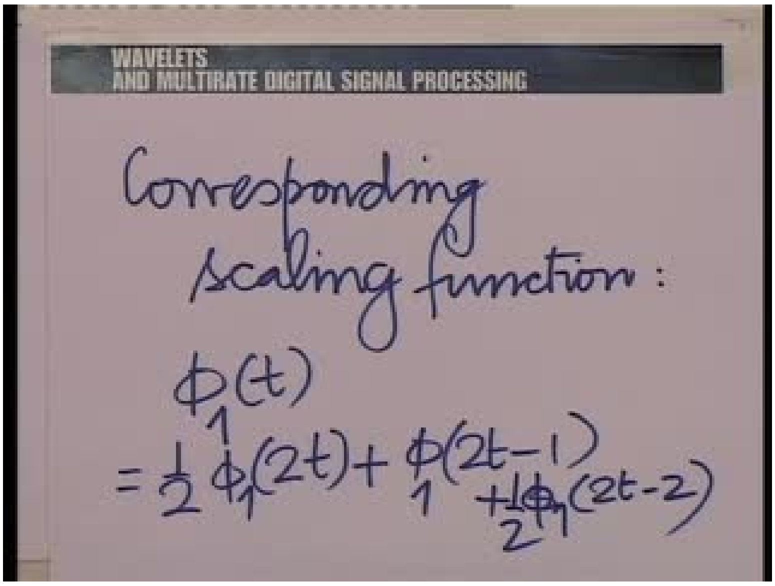

Now, we know what scaling function is generated out of this? So, in fact, let me not repeat all that discussion. We know that, the corresponding scaling function is phi 1 t, where of course, as you know, phi 1 t obeys this dilation equation, multiplied by half there and phi 1 t has an appearance like this.

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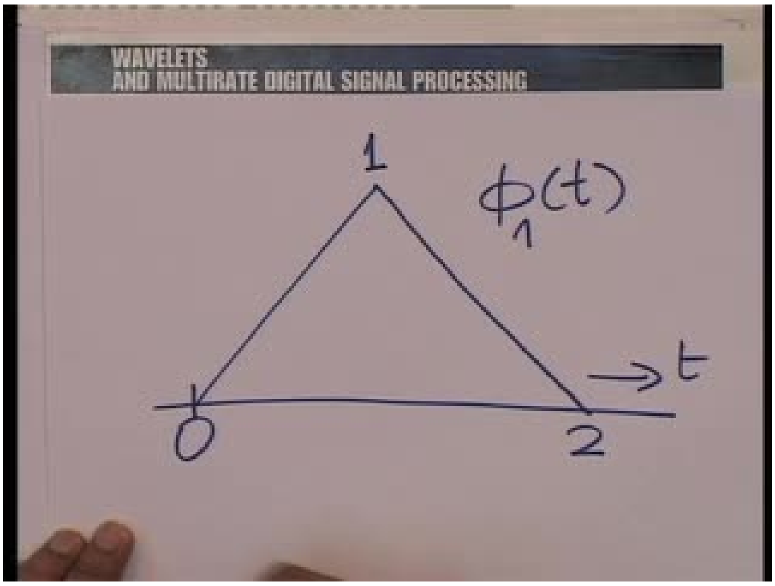

Now, our main problem and the reason why we needed to go to a bi-orthogonal filter bank as opposed to a bi-orthogonal filter, as opposed to an orthogonal filter bank, was that, this is not orthogonal to its translate by 1 unit.

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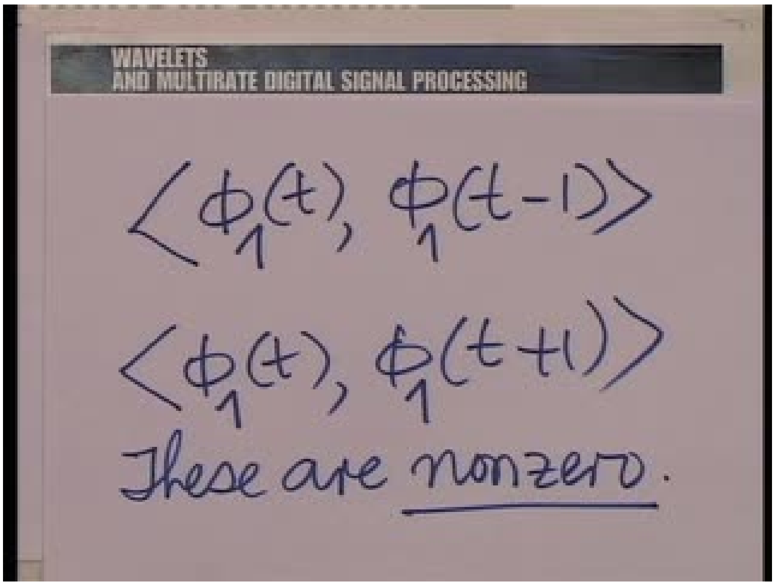

So, if I translate this by 1 unit and I take the dot product, essentially between phi 1 t and phi 1 t minus 1 or phi 1 t plus 1. That is, if I consider these two dot products, these are nonzero and that was our main bone of contention, because of which, we could not be satisfied with this phi 1 t to construct an orthogonal multiresolution analysis out of it.

Now, what we wish to do today, is to explore that possibility. Even though phi 1 t is not orthogonal to its integer translates, can we build an orthogonal multiresolution analysis, no doubt of phi 1 t, but, perhaps, out of a function that looks similar to phi 1 t. In other words, out of a function which is piecewise linear.

So, what we are trying to do… So, let me put it down in clear terms, what we are trying to do today, is to build an orthogonal multiresolution analysis with scaling functions, which are piecewise linear, even though they are not exactly phi 1 t.

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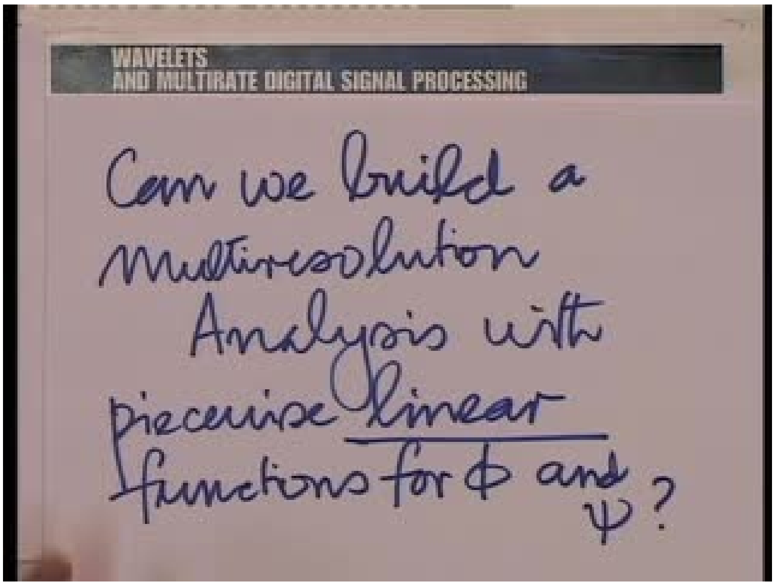

So, we are saying, can we build a multiresolution analysis with piecewise linear functions for phi and psi? That is the question which we are trying to answer today. And to answer that question, we must first relax the requirement of orthogonality as we have seen.

So, you know, if you look at the notion of orthogonality of phi t to its own integer translates, one can express this requirement, in terms of the autocorrelation of phi. So, what we are saying is, if we consider the autocorrelation function, I mean the autocorrelation of the continuous function phi t, here it is phi 1 t and if we sample this autocorrelation at the integers, all except the zero at sample must be 0, that is what orthogonality means. Let us put that down clearly.

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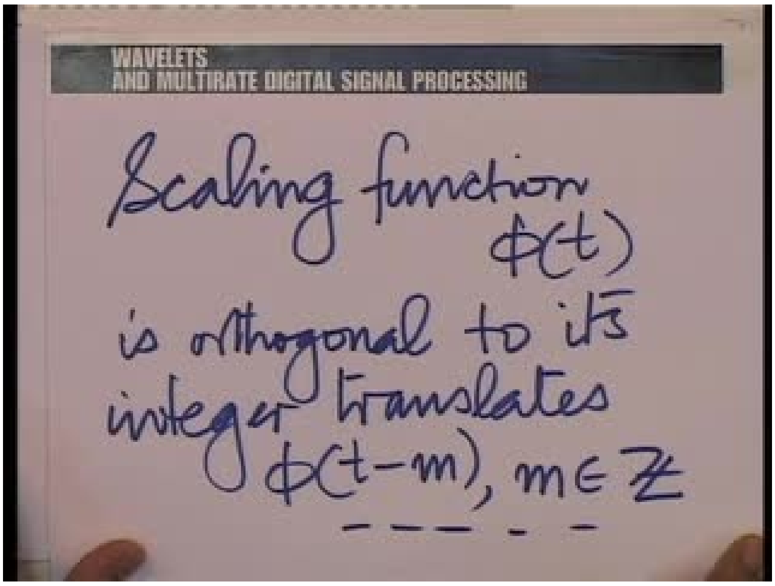

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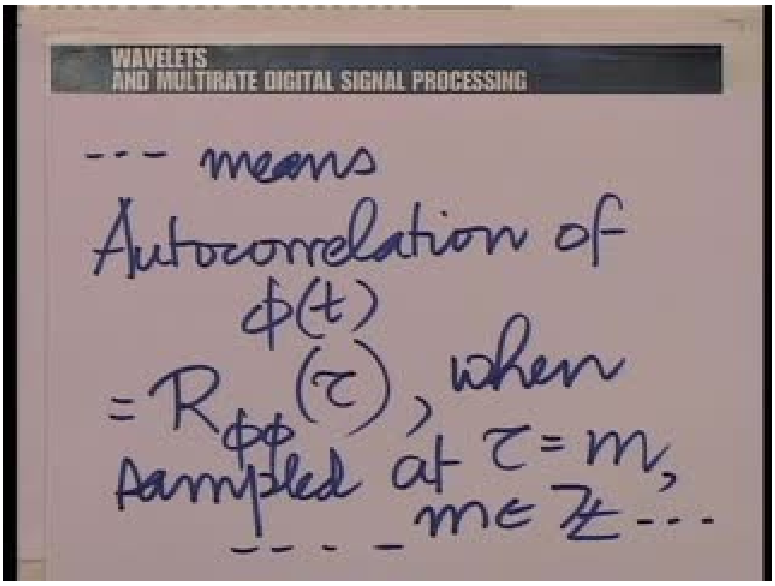

So, scaling function, let the scaling function be phi t. Scaling function phi t is orthogonal to its integer translates phi t minus m, m over the set of integers essentially means that, the autocorrelation of phi, let us denote it by R phi phi evaluated at the shift tau.

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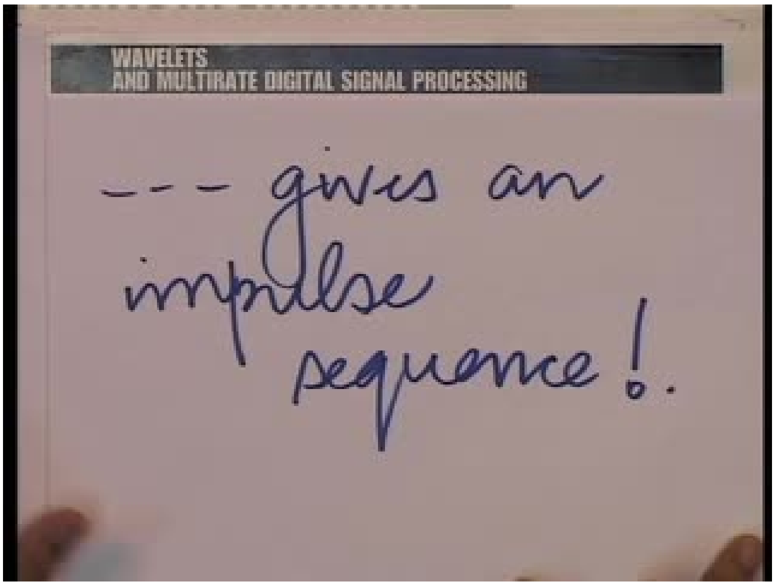

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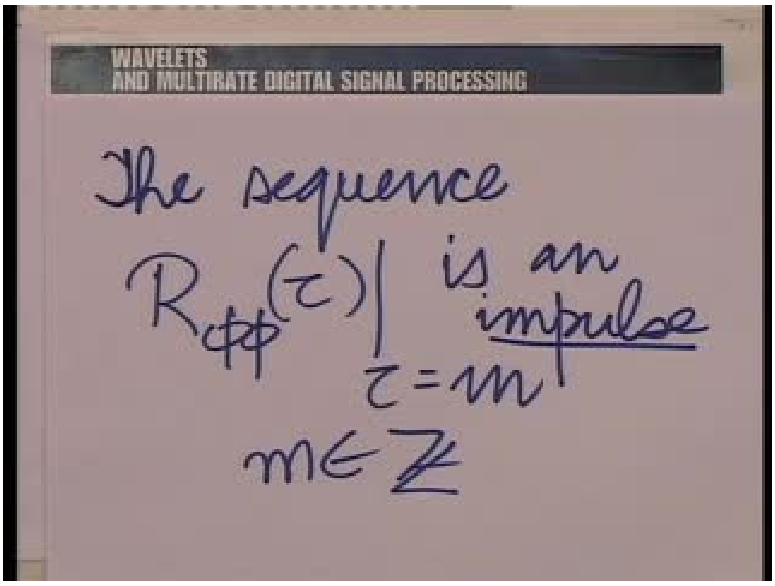

When sampled at tau equal to m, m integer, that means sample with a rate of 1 gives an impulse sequence. ((of a saying an effect)), let me put it down mathematically is, the sequence R phi phi tau, tau evaluated at m over all integers, m is an impulse sequence, discrete impulse, of course.

Now, we are ready to deal with this, in the frequency domain. So, when we sample an autocorrelation, the Fourier transform of the autocorrelation is going to get aliased. In fact, we know, what is the Fourier transform of the autocorrelation, from our basic understanding of signals and systems theory.

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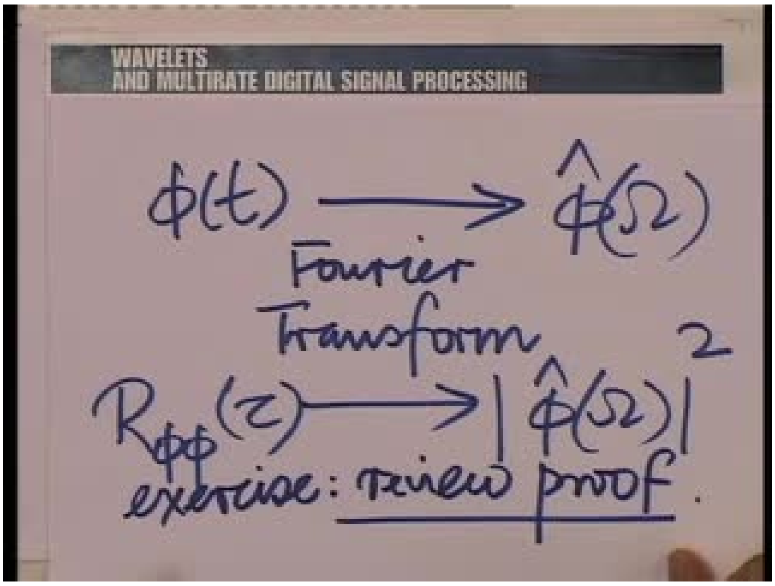

So, if phi of t has a Fourier transform, phi cap omega, then R phi phi or the autocorrelation of phi has a Fourier transform given by the squared magnitude of phi cap omega. This is a basic result in signals, systems and transforms. If you recall, the Fourier transform of the autocorrelation function is the power spectral density, in the Fourier domain. Anyway, I leave it as an exercise for you to prove this. It is the basic result in signals systems theory. Exercise: review the proof. But what we intend to do, is to use this result to our advantage.

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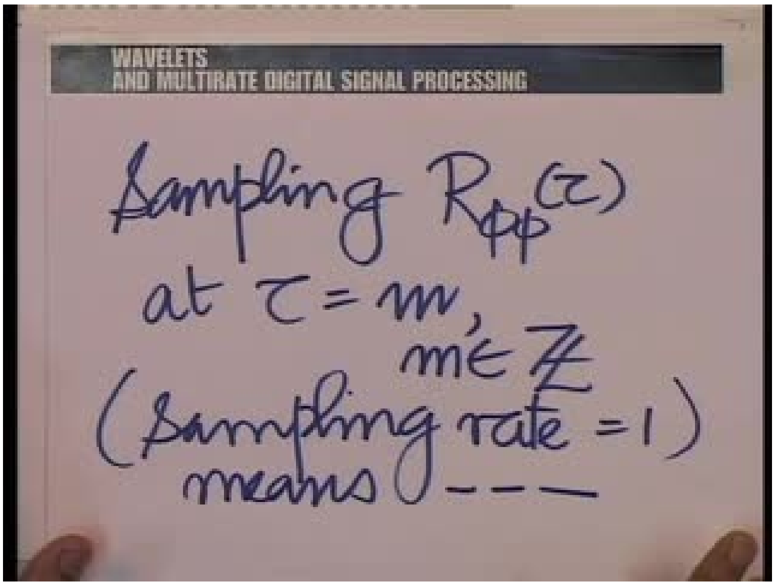

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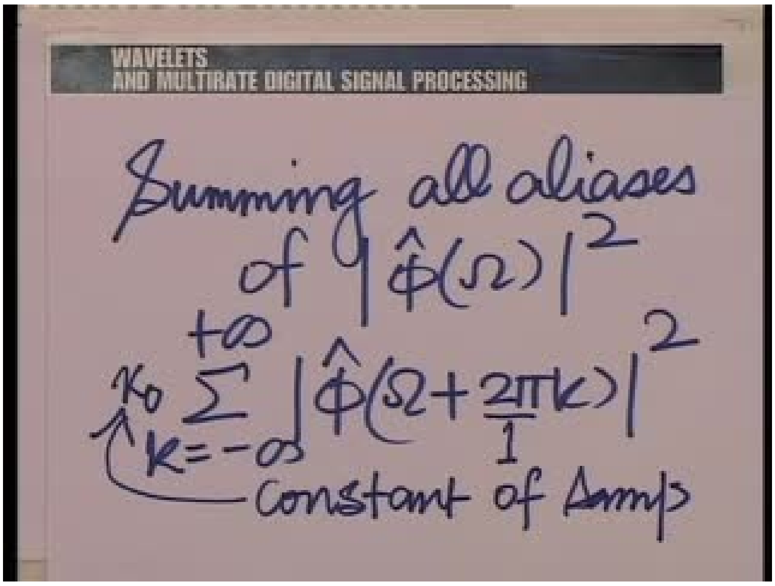

Now, when we sample, sampling R phi phi tau at tau equal to m, sampling rate of 1 essentially, means, in the Fourier domain, summing up all aliases; in other words, constructing the sum, summation k going from minus to plus infinity, phi cap omega plus 2 pi k mod squared. You know, recall what you need to do when you sample, is to shift the Fourier transform on the frequency axis by every multiple of the sampling frequency. The sampling frequency on the angular frequency axis is 2 pi divided by 1.

So, if you like, I can write 2 pi divided by 1 here, to make matters very clear and every multiple of this, every integer multiple, so, 2 pi by 1 times k for all integer k. Shift the original Fourier transform by all these multiples of the sampling frequency on the angular frequency axis and add up all these translates, add up all these aliases.

So, this is the Fourier transform, of course, there is a constant. So, you know there could be a constant here. Let us call that constant kappa 0. Constant associated with sampling. And we can just ignore that constant for the moment, even if that constant is there, we can, you know, take care of it appropriately in the rest of our discussion. So, we shall not pay too much of attention to this constant, when we discuss for that, but, we know that it is there.

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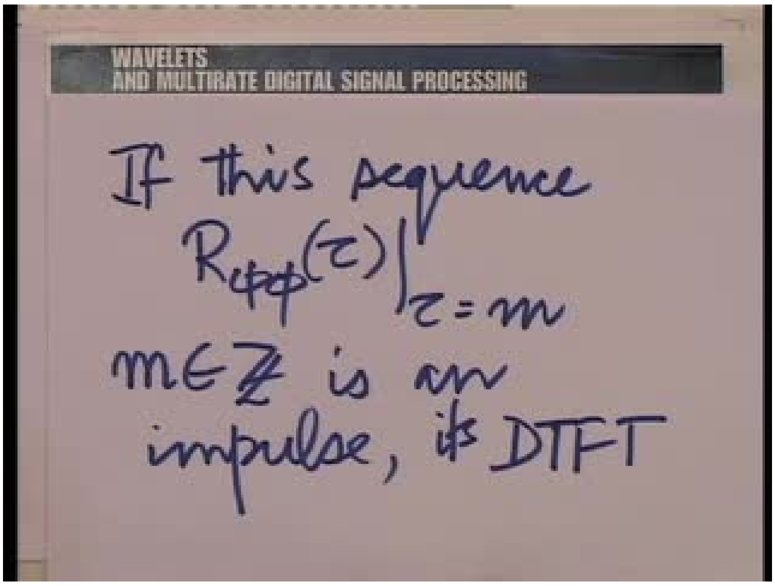

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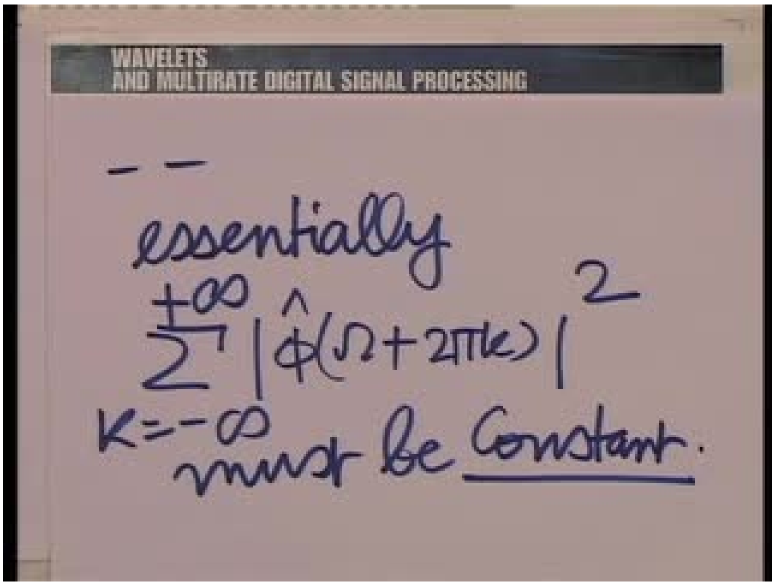

Anyway, you know, if this, if this sequence is an impulse, then its discrete time Fourier transform, we are using the abbreviation DTFT for Discrete Time Fourier Transform, its discrete time Fourier transform, which is essentially, essentially this, essentially the Fourier transform of the sampled autocorrelation must be a constant.

So, the Fourier transform, the discrete time Fourier transform of an impulse is a flat, a constant function and therefore, we now have a clear cut criterion in the Fourier domain. In order that the function phi t be orthogonal to its integer translates, we require that, this quantity, this, some of the aliases of the power spectral density, must be a constant.

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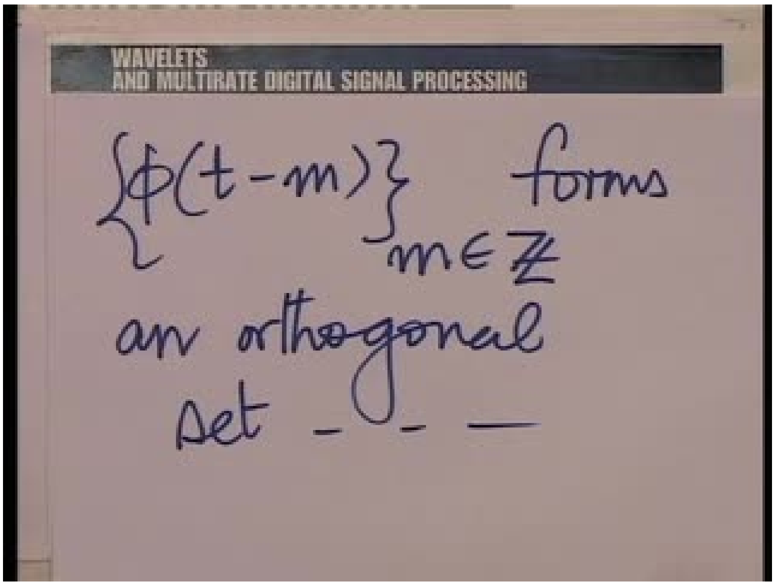

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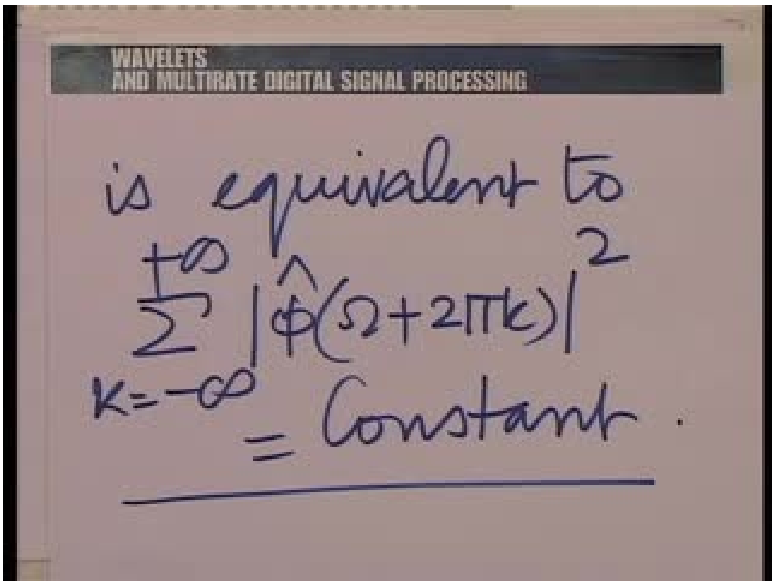

So, in other words, what we are saying, is the following, we are saying phi t is orthogonal or phi t minus m, for all integer m, forms an orthogonal set, is equivalent to summation k going from minus to plus infinity phi cap omega plus 2 pi k mod whole squared is a constant.

Well, it is not too difficult to prove this both ways. So, if this is a constant, then essentially, what we are saying is, when you take mod phi cap omega the whole squared and add it to its aliases, that means, you sample the autocorrelation at the integers, you get the Fourier transform of an impulse. Therefore, the discrete time Fourier transform is invertible. So, this result works both ways.

Anyway, the requirement of orthogonality to integer translates amounts to a requirement of, now, we introduce the term sum of translated spectrum. See, you know, remember when we were trying to discretize the scale parameter, we had brought in the sum of dilated spectrum. Here we have a sum of translated spectrum.

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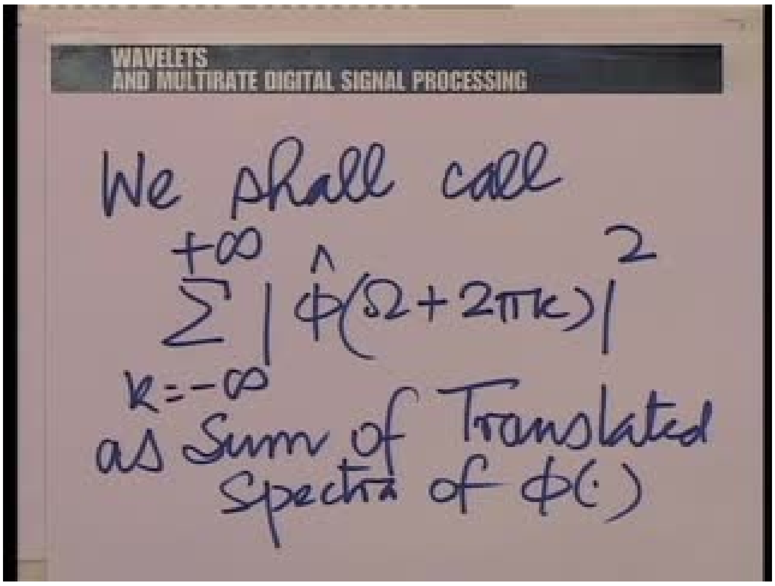

We shall call summation k going from minus to plus infinity phi cap omega plus 2 pi k mod whole squared as the sum of translated spectra of phi. You know, if you look at it, it is indeed that, you are taking the original spectrum, translating it by every multiple of 2 pi and adding up these translates, summing.

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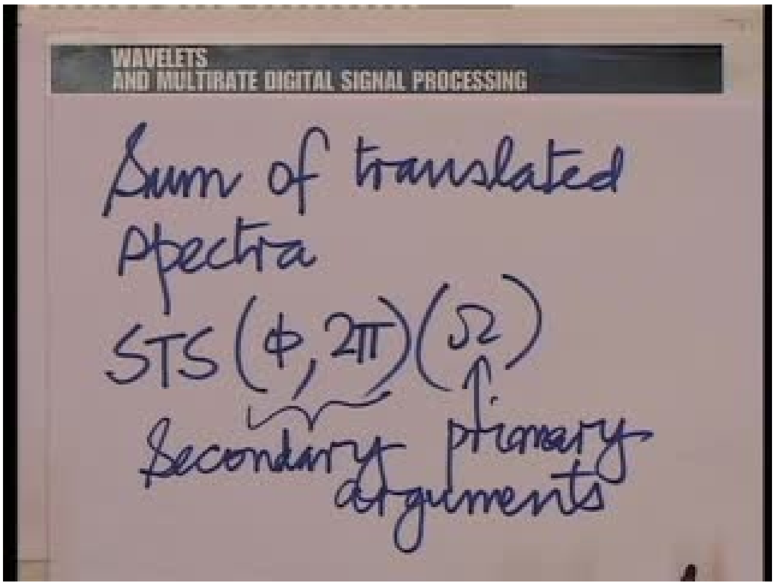

So, the name is very clear and we shall abbreviate sum of the translated spectra by S T S. So, S T S and again, we are going to have primary and secondary arguments. The secondary argument here is phi and 2 pi, and the primary argument is essentially omega, because, you are taking a sum of translated spectra of the spectrum of phi cap and the translations are all multiples of 2 pi. Secondary arguments and primary argument. The primary argument of course, is frequency, as expected.

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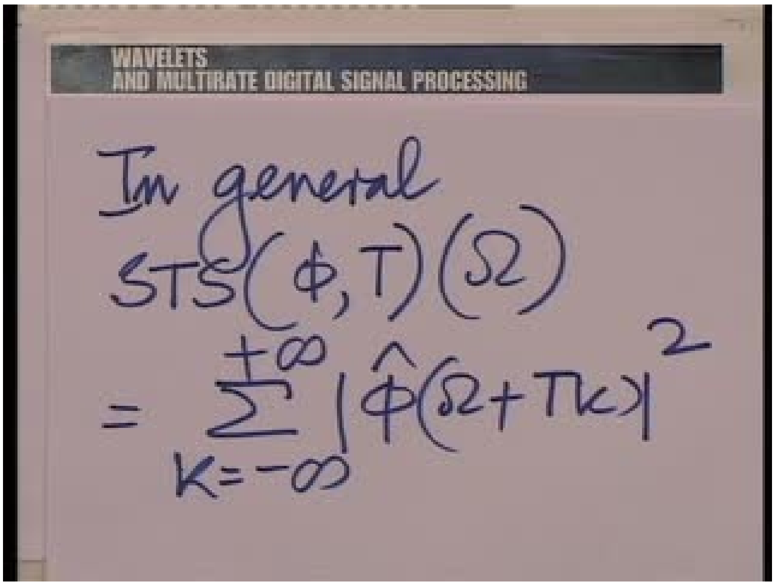

So, in general, to define this term clearly, S T S of phi with a translation T and primary argument omega is essentially, sum k going from minus to plus infinity phi cap omega plus T times k mod squared. Anyway, with this little notation introduced, we take the same strategy as we did, when we relaxed the condition for the sum of dilated spectra.

So, you know, when we talk about discretizing the scale, we need to, essentially relax the requirement of the sum of dilated spectra to be a constant, to where it is between two constants, between two positive constants. So, we said, well, even if we cannot quite get the sum of dilated spectra to be a constant, we will be happy if it is between two positive strictly nonzero and finite constants.

Now, something true, something similar will be true for this case. And in fact, now, we will also bring out a beautiful relationship between relaxation of this requirement in the tau domain or in the shift domain and in the frequency domain. Now, if we look back, it is easier to start from the tau domain. If we look back at the function phi 1 t for example, here.

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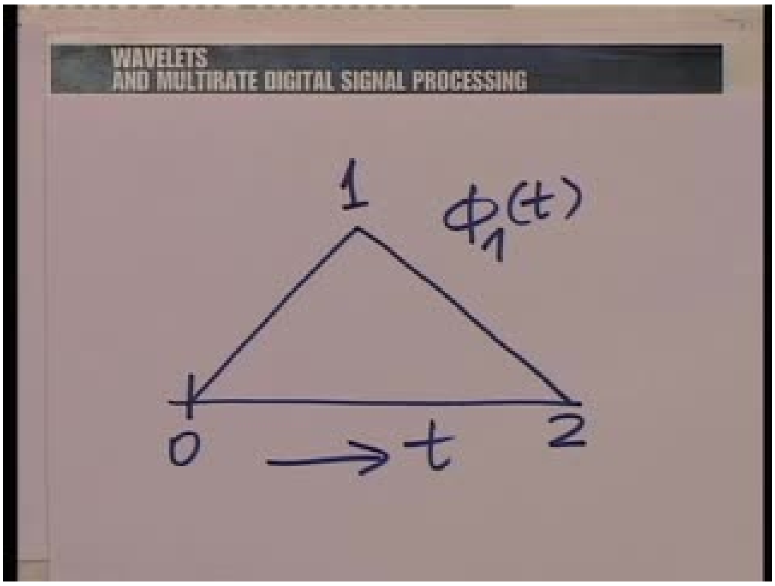

So, remember, phi 1 t looks like this. Now, if you ask me about the dot product of phi 1 t with its integer translates, the relaxation that we are asking for, is at the dot product, is of course, 0, definitely from shifts of 2 and larger in magnitude, that is shifts of 2, 3, 4 and so on and minus 2, minus 3, minus 4 and so on. But then, you know, it is only for 1 and for minus 1 that we are asking for a relaxation here. And, we can even actually calculate those 2 dot products. The dot product of phi 1 t with itself would have a certain value. It is of course, going to be the energy in phi 1 t. And if you take the dot product of phi 1 t with its translate by 1 and translate by minus 1, they are expected, intuitively, you can see, they are expected to have a smaller value.

So, in other words, the relaxation that we are asking for, is that, this autocorrelation sequence, the autocorrelation sample at the integers is not quite an impulse, but, close to an impulse. That means, it is non, it is a sequence, which is nonzero for very few values, around n equal to 0. And that manifests in the frequency domain as the sum of translated spectra, not quite being a constant, but, being between two positive constants. We shall exactly calculate these quantities now and prove what we are saying, mathematically.

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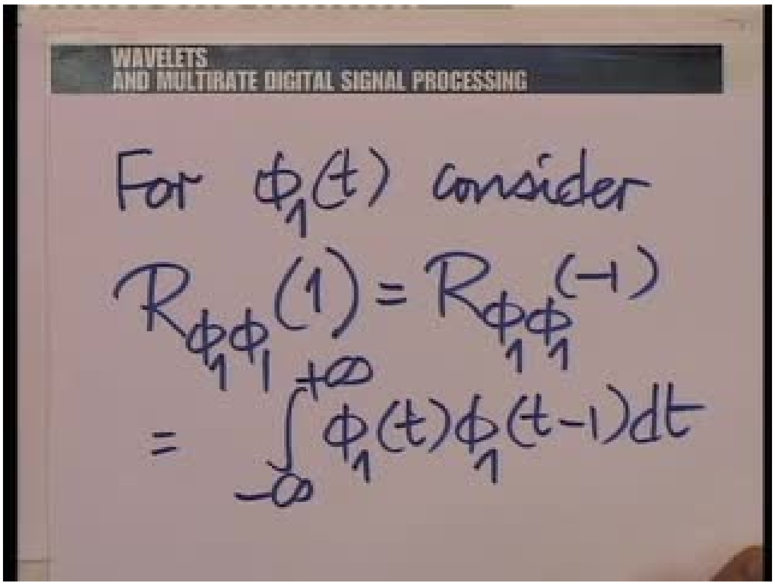

So, consider, for phi 1 t in this case. Consider the autocorrelation R phi 1 phi 1 evaluated at 1 and minus 1. It is not at all difficult to see that, they are equal, shifting by plus 1 and taking the dot product or shifting by minus 1 and taking the dot product, they give you the same answer. And this is essentially equal to, integral from minus to plus infinity phi 1 t phi 1 t minus 1 d t, which I shall calculate graphically now.

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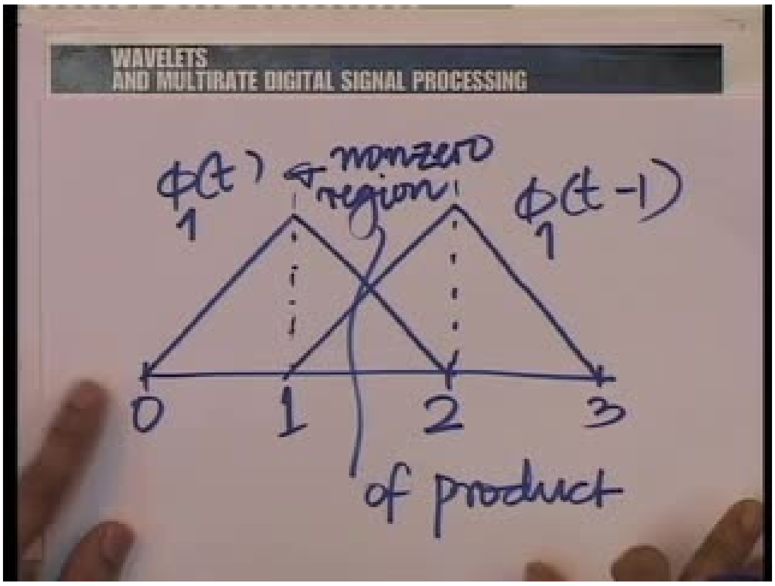

Now, graphically this amounts to finding the area under the product of the following two functions. So, this is what phi 1 t looks like and this is what phi 1 t minus 1 looks like.

So, the product is nonzero only in this region, between 1 and 2. And in fact, when we take the product and integrate, we do not really have to worry about its being between 1 and 2, I mean, it will be as well, if this is, the same thing is shifted to lie around 0.

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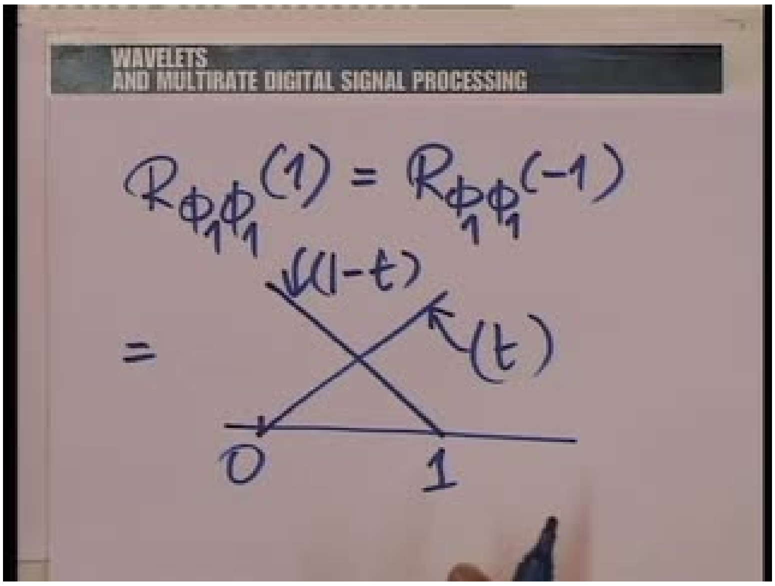

So, this integral R phi 1 phi 1 of 1 or R phi 1 phi 1 of minus 1 is essentially the following integral. You know, you could look at a function of the form 1 minus t, between 0 and

1and the function t, between 0 and 1and you could take their product, integrate between 0 and 1 and that could be, essentially this autocorrelation point.

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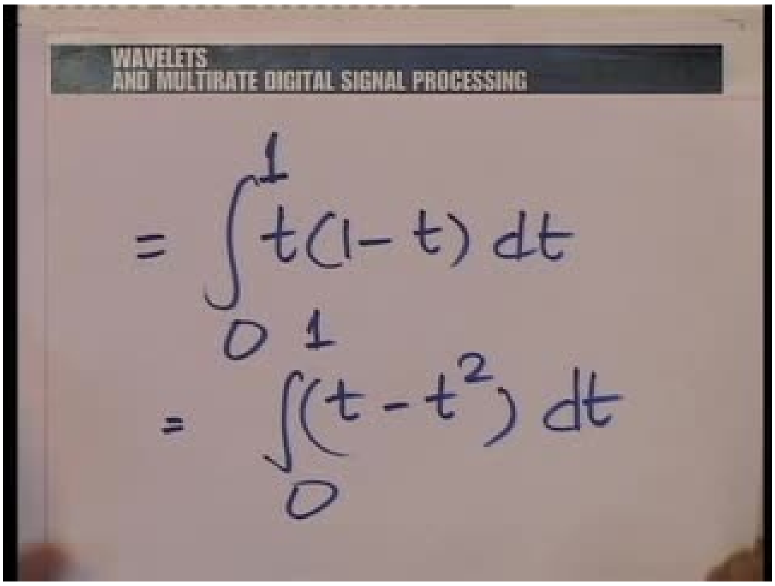

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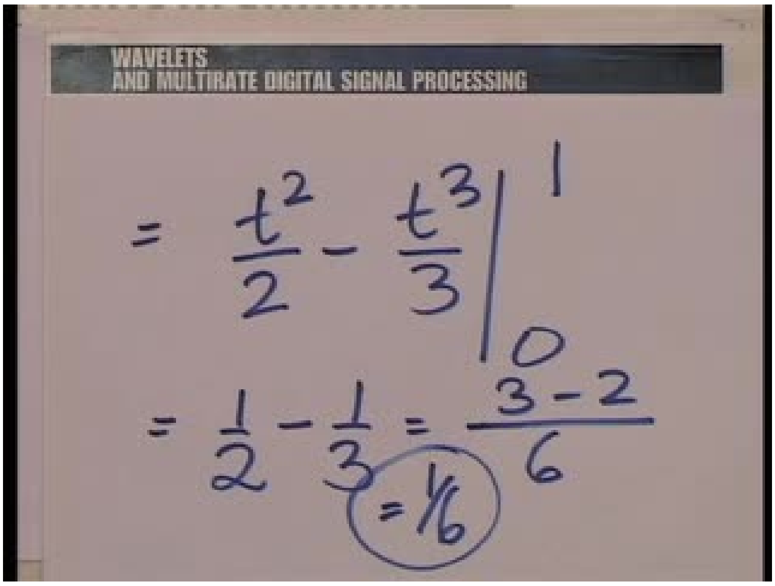

So, it is essentially, integral t times 1 minus t d t between, integrated between 0 and 1, which amounts to integrating t minus t square between 0 and 1 and that is a very easy integral to evaluate. That is half minus one-third and that is easy to see to be, one-sixth.

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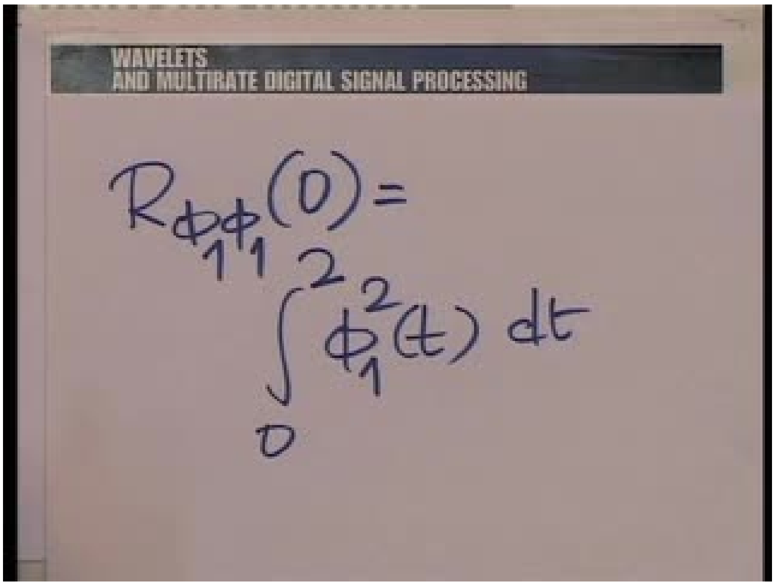

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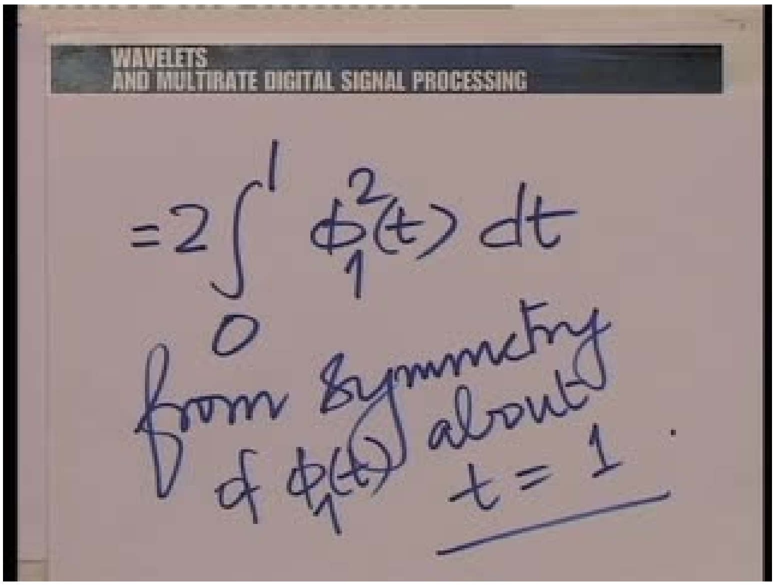

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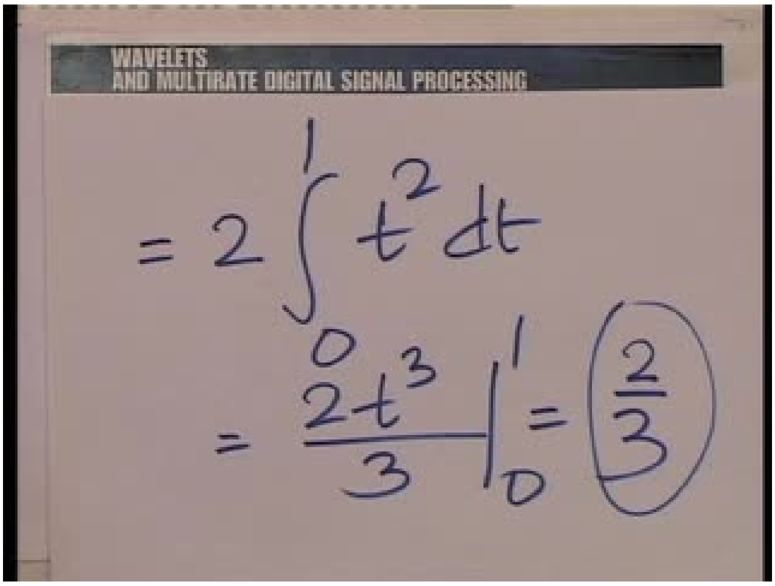

So, we have the autocorrelation at 1 and minus 1. Now, let us find the autocorrelation at 0 to complete the discussion. So, R phi 1 phi 1 at 0 is essentially integral phi 1 t into phi 1 t. So, phi 1 squared t d t, integrated of course, from 0 to 2. And that is very easily seen to be, I mean, when I, if I look at it graphically, it is very easy to see that, this amounts to integrating the 2 halves, so to speak, so it is 2 times the integral from 0 to 1 of phi 1 squared d t, from the symmetry, about t equal to 1. And this is an easy integral to evaluate. This is 2 times integral from 0 to 1 t whole squared d t.

So, that is t cubed by 3 into 2, integrated from 0 to 1, clearly equal to 2 by 3. Therefore, we have a very clear set of autocorrelation values now and we can actually find out the discrete time Fourier transform of the autocorrelation sequence.

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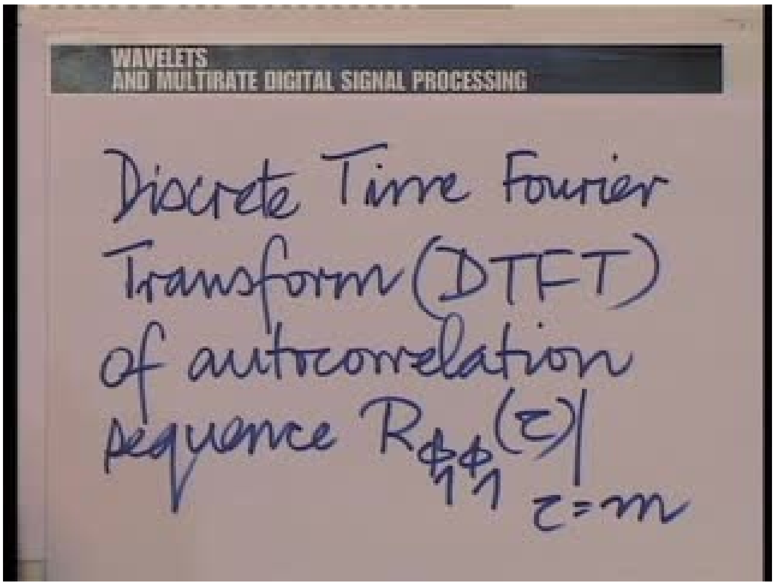

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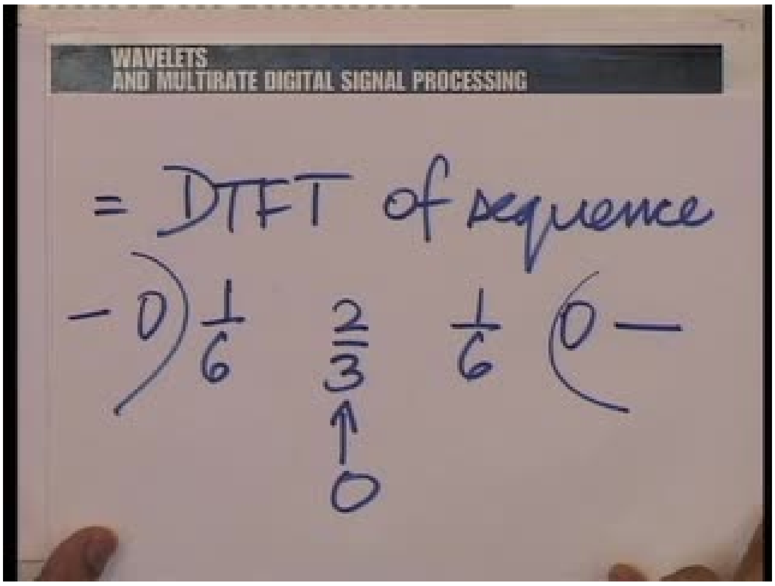

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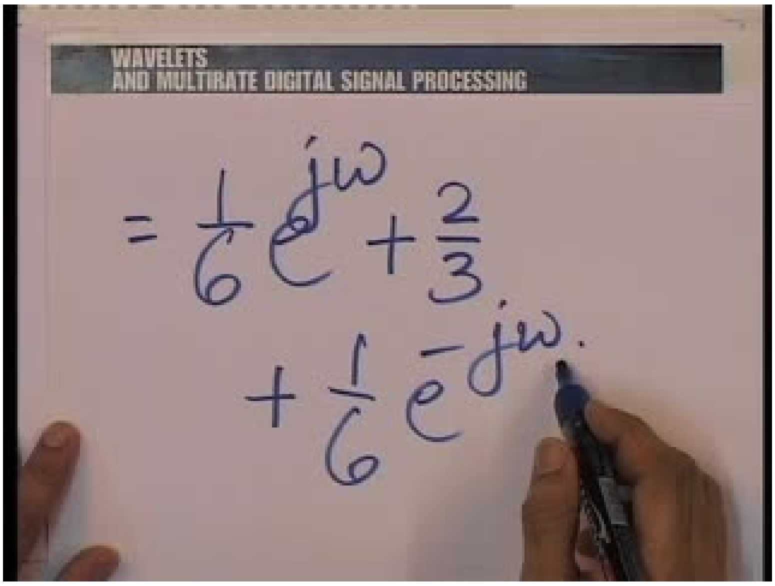

So, the discrete time Fourier transform or the DTFT of the autocorrelation sequence R phi 1 phi 1 tau, tau at the integers, is essentially the DTFT of the sequence 2 by 3 at the point 0, 1 by 6 at the point 1 and 1 by 6 at the point minus 1. And, outside that interval of course, it is 0. I would not, in fact, need to even write this. That is understood when we use a notation like this. And, this is a very easy discrete time Fourier transform to evaluate. This is 1 by 6 e raised to the power j omega plus 2 by 3 plus 1 by 6 e raised to the power minus j omega.

Now, please note, I will use small omega here, because we are talking about discrete time Fourier transform. So, I should be using the normalized frequency, but, then, I could as well, here, you know, we are interchanging the ideas of analog and discrete time and therefore, we can as well replace this by capital omega.

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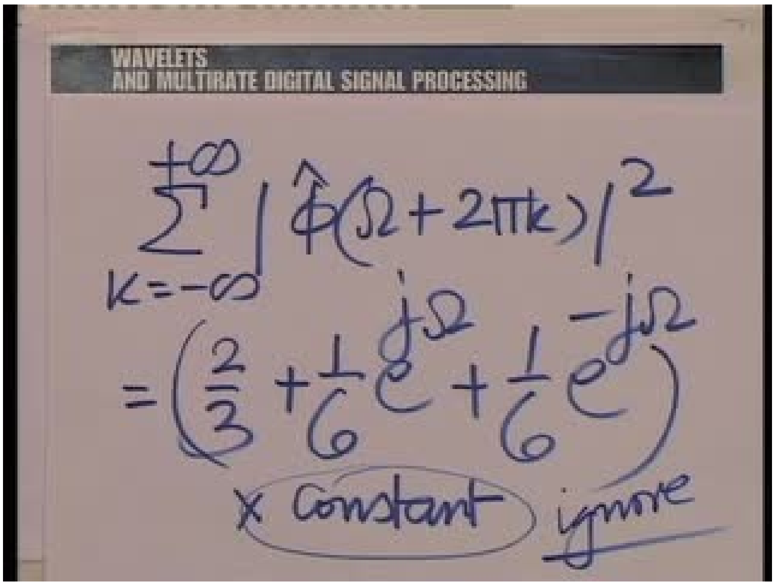

So, in fact, what we infer is that, summation k going from minus to plus infinity, phi cap omega plus 2 pi k mod whole squared is essentially 2 by 3 plus 1 by 6 e raised to the power j omega plus 1 by 6 e raised to the power minus j omega, possibly to within some constant.

So, you know, you may have to multiply this by some constant, depending on the, you know, scaling of phi 1 t, if that is the case. So, let us forget about this, ignore this, that is not of great consequence to us.

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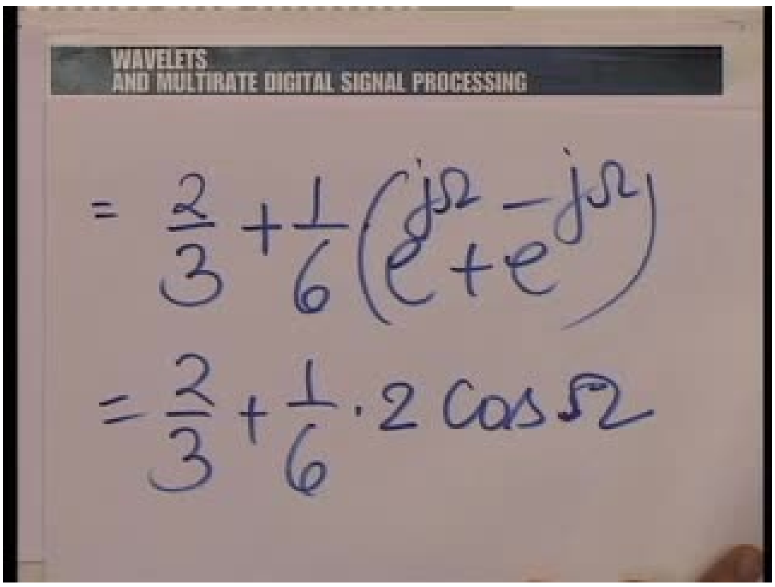

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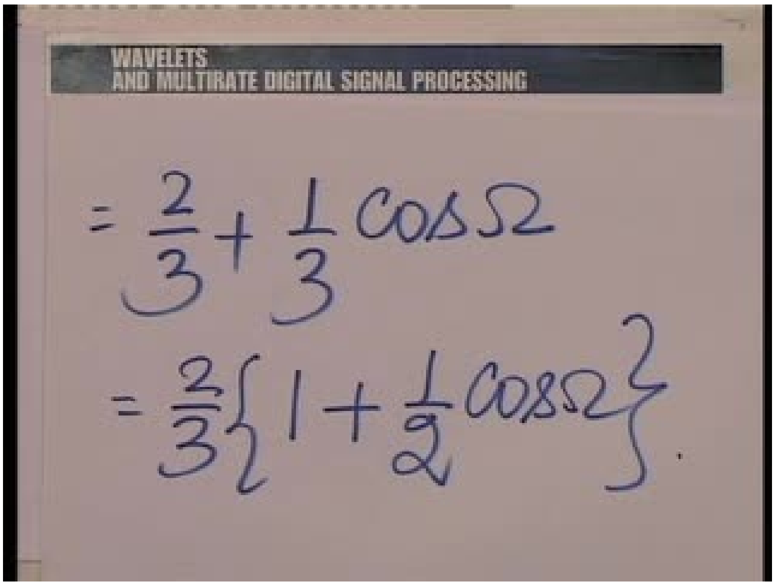

What is of consequence is only this and this is easy to expand. This is essentially 2 by 3 plus 1 by 6 into e raised to the power j omega plus e raised to the power minus j omega, which is 2 by 3 plus 1 by 6 into 2 cos omega. And, that is 2 by 3 plus 1 by 3 cos omega. Or in other words, 2 by 3 into 1 plus half cos omega.

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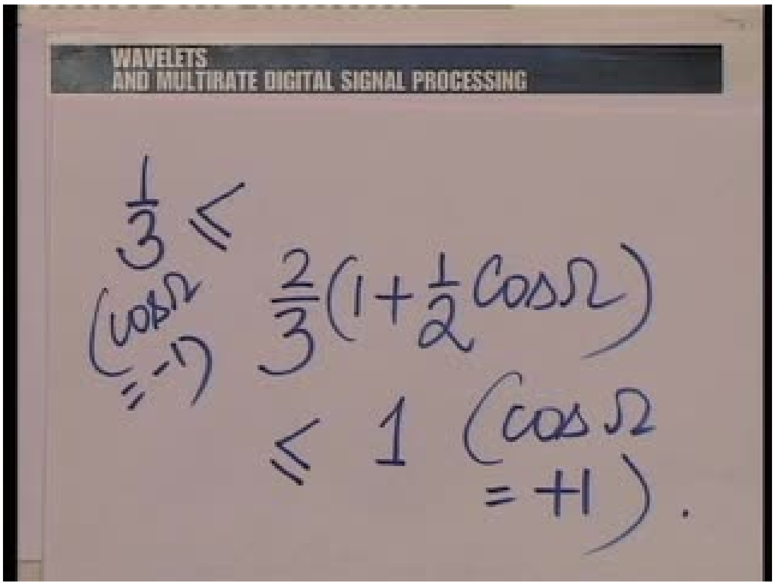

Now, as expected, this is always non negative. What is more, it is also very clear, that the sum of translated spectra, namely 2 by 3 into 1 plus half cos omega strictly lies between

2 positive bounds. The lowest possible value that this can take, lowest positive value is obtained when cos of omega is minus 1. In other words, this is 1 minus half.

So, this is bound to lie between 1 by 3 when cos omega is minus 1 and when cos omega is plus 1, this becomes 1.

So, you know, a relaxation of the requirement in time or in tau, has also led to a corresponding relaxation in the frequency domain. And now, we could employ the same strategy as we did, when we relaxed the requirement of the sum of dilated spectrum. Now, that we can see the sum of translated spectra lies between 2 positive bounds, we could say, well, even though phi 1 t by itself cannot give us an orthogonal, multiresolution analysis can be construct out of phi 1 t by using the sum of translated spectra, another function, let us call it phi 1 tilde, in such a way that phi 1 tilde gives us an orthogonal multiresolution analysis.

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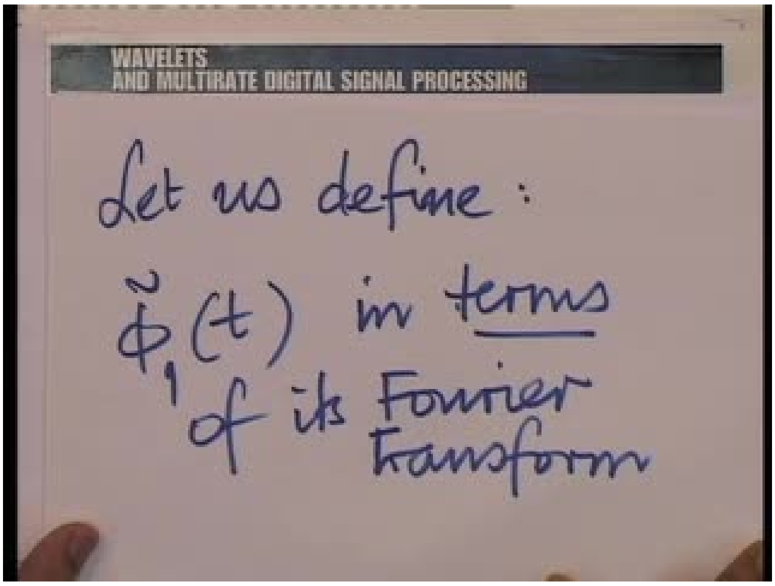

So, let us explore that possibility. In fact, let us strategically define such a phi 1 tilde, as we did, taking inspiration from the sum of dilated spectra. So, let us define phi 1 tilde t, in terms of its Fourier transform. Unlike the case of some of dilated spectra, here we will also be able to give a meaning to the definition that we make.

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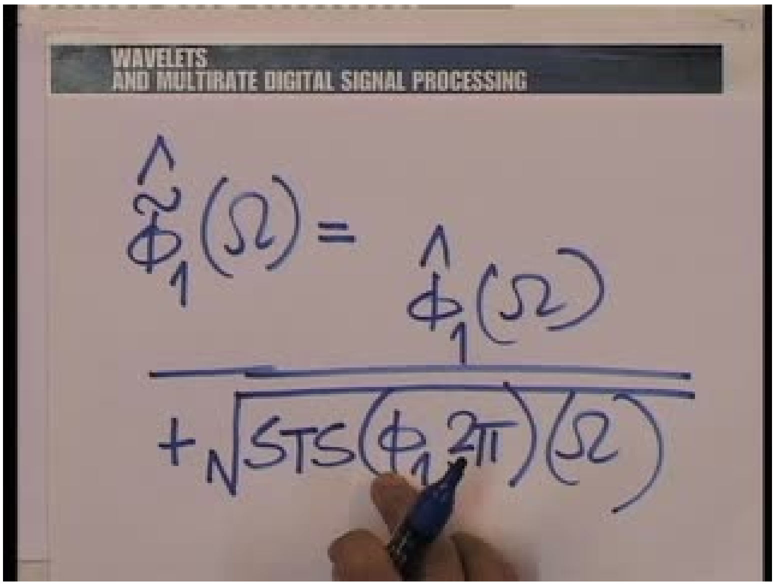

So, we will define phi 1 tilde cap of omega to be phi 1 cap omega divided by the square root, positive square root of the sum of translated spectra of phi 1, with translation of 2 pi as a function of omega. And, we justify this definition by noting that the denominator is between 2 positive bounds. In fact, the denominator is known to be between, let us put that back here, denominator is known to be between one third and 1.

So, we are justified in making this definition here. This division will not blow up towards infinity and neither will it go all the way down to 0. So, no frequency would get annulled, by this going to infinity and there would be no blow up of this definition, when the denominator goes to 0. This definition is meaningful.

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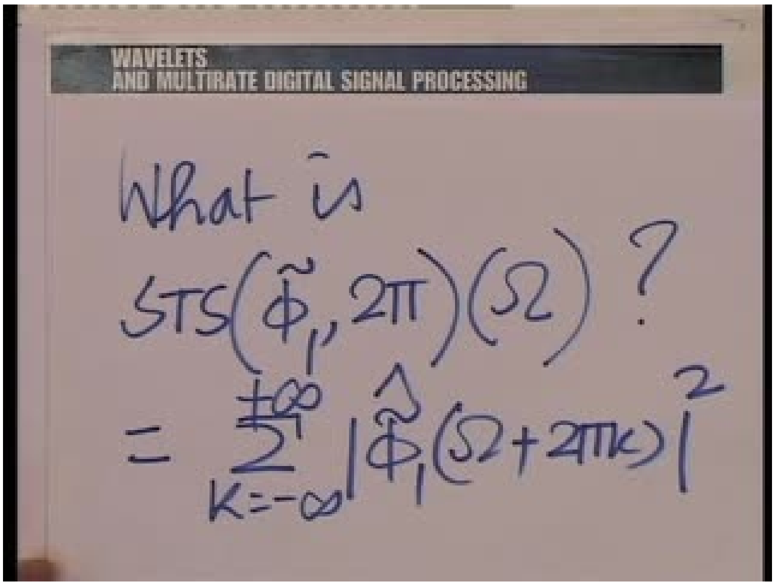

And, let us explore the sum of translated spectra of phi 1 tilde with of course, 2 pi as a translation parameter and omega as the primary argument. Of course, by definition, this is, summation k going from minus to plus infinity phi 1 tilde cap omega plus 2 pi k mod squared. And let us substitute this. Now, again here, before we directly make a substitution, we would like to establish a property, an important property of the sum in the denominator.

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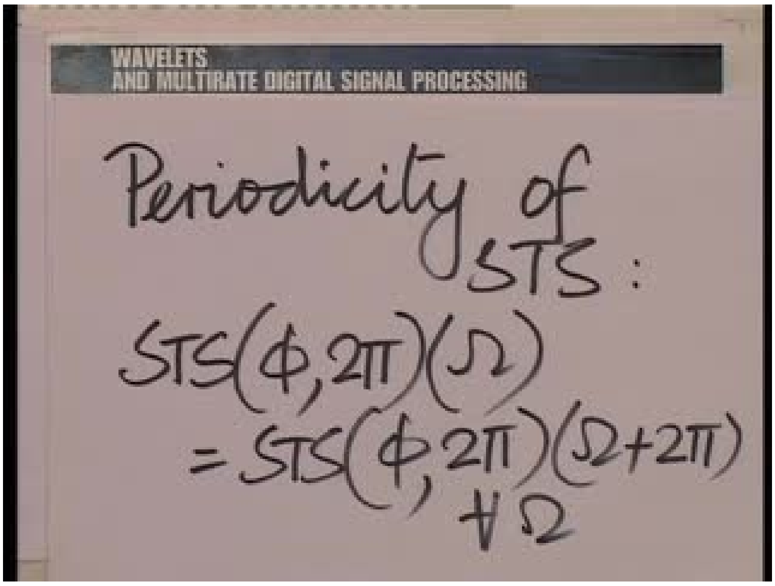

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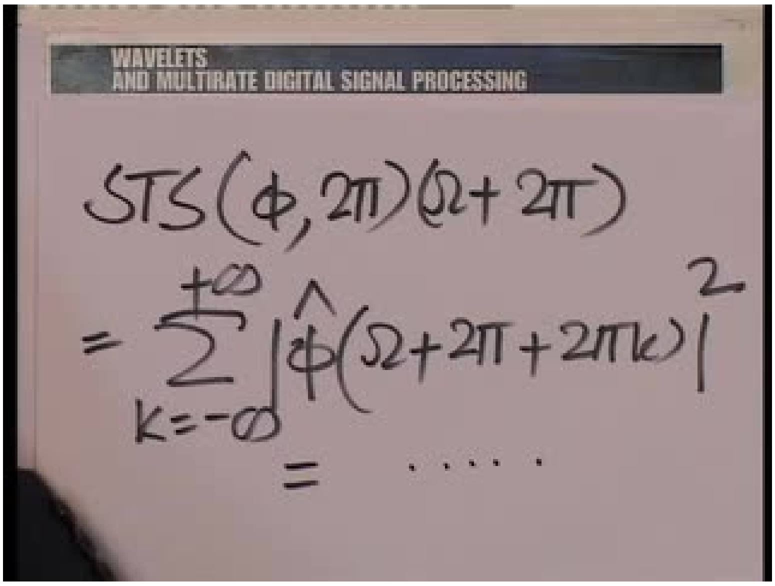

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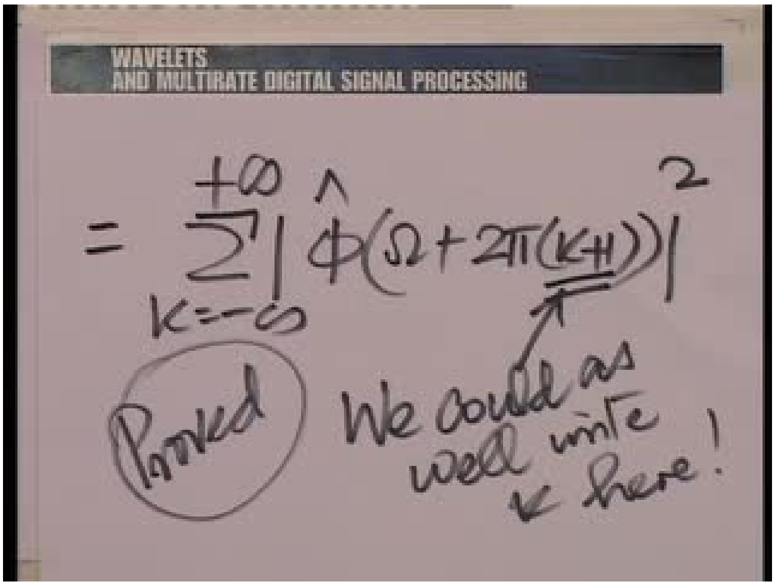

So, the Periodicity, of the sum of translated spectra. That is easy to establish. What we will show, is that the sum of translated spectra of any function phi, with translation parameter 2 pi, is periodic with a period of 2 pi. That is very easy to show. Indeed, by very definition, the sum of translated spectra phi with a translation of 2 pi and the argument replaced by omega by 2 pi is essentially sum k going from minus to plus infinity phi cap omega plus 2 pi plus 2 pi k mod squared. And, this as you can see, is going to be equal, to summation k going from minus to plus infinity phi cap omega plus

2 pi k plus 1 mod squared. And, we once again note, when k goes from minus to plus infinity, k plus 1 also goes from minus to plus infinity.

So, you could have as well replaced k plus 1 by k. We could as well write k here. And therefore, proved. This is the same as S T S phi 2 pi evaluated at omega.

So, that was a little aside, it was a kind of corollary, that we needed to prove. Now, we will prove the main or we will establish the main result, the sum of translated spectra of phi 1 tilde.

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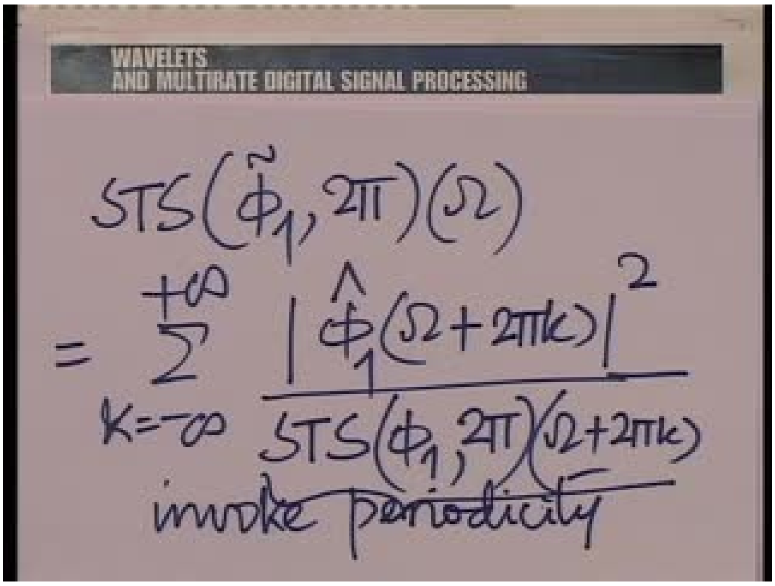

So, you know, the sum of translated spectra phi 1 tilde is now going to be, summation k going from minus to plus infinity phi 1 cap omega plus 2 pi k mod squared divided by, the numerator has the square root of the sum of translated spectra, when you square, it will become the sum of translated spectra of phi 1 2 pi, evaluated at omega plus 2 pi k.

Now, we will invoke the periodicity of this. So, this omega plus 2 pi k is redundant here. And, in fact, this can be replaced by just omega and once this is replaced by omega, then this S T S has nothing to do with the summation of index, the summation index k. So, it can be brought outside.

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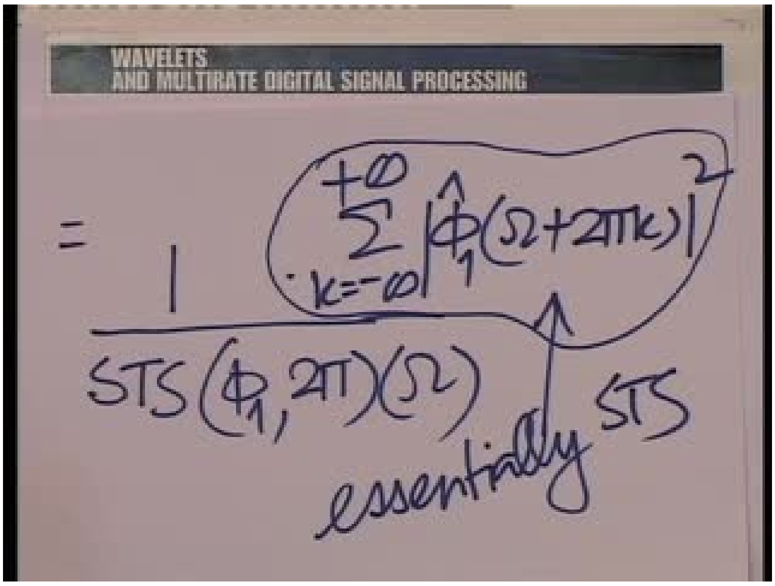

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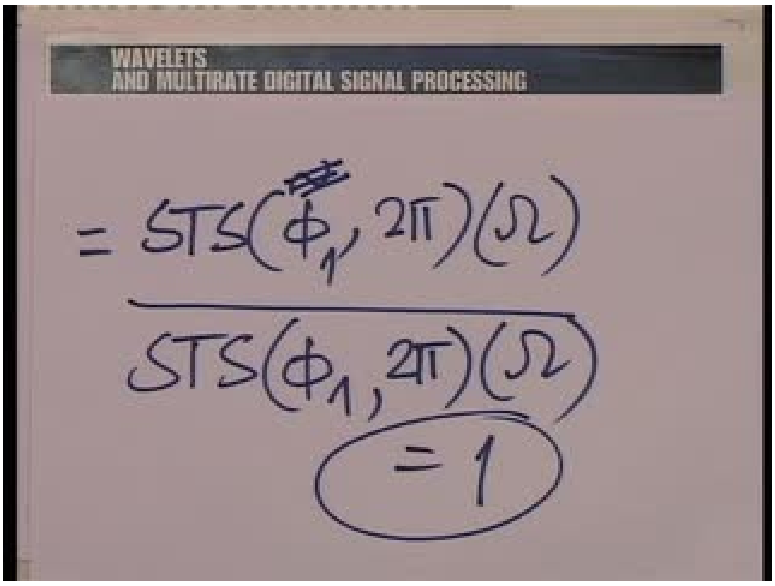

So, this, S T S becomes essentially, 1 divided by S T S phi 1 2 pi, evaluated at omega and the numerator we have, summation k from minus to plus infinity phi 1 cap omega plus 2 pi k the whole squared. But, this is familiar. This is essentially S T S. In fact, this is the same as the denominator, as you can see. And therefore, this is clearly, S T S phi 1 tilde, I am sorry, S T S phi 1 2 pi evaluated at omega divided by S T S phi 1 2 pi evaluated at omega. And once again, invoking the fact that, S T S, this quantity lies between one third and 1, it is alright to cancel this quantity from the numerator and the denominator and to obtain, that this is equal to 1. This is justified, because this does not go to 0 or infinity.

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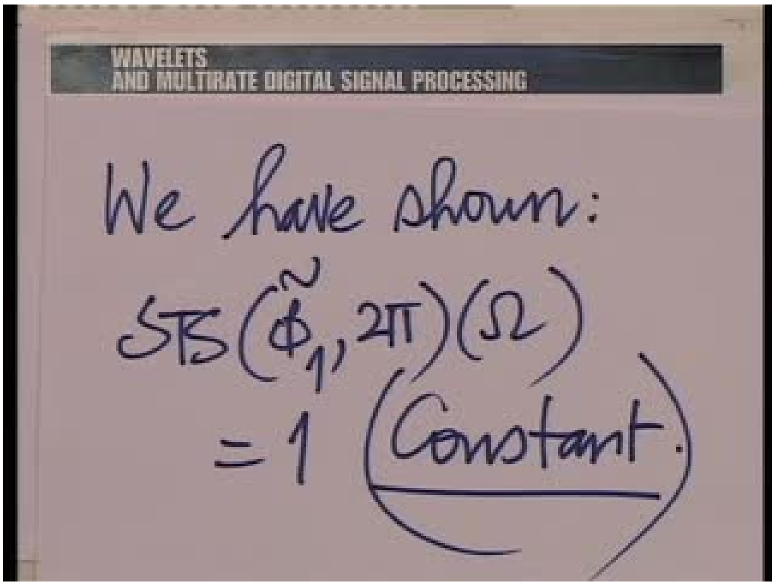

So, we have shown a very important result. We have shown the sum of translated spectra of phi 1 tilde evaluated at omega is a constant. In fact, that constant is 1. That is interesting. And we know what that means, that means that, phi 1 tilde is now orthogonal to its integer translates. Phi 1 was not orthogonal to all its integer translates. The trouble was with 1 and minus 1, but, phi 1 tilde is orthogonal to all its integer translates. And now, we shall look at the nature of phi 1 tilde.

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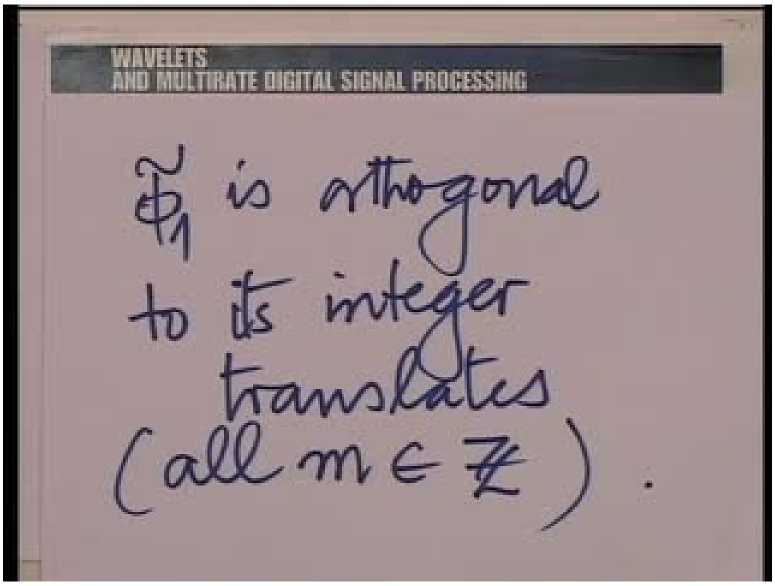

So, let us make that remark very clearly. Phi 1 tilde is orthogonal to its integer translates, all integer translates, unlike phi 1. Now, you know, that is, so far so good. I mean, if one cannot describe phi 1 tilde, what is the point in talking about its orthogonality.

So, we must able to get a way of constructing and getting a feel of what this phi 1 tilde looks like. So, let us do that, before we do anything else. If we cannot do that, then, all the rest of the discussion is meaningless.

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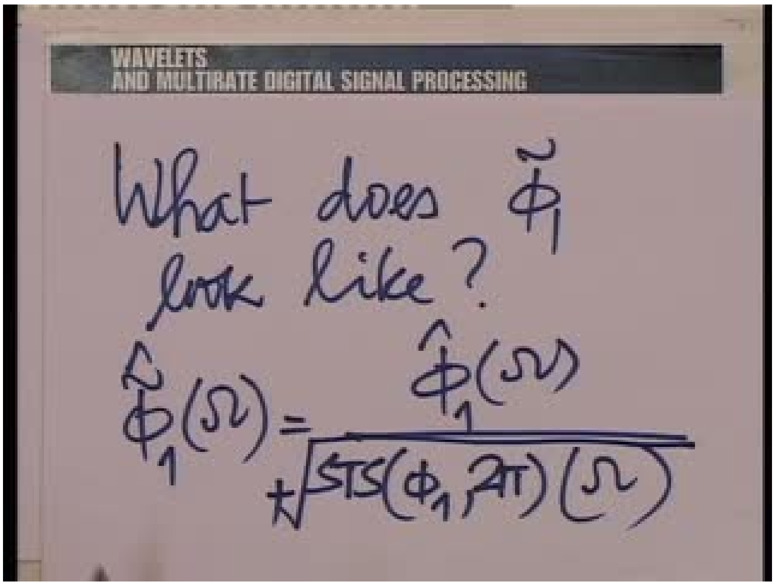

So, let us try and obtain the nature of phi 1 tilde. What does phi 1 tilde look like? In fact, let us go back to the Fourier domain. So, we have phi 1 tilde cap omega is phi 1 cap omega divided by the sum of translated spectra of phi 1 with the translation parameter of 2 pi evaluated at omega, but, with the square root here. Let us write this down explicitly.

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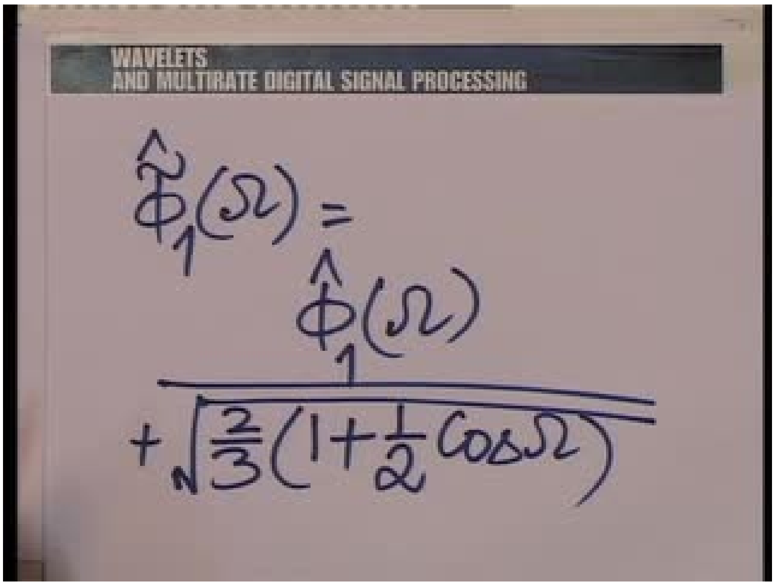

So, let us write the denominator down explicitly. So, we have phi 1 tilde cap omega, this is essentially phi 1 cap omega divided by the square root, positive square root of 2 by 3 into one plus half cos omega. Now, let us write this down in the form of an exponential or binomial expansion.

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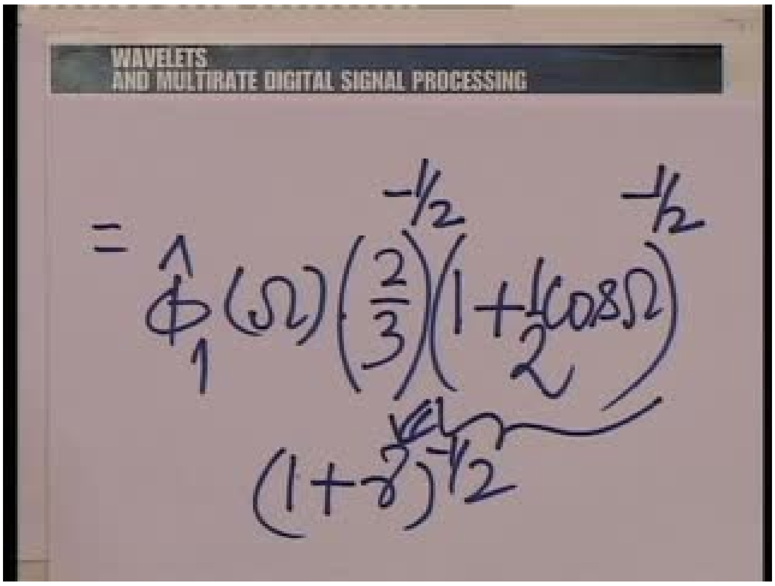

So, let us rewrite this. 2 by 3 to the power minus half into 1 plus half cos omega to the power minus half. Now, look at this expression. This is of the 1 plus some gamma to the power minus half.

Now, 1 plus gamma to the power minus half can be expanded, with our knowledge, either of the Taylor series or of the generalized binomial theory.

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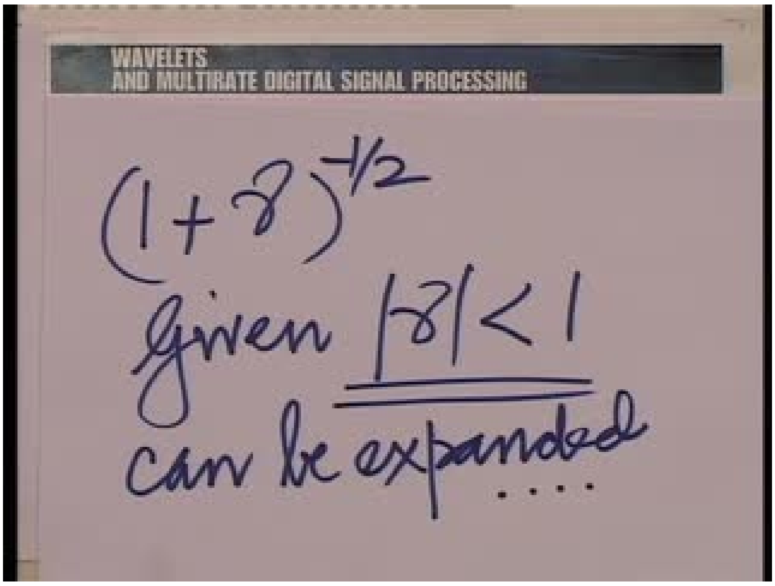

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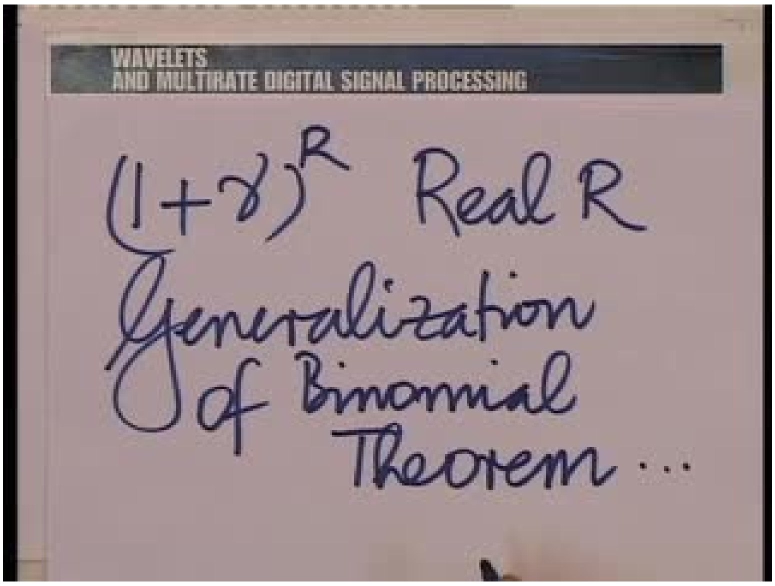

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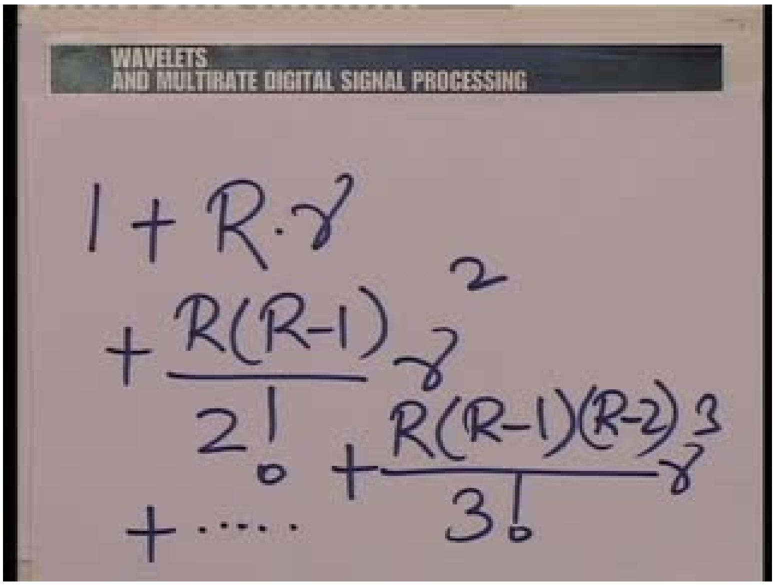

So, 1 plus gamma to the power minus half, given mod gamma is strictly less than 1, please note, can be expanded as follows, where you know 1 plus gamma to the power R, in general, for real R, using what is called a generalization of the binomial theorem, can be expanded as 1 plus R times gamma plus R into R minus 1 by 2 factorial times gamma squared plus and so on. So, I will write one more term, R into R minus 1 into R minus 2 by factorial 3 times gamma cubed and then continue.

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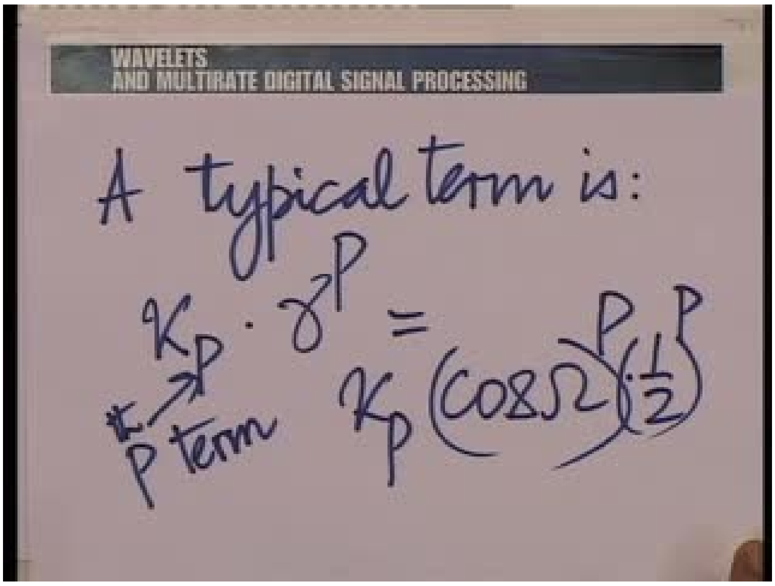

Now, you know the precise terms in the expansion are not so critical. What is critical is, the nature of the terms. A typical term here is of the following form. It is of the form, some constant, let us call it, suppose the p th term, some constant kappa p times gamma to the power p and this is essentially, kappa p times cos omega to the power p into half to the power p. That is interesting.

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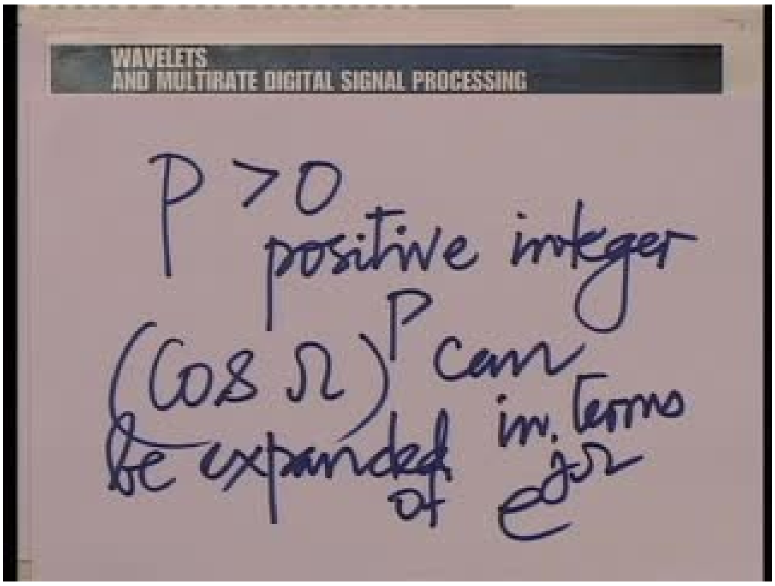

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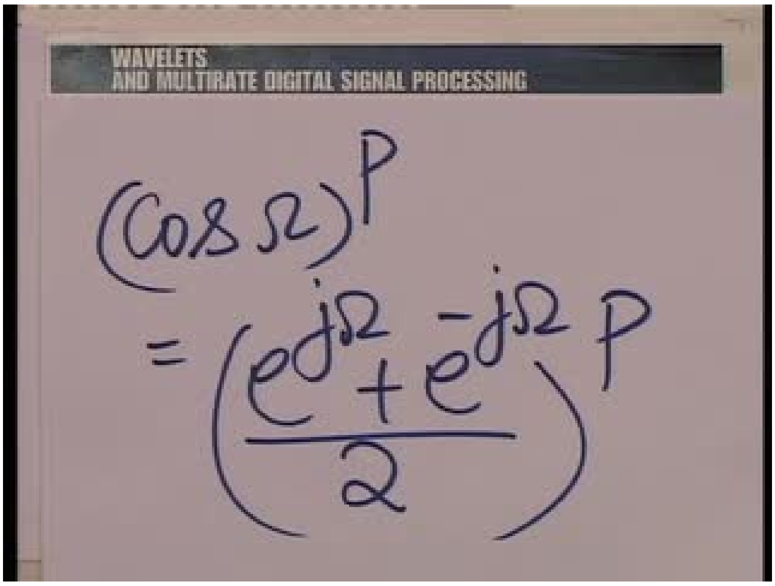

Now, let us once again recall, of course, here p is a positive integer. Now, you know it is a basic result in trigonometry, or for that matter, even in complex analysis, that cos omega to the power of p can be expanded in terms of e raised to the power j omega. In fact, we can easily do that. We can simply expand. Cos omega to the power p is essentially e raised to the power j omega plus e raised to the power minus j omega by 2 to the power p.

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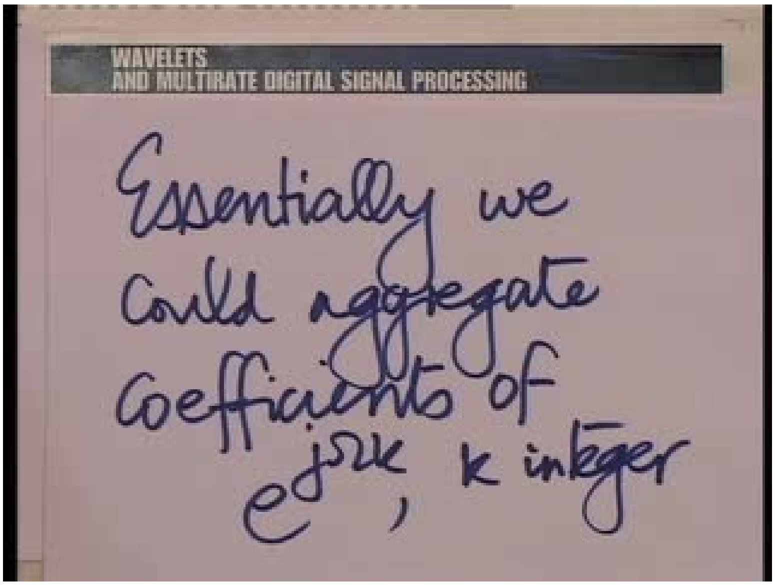

So, it can be expanded as a series in e raised to the power j omega, e raised to the power j omega k, for integer k. Now, if we take each of these terms, we can see that, essentially there is a contribution of the form e raised to the power j omega k, where k is an integer. We can aggregate the coefficients of e raised to the power j omega k, coming from each of these terms. And therefore, essentially with this expansion, we could aggregate coefficients of e raised to the power j omega k, k integer..

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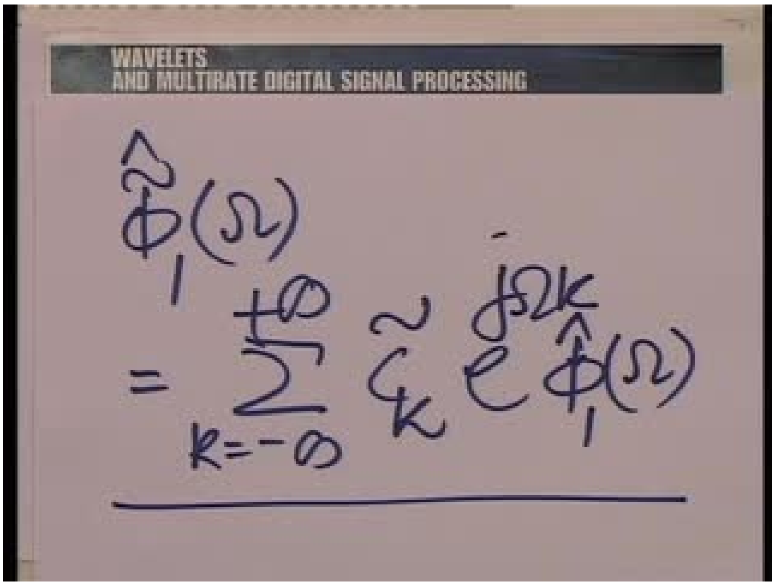

And we could show that, phi 1 tilde cap is of the form, some summation k going from minus to plus infinity, some C k tilde, these coefficients times e raised to the power j omega k times phi 1 cap omega. And this is very easy to invert in time. In fact, if you look back at this expression, what we are saying essentially is, the Fourier transform of phi 1 tilde, is the Fourier transform of phi 1 multiplied by, essentially the discrete time Fourier transform of the sequence C k tilde here.

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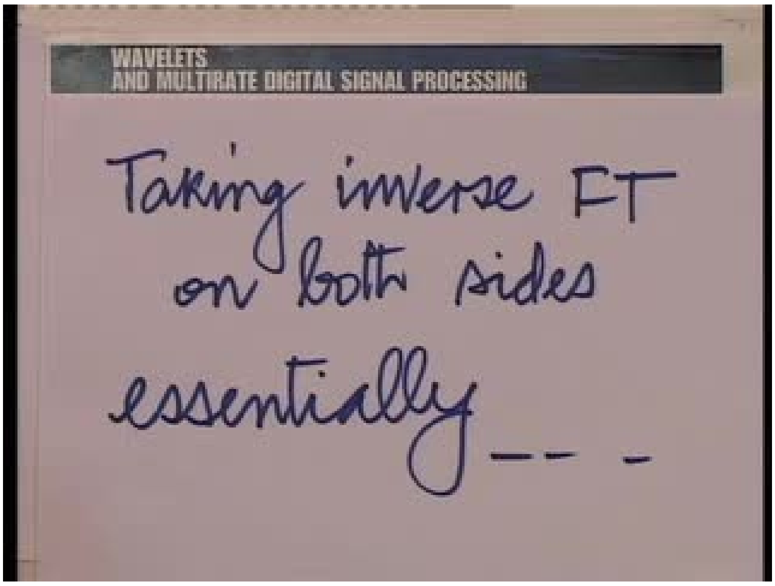

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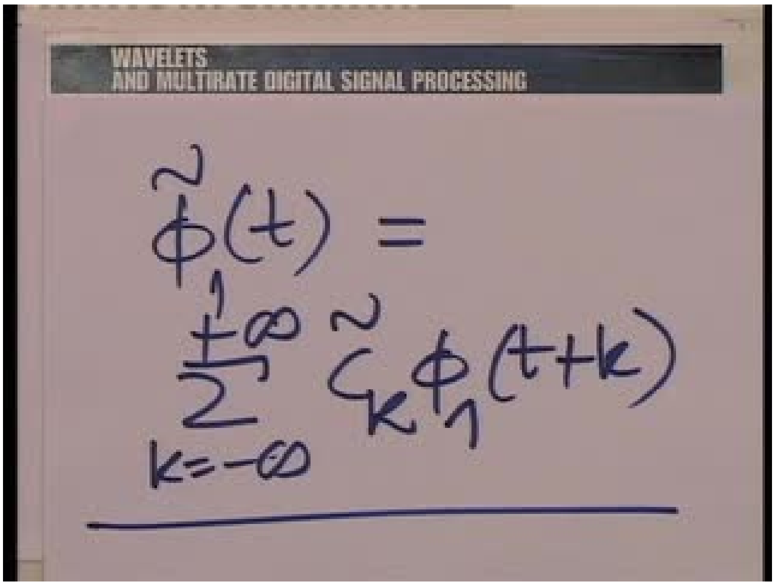

Now, if you take any one term here, in this expansion e raised to the power j omega k times phi 1 cap omega, it is the Fourier transform of phi 1 shifted by k. So, if we take the inverse Fourier transform on both sides, essentially we have, phi 1 tilde t is of the form summation k going from minus to plus infinity C k tilde phi 1 t plus k.

So, phi 1 tilde is essentially a linear combination of phi 1, shifted by integer translates. And even when you shift phi 1, which is a piecewise linear function, by integer translates, it still remains piecewise linear. When you sum together piecewise linear function, the sum is piecewise linear.

So, it is very clear at this point, that phi 1 tilde is going to be a piecewise linear function. What else is it going to have? What are the, what is the nature of these C k tildes and how do we construct the multiresolution analysis, we shall see in the next lecture. So, we will proceed from this point in the next lecture.

Thank you.