Advanced Digital Signal Processing –Wavelets and Multirate
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Lecture No.#26
Proof Of The Theorem Of (DYADIC) Multiresolution Analysis

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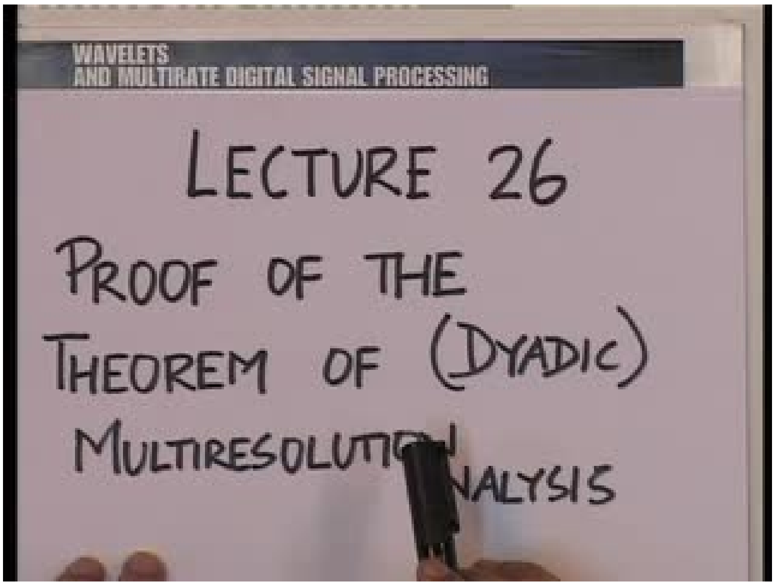

A warm welcome to the twenty sixth lecture on the subject of wavelets and multirate digital signal processing. This lecture is intended for a proof of the theorem of multi resolution analysis, - dyadic multi resolution analysis - and therefore, we have essentially focus today on continuing from where we were the last time, where we had informally introduced the meaning - code un-code meaning - of the theorem of multi resolution analysis, and today, we set out to prove it.

Let me once again put before you the thoughts, with which, we concluded the previous lecture. We said that one wave to interpret this whole question of discretization of the translation parameter is to raise the issue of sampling, but generalize sampling noting that we would dealing with band pass functions instead of band limited functions.

We noted in brief that if you look at the discretization of the translation parameter in the spaces v v zero v one v two in the ladder, it amounted to a version of the band limited sampling theorem, the conventional sampling theorem, because if you looked at the spaces v zero contain and v one contain and v two and then v minus one contain and v zero and so on in the ladder, we are talking of spaces of band limited functions with the band doubling each time, but of course the bands are all around the frequency zero. So, it is the all inclusive band up to a certain frequency, and then double, and then four times and eight times, as you go up the ladder.

So, naturally the sampling frequency needs to double each time as well and that is what we observed. On the other hand, we looked at the ideal case of a band pass function, and we said that if the band verse strategically placed, for example, if you looked at the band between phi and two phi and sampled such a band pass signal, a signal contained in this band, only in this band, restricted to this band at a sampling rate of two times phi instead of two times two phi. You know, we got a band pass version of the sampling theorem. We saw that translates of the spectrum created by a sampling this band pass function at a sampling. An angular sampling frequency of two times the band, two times phi instead of two times two phi, which is the highest frequency. We still got the translates of the original spectrum being non-overlapping with the original spectrum, and therefore, we could put a band pass filter and reconstruct the signal even after sampling.

Now, sampling functions in the ideal sense; I mean the ideal band pass reconstruction filter is going to have an impulse response which is unrealizable, and therefore, wavelets is a way of band pass sampling and reconstruction practically. That is the interpretation we had given it, and therefore, we were asking the question - can we proof that under the axioms of an m r i? We can extract this wavelets function psi t which will allow band pass sampling. This was the interpretation that we are given it and we set forth to prove it today.

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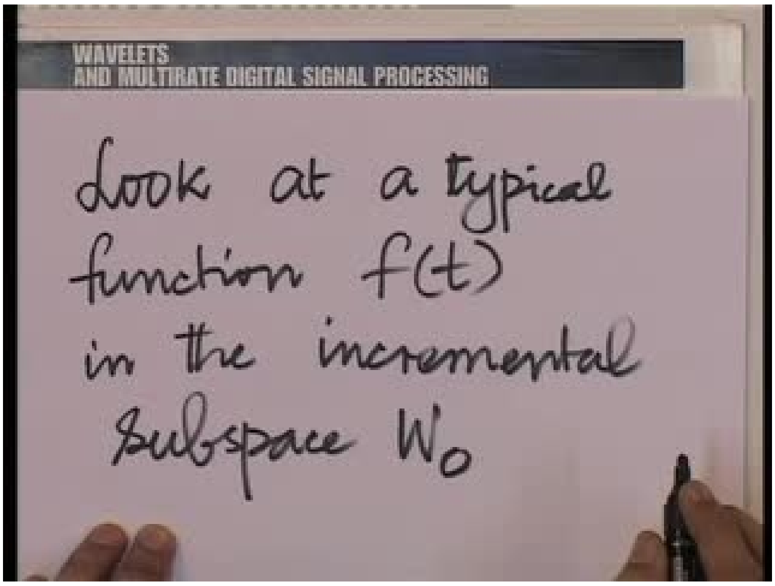

Well, the strategy that we would employ in the proof is to look at a typical function in the incremental subspace. Let us call that function f t.

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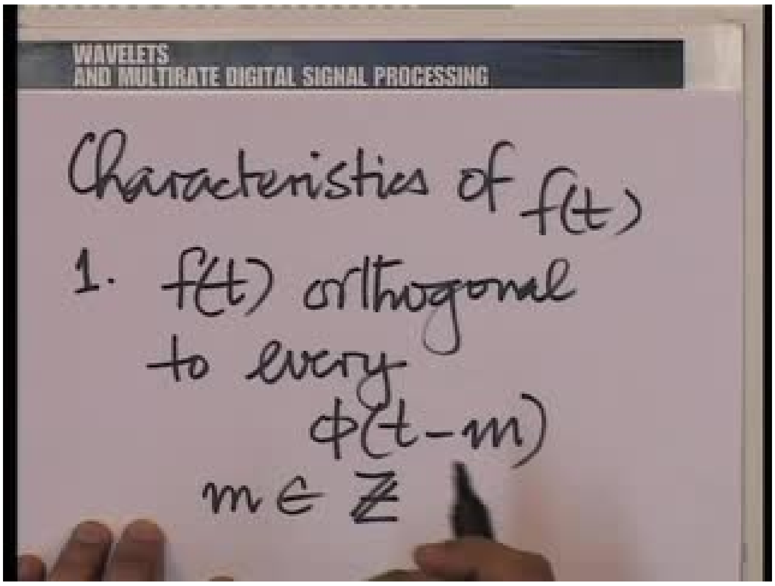

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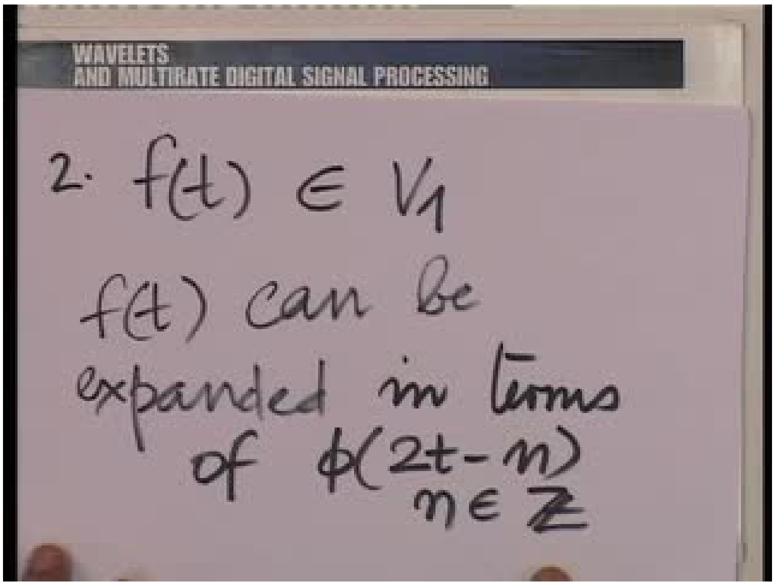

Now, what is it that characterizes this function f t? There are two things that characterize: one - f t is orthogonal to every translates of phi t m integer; that is because phi of t minus m for all integer m forms an orthogonal basis for the space v zero, and since f t is orthogonal to every function in v zero, remember, that is how we describe this incremental subspace has being the novelty in v one over v zero; novelty in the sense of dot products. So, f t is orthogonal to every phi t minus m m over the set of integers, which of course gives the complementary property, but the inclusion property, namely that f t belongs to v one is captured by the fact that f t can be expressed. It can be expanded in terms of phi two t minus n for integer n. Essentially the inclusion property, the fact that f t is a part of v one.

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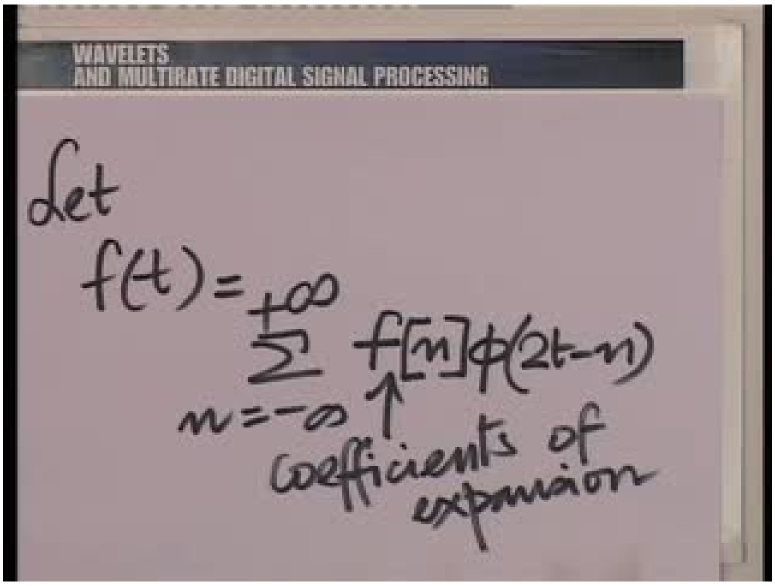

Now, let us use these two properties together and come up with some interesting observations. So, let f t in particular be the sum for n going from minus to plus infinity f square bracket n. Now, this is the set of coefficients in the expansion of f t with respect to phi two t minus n, coefficients of expansion.

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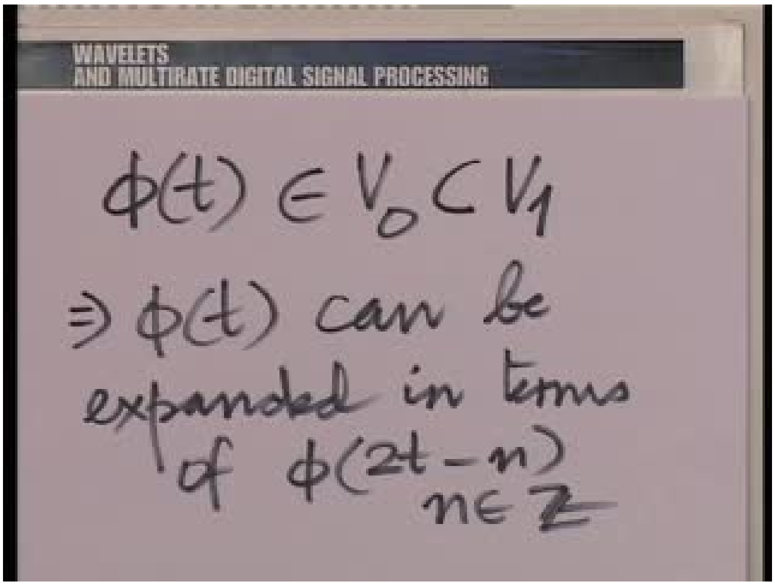

And of course, we must not forget that phi t itself belongs to v zero, and v zero is contained in v one. Therefore, phi t could be expanded in terms of phi two t minus n integer. You must not forget this, and we also know what helps us expand five t in terms of five two t minus n. Remember that when we have a discrete time filter bank which on a iteration leads to the wavelet, it is the coefficients of the low pass orthogonal filter, you know, the orthogonal filter bank, the discrete filter bank which leads on a iteration to a wavelet. For example, in the case of the have wavelets or the Daubechies wavelet, the low pass filter coefficients on iterations give you function phi t, and in fact, there is a recursive dilation equation that relates phi t to its own dyadic dilates and translates, and in that recursive dilation equation, it is the impulse response or the coefficients of the low pass filter which come in to picture.

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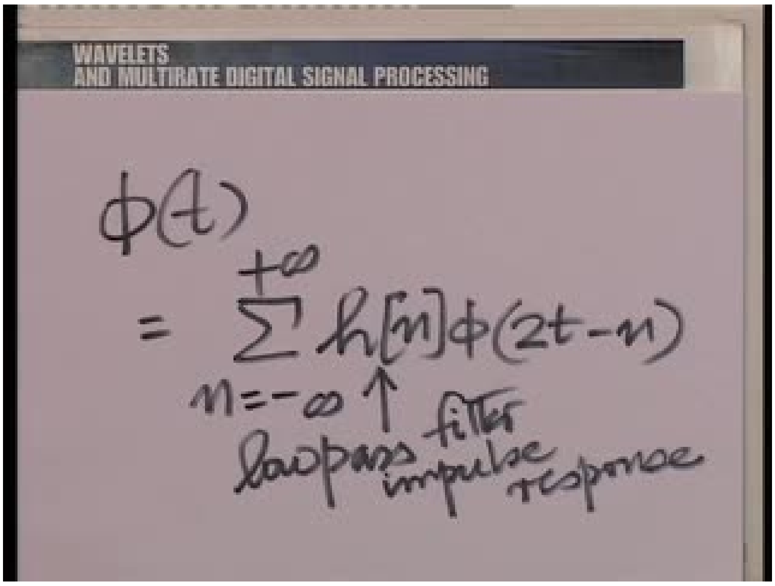

Therefore, we can write down phi of t is also summation on n from minus to plus infinity in principle in general h of n phi two t minus n. So, these are the low pass impulse response coefficients.

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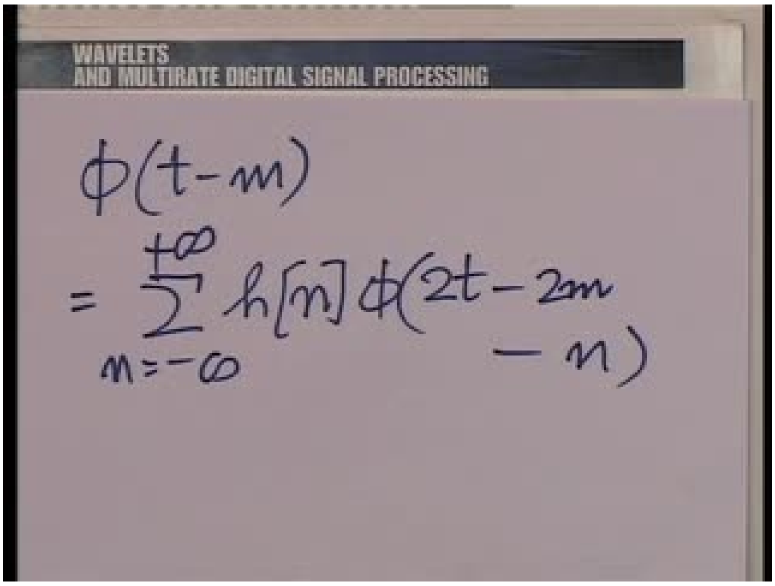

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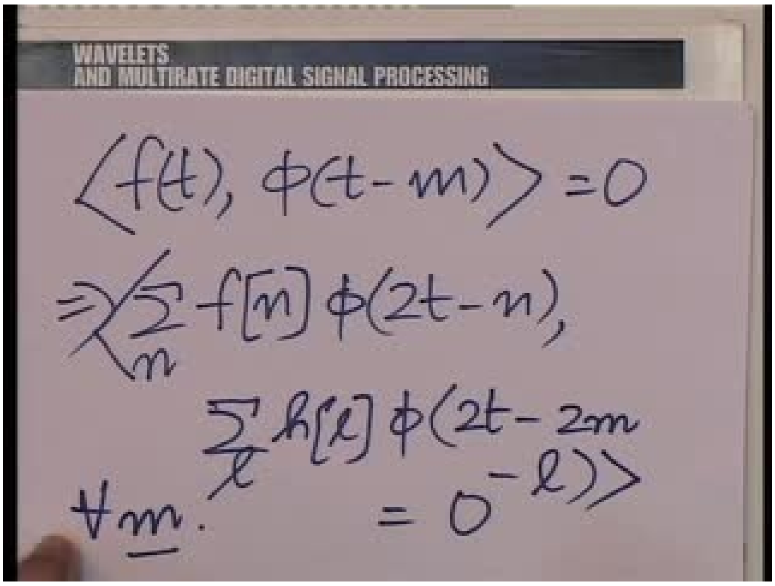

Now, we shall use the orthogonality of f t to phi t minus m to establish something interesting. So, we note that if phi t is expandable in this way, then phi t minus m shall be expandable in the following way, and now, using the orthogonality of f t with phi t minus m, we get summation on n f n phi two t minus n in a product with summation on n. Now, here, the summation is over all the integers in both cases, but we will use a different variable here to distinguish it from n. So, we could use summation on l here - h l phi two t minus two m minus l is zero for all m.

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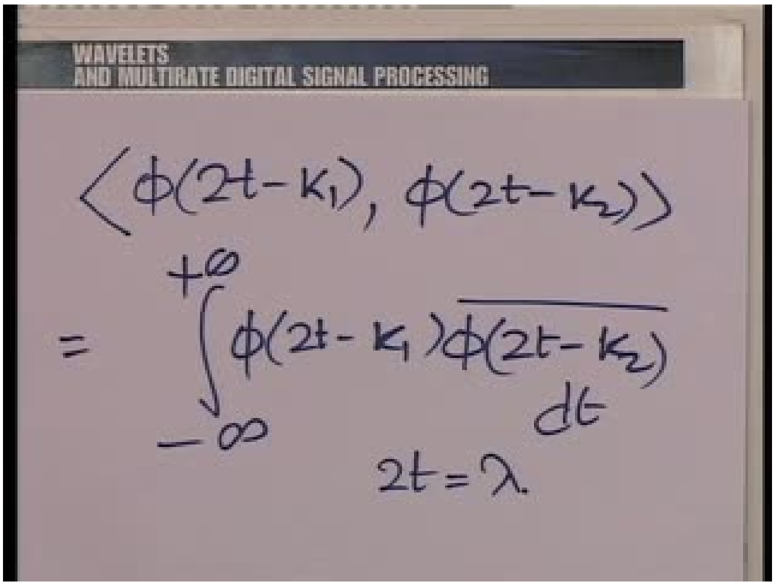

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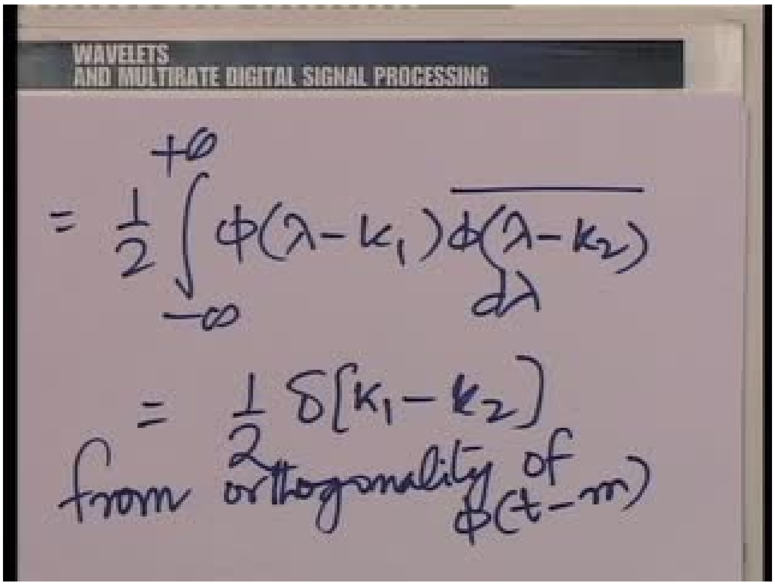

Now, we again invoke the orthogonality of phi with respect to its own translates. So, we not that the dot product of phi two t minus say k one and phi two t minus k two is essentially phi two t minus k one phi two t minus k two bar d t integrate it, and if you only k are to put two t is equal to lambda, we would get this integral to be half phi lambda minus k one phi lambda minus k two bar d lambda a factor of half outside, no other change, and this is of course, half delta k one minus k two from the orthogonality of phi.

So, phi is orthogonal to its own translates. In fact, its forms an orthogonal basis of v zero your orthogonality of five to its own integer translates, of course, guarantees the orthogonality of contracted version of the same phi t contraction must be the same, that is, in this case two.

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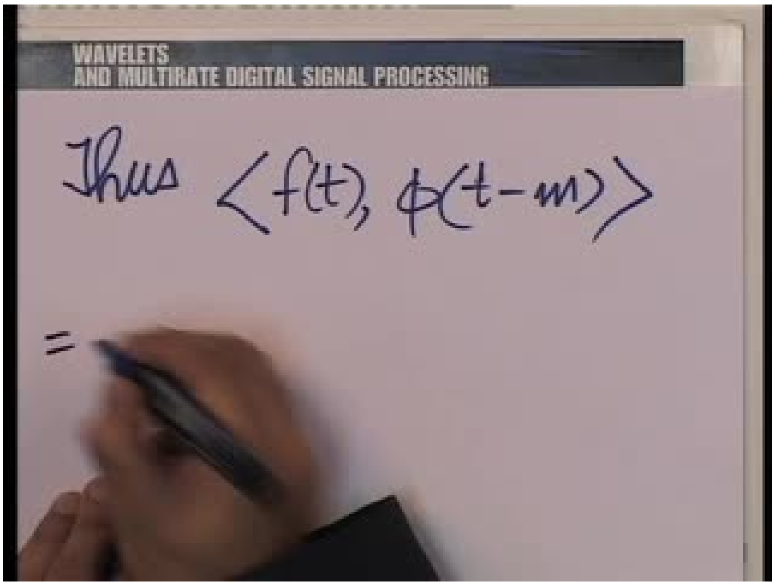

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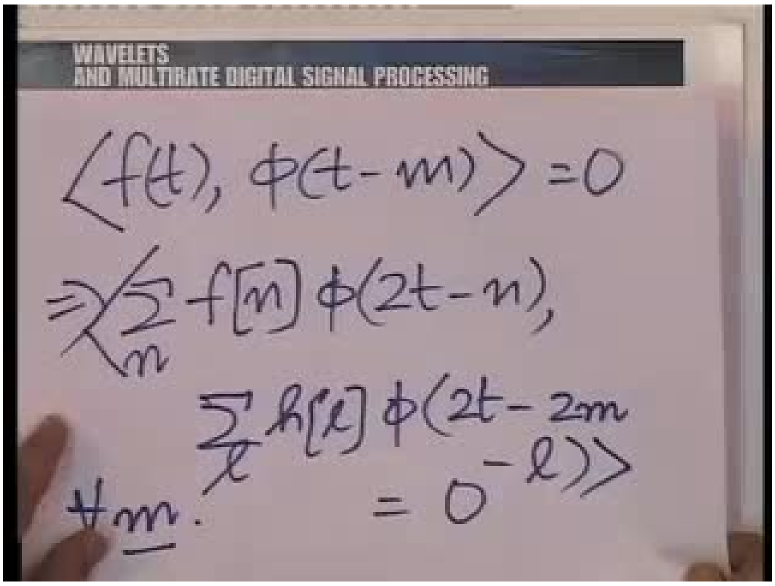

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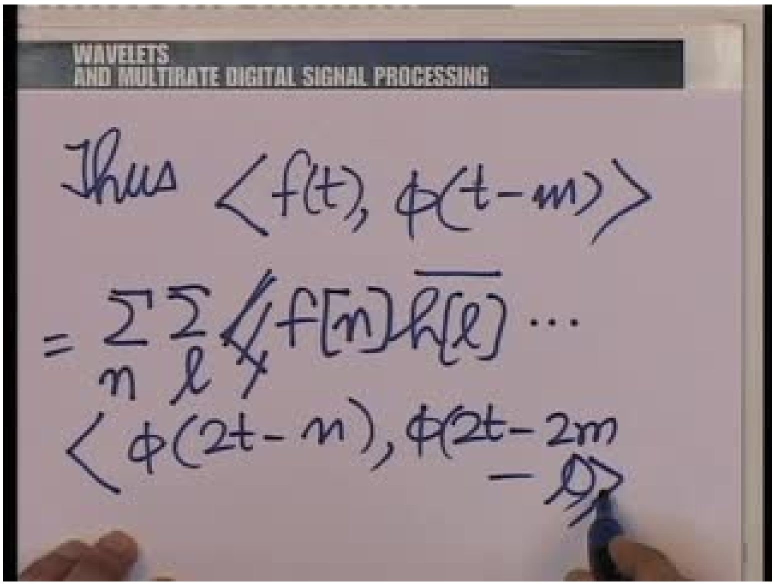

Well, using this, we have the dot product of f t with phi t minus m is therefore, you know, if I go back, let me put back that summation for you. In this summation, we would expands; so, summation on l n summation on n f n times this term and summation on l h l times this term. So, I can bring the summations outside the dot product expression. It summation on n, summation on l in a product f n. Well, in fact, we could even take the coefficients outside. So, this inner product could be operated last f n 0h l bar, and then, an inner product phi two t minus n phi two t minus two m minus l.

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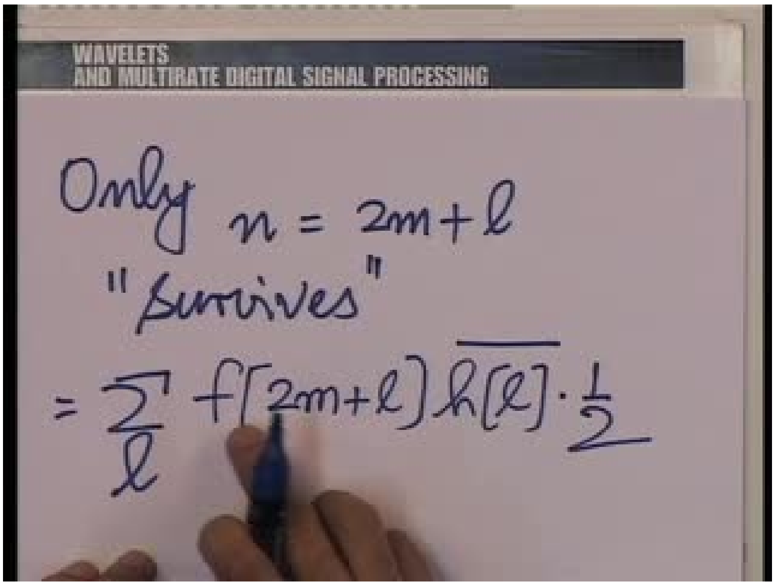

And using the orthogonality of these phi's, we notice that only when n is equal to two m plus l, thus this dot product survive. So, we can eliminate the summation on n and leave only a summation on l, and that leaves us with summation on l. For that particular n equal to two m plus l, we have f two m plus l h l bar, and then, the dot product reduces to half and this quantities familiar to us. This is the familiar quantity with which we deal.

Essentially, it is the cross correlation of the sequences f and h. So, if we come back here, this summation, summation on l f two m plus l h l bar, forget for the matter, for the, for the time being about the factor half. It is not that terribly important, because ultimately going to equate this to zero. What is important here is that there is a dot product between the sequences f and h with an appropriate shift - mutual shift - and that is essentially a cross correlation.

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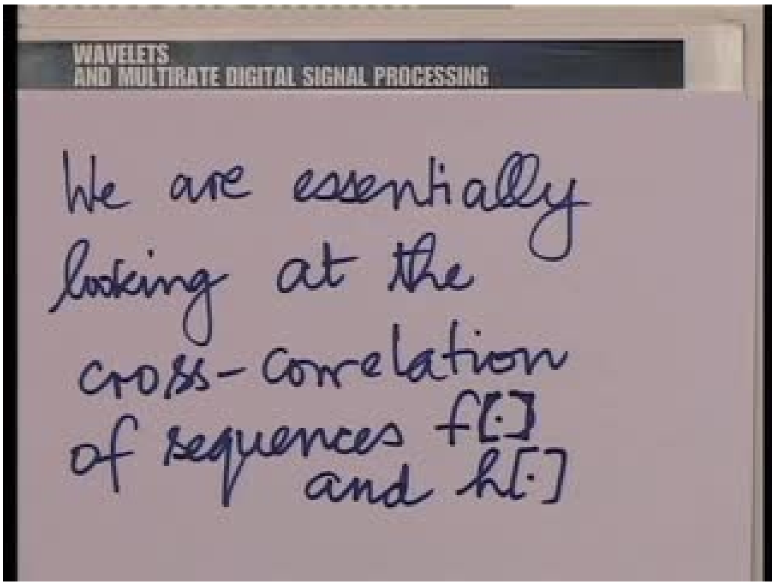

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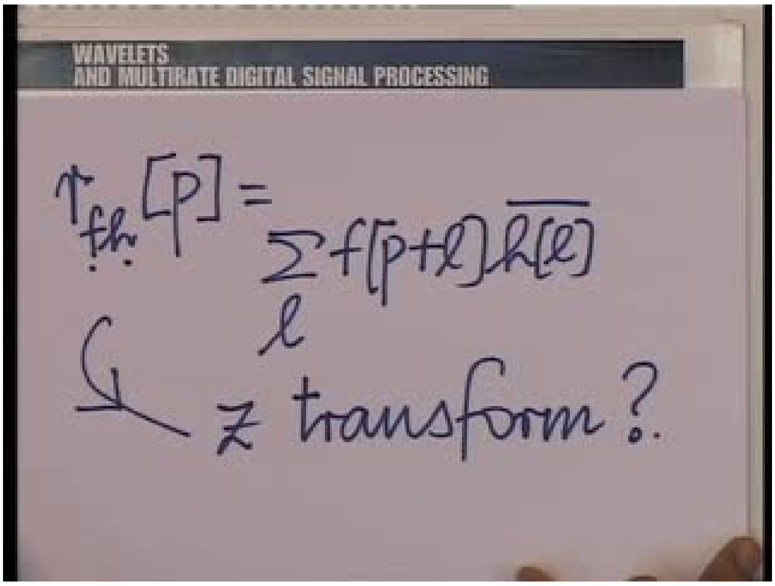

So, we are essentially looking at the cross correlation of sequences f, maybe we should use square brackets and h, and how is this cross correlation defined? Well, this cross correlation is often denoted by r f h. This denotes the secondary arguments. Evaluated at a shift, let us say of p summation on l f p plus l h l bar. The complex conjugate is important only if a dealing with complex responses. Otherwise, it is not so terribly important.

Now, I leave it as an exercise to find the z transform of this. I provide a few hints. We notice that this cross correlation as a function of p is by itself a sequence. In a way, this is the sequence almost obtained by convolution of f and h but with a difference. You know, in fact, had this been minus l here. It would precisely have been a convolution of f and h complex conjugate, but the presence of plus instead of minus, makes it slightly different from a convolution. As we expect the z transform of this sequence is going to be related to a product of the z transforms of f and h.

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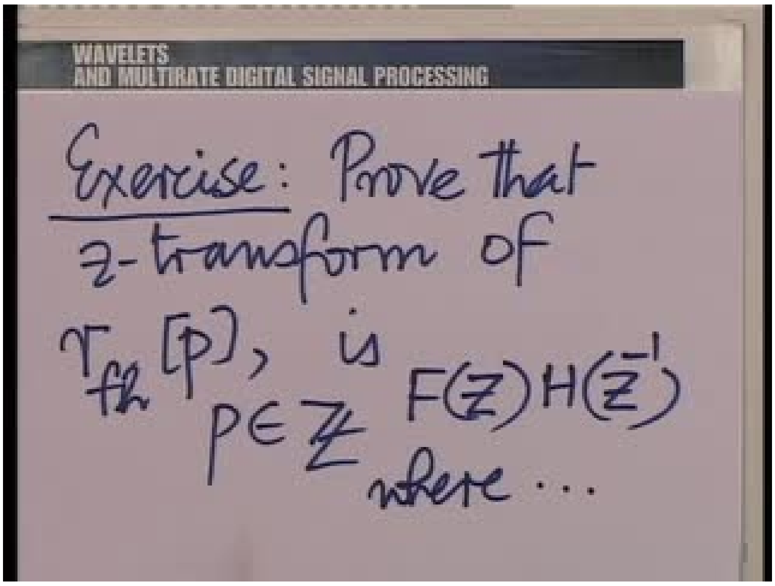

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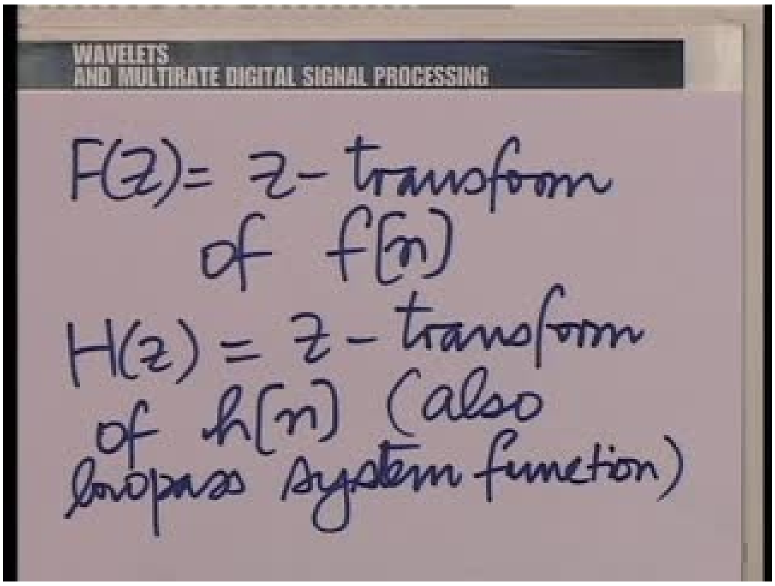

Now, I leave it as an exercise to prove the following: prove that the z transform of the sequence - the cross correlation sequence - is f of z times h of z inverse, where then I continue f of z is the z transform of f n, and h of z is the z transform of h n also happens to be the system function of the low pass filter. In fact, if you recall this low pass system function, characterizes the whole filter bank. It is very central to the filter filter, to the orthogonal filter bank. Once you know h z, the system function of the low pass filter. You know the high pass filter on the analysis side, and you know the low pass and high pass filter on the synthesis side.

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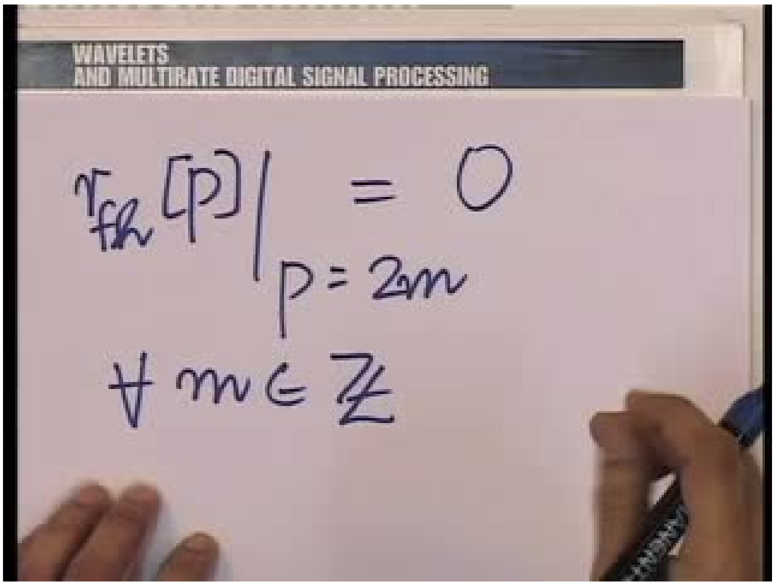

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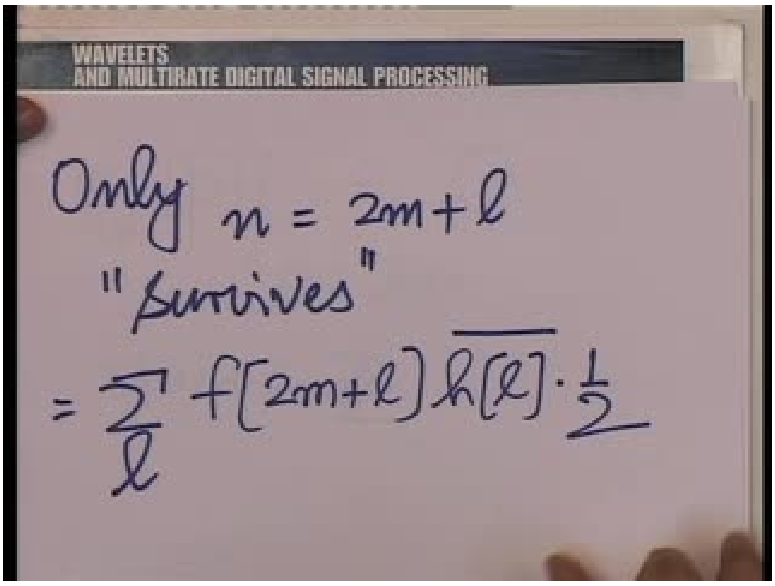

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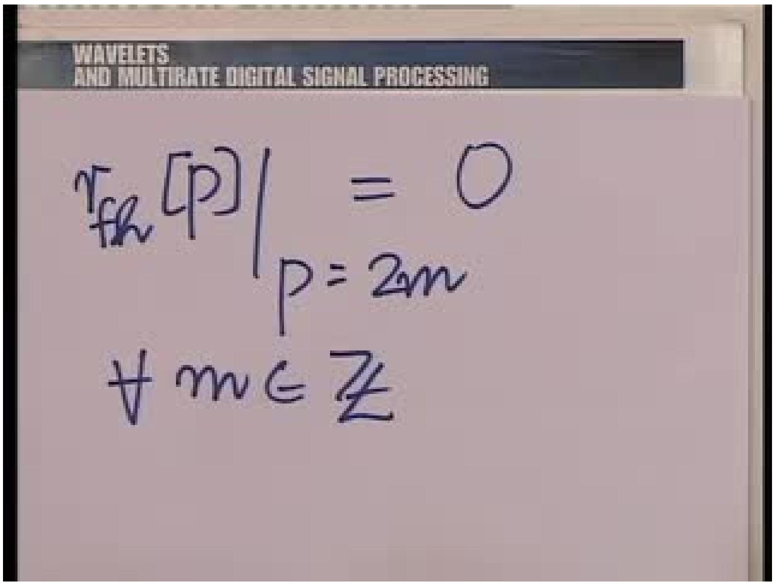

So, without observation, we notice that the cross correlation r f hp p evaluated at two m for all integer m. In other words, that all even locations is zero. That is what we are stating in the result that we delight of a few minutes ago. Recall, when we put this here and when we said that, this must be equal to zero for all m. What we are saying in a effect is that the cross correlation of f and h. When evaluated at all even locations, it must be zero. Let us in fact make a note of it. It is a very important conclusion we are draw.

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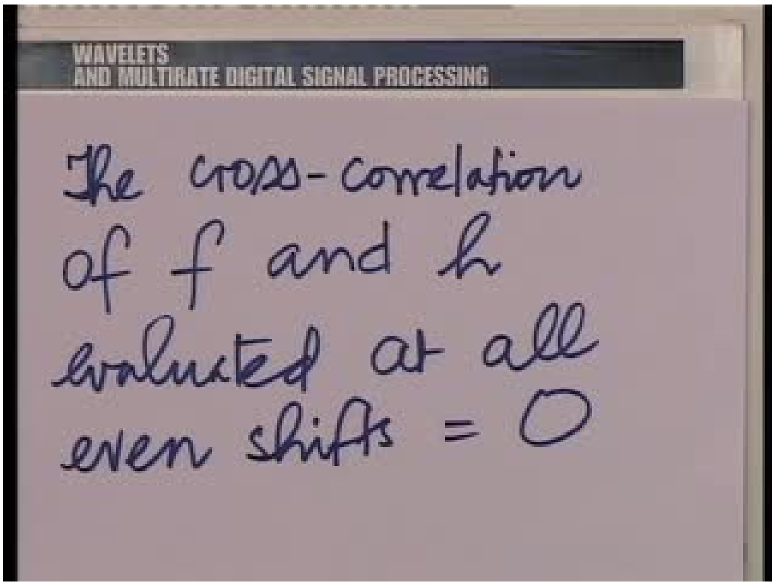

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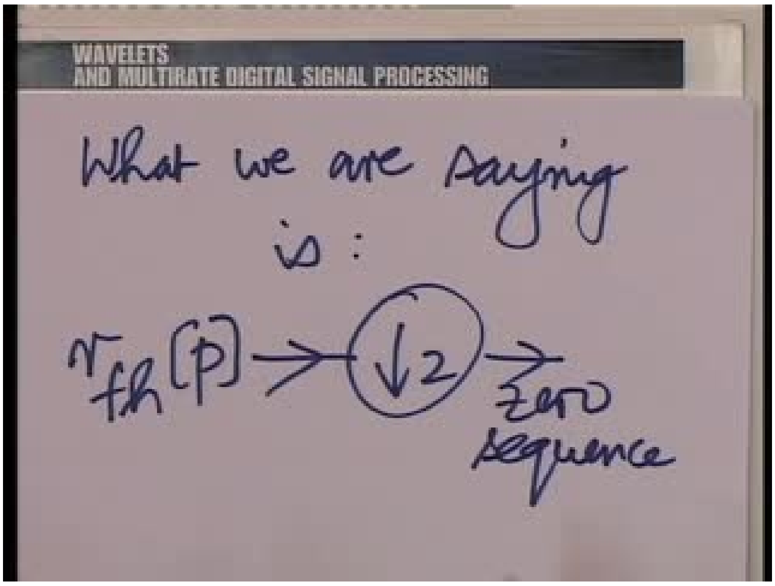

This is the orthogonality requirement. Now, we know how to deal with the situation when we wish to look only at even locations. In fact, we know the operation that does that, down sampling by two. So, suppose you were to notionally take this cross correlation sequence and down sample it by two, you would get and all zero sequence. That is what we are essentially saying. Let us put that down graphically. What we are saying is - if we take r f h p and subjected to down sampling by two, we get a zero sequence, and therefore, if we look at this situation in the z domain, we of course have a zero z transform here also. So, let us interpret in the z domain.

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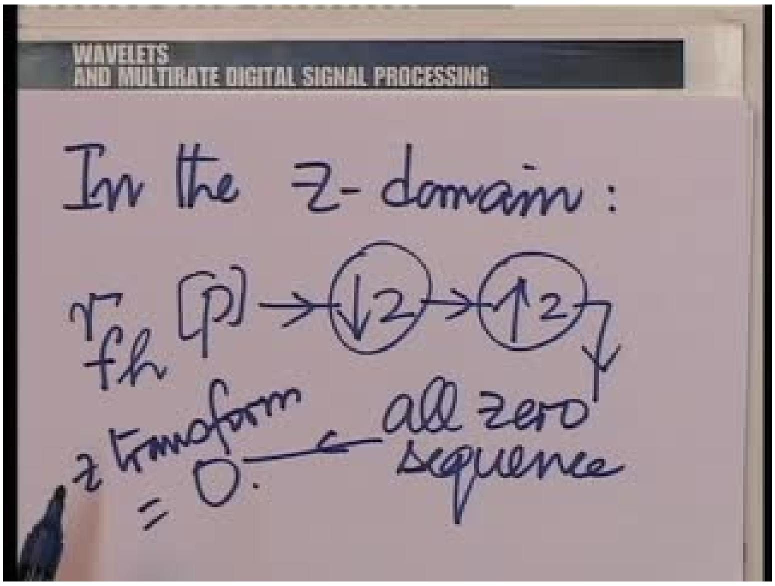

Now, you know, it is easier for us to do the following. We could say take r f h p down sample by two and up sample by two and that would also be an all zero sequence, and of course, the z transform is zero here.

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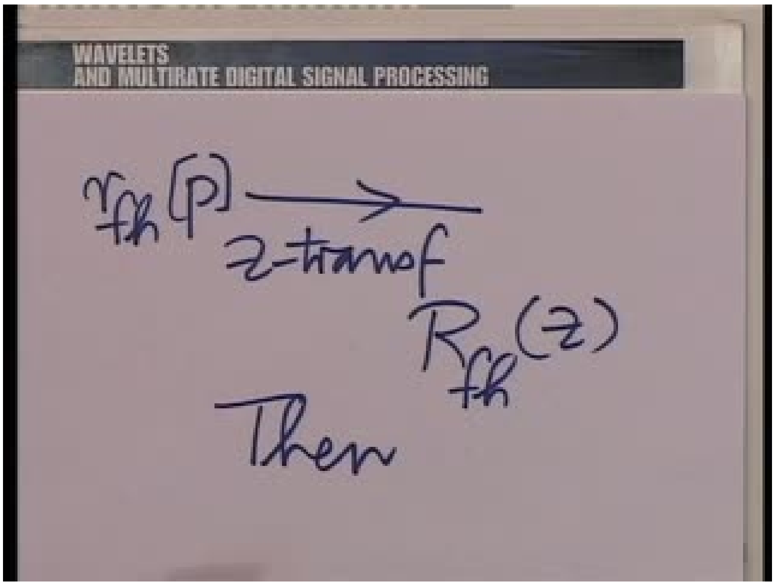

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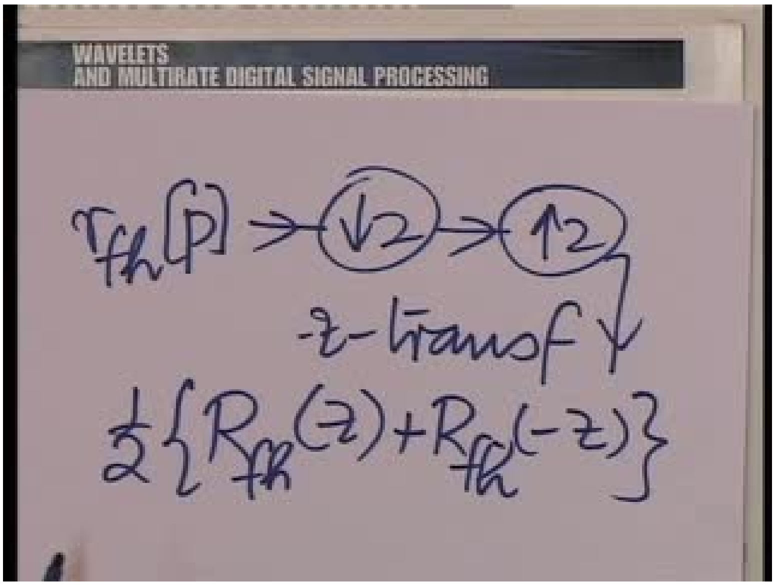

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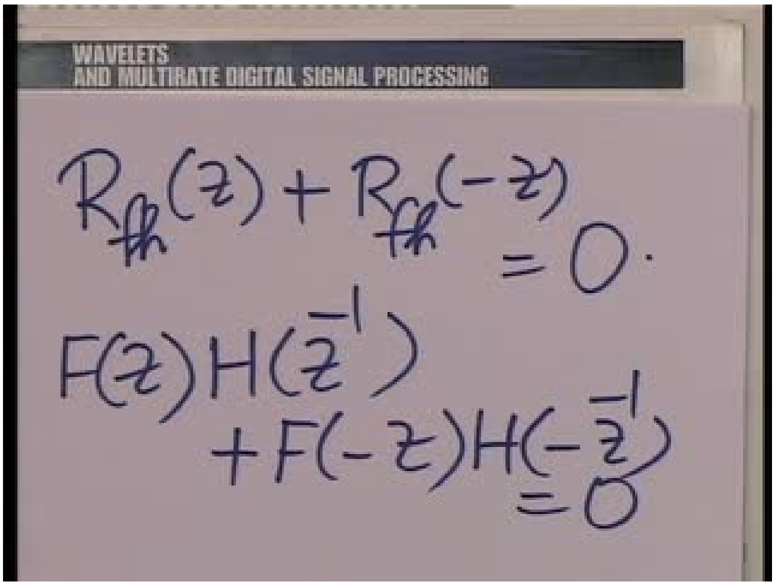

How do we deal in the z domain with this pair of operations? You recall that if r f h p has the z transform capital r f h z. What I am saying is - if r f h p on z transformation results in capital r f h of z, then r f h p subjected to down sampling by two, and then, up sampling by two results on z transformation into half or f h z plus r f h minus z, and therefore, what we has then is r f h z plus r f h minus z must be identically zero, or in other words, f z times h z inverse plus f minus z times h of minus z inverse is identically zero, and in fact, we can now rearrange this. We can rearrange this to get some very interesting insides in to f z.

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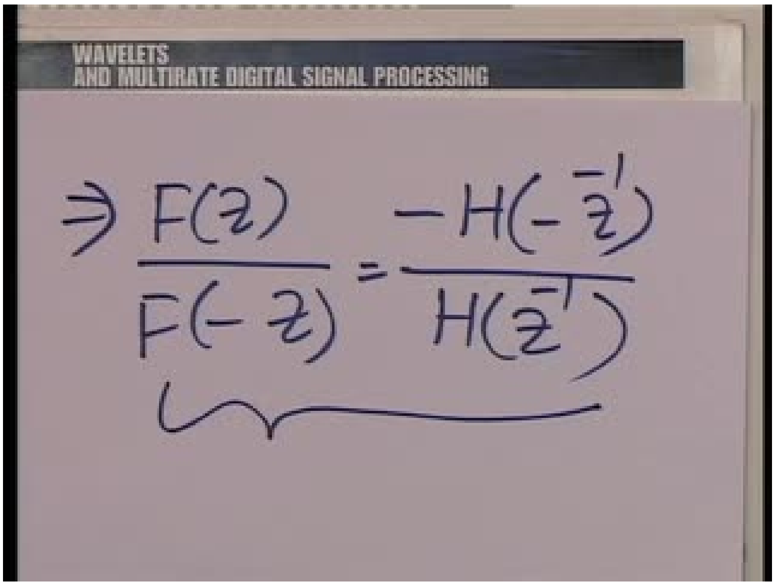

So, that would give us the condition f z by f of minus z is eventually minus h of minus z inverse divided by h of z inverse. Now, this where we have an interesting inside in to f, this typical function f, which belongs to the incremental subspace. You know, in this ratio f of z by f of minus z, we managed to eliminate what is specific to f. What I mean by that is - if you think of w zero as a country with citizens comprised of function in w zero and each citizens having a passport, then you could think a first the blank passport and then a passport with entries made for that purse. What we want to identify is the nature of the blank passport in that country, because that characterizes the citizen of that country in general. The specific entries corresponding to that person are not of (( )) at very important to us at the moment. We wish to characterize the space w zero in general.

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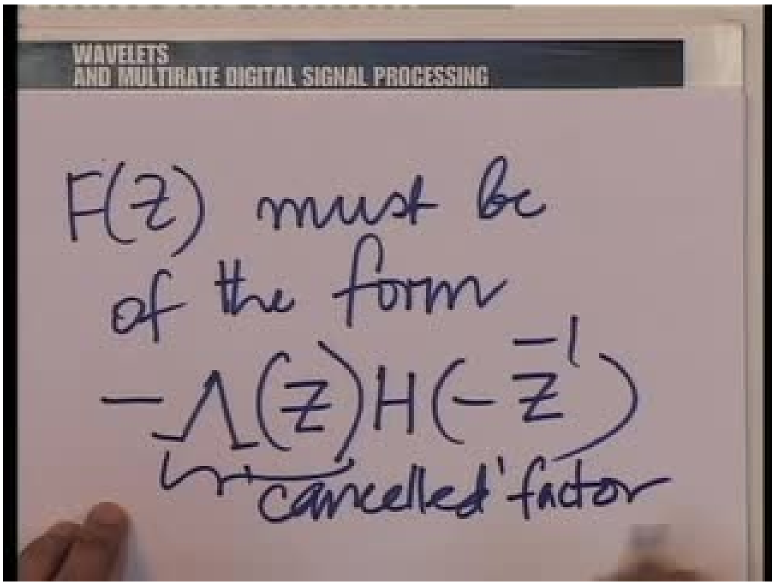

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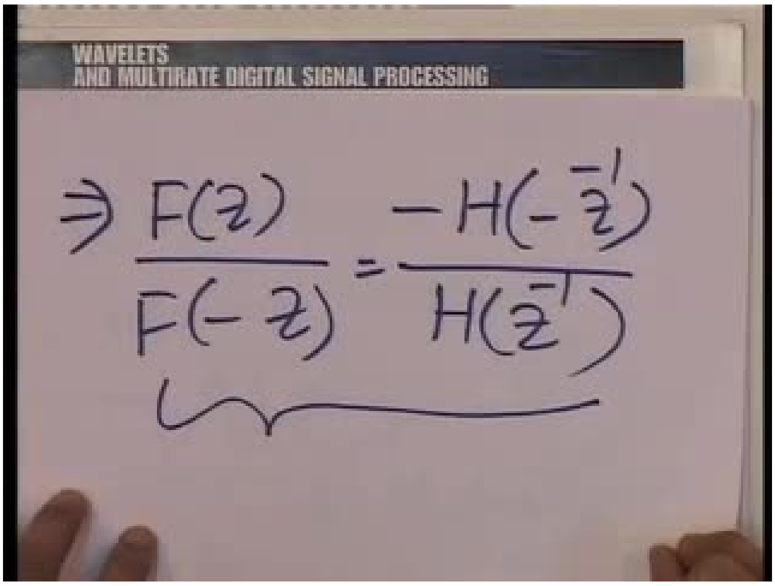

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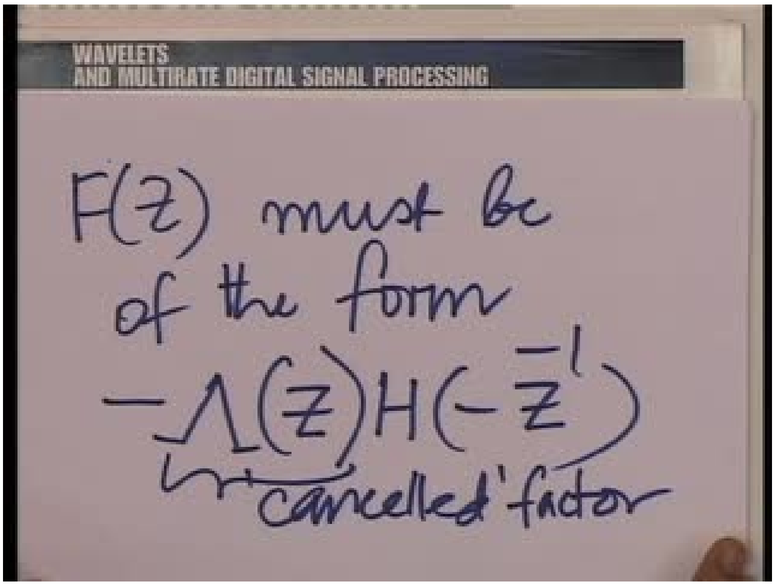

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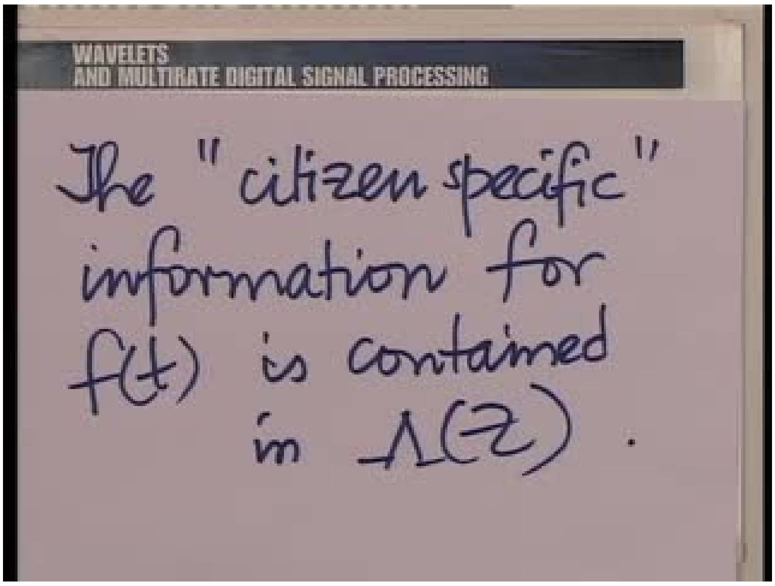

So, in this ratio, f of z by f of minus z we got on the right hand side something that has no specificity at all. It depends on the low pass filter essentially of the filter bank, and therefore, we can note that when we take a ratio the difference between the numerator on the left hand side and the right hand side must in general be a function of z. So, we could in general say from this that, f of z must then be of the form. Some capital lambda z, a function of z times minus h of minus z inverse. So, this is the factor which got cancelled; cancelled in that ratio. What I mean is - when we took this ratio f z by f of minus z, some factor must have got cancel to result in something independent of that particular function, and that cancelled factor is lambda z, and therefore, lambda z capital lambda z contains information specific to that particular citizen f t in w zero. So, the citizen specific information as you might call it for f t is contained in lambda z.

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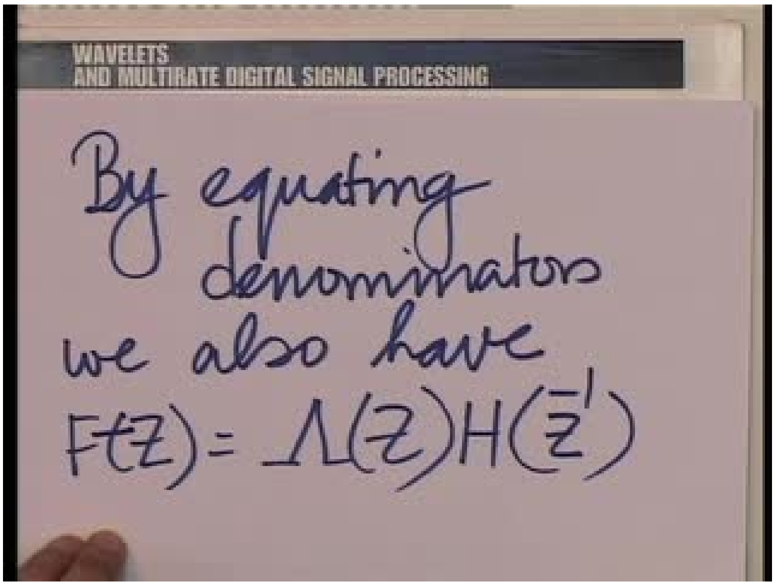

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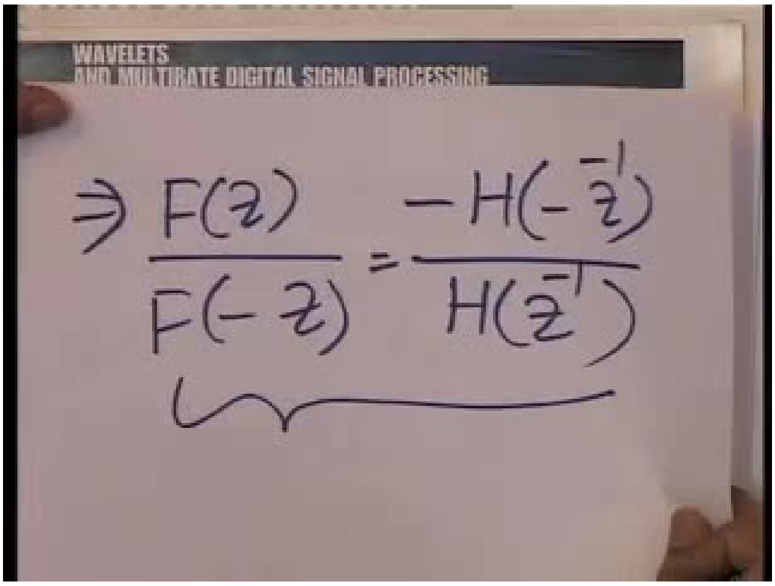

I am not saying that all of lambda z must be citizen specific, but whatever citizen specific information there is, is all captured in that lambda z. Now, let us also make an observation. By equating the denominators, we also have f of z is lambda z times h of z inverse or other f of minus z, I am sorry, and I put back the ratio before you to explain. What I am saying is if lambda z is the factor that has got cancelled then f of minus z must also be lambda z times h of z inverse, and therefore, we have two equations now.

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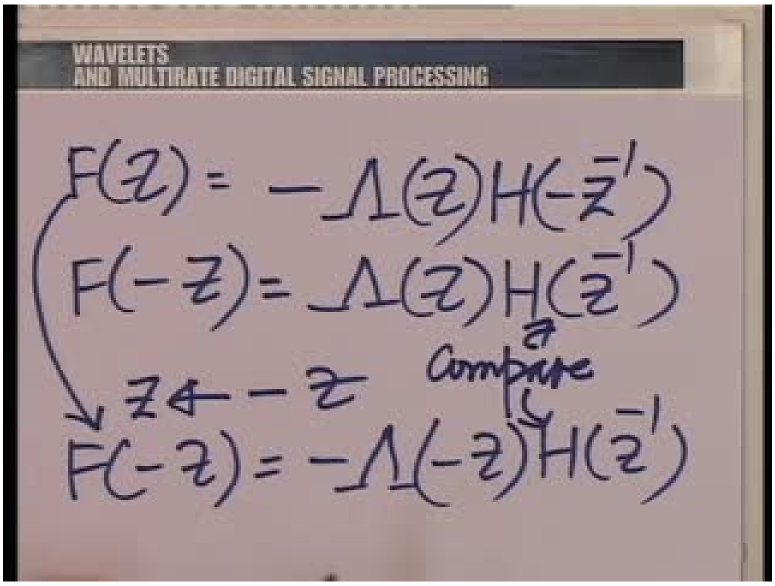

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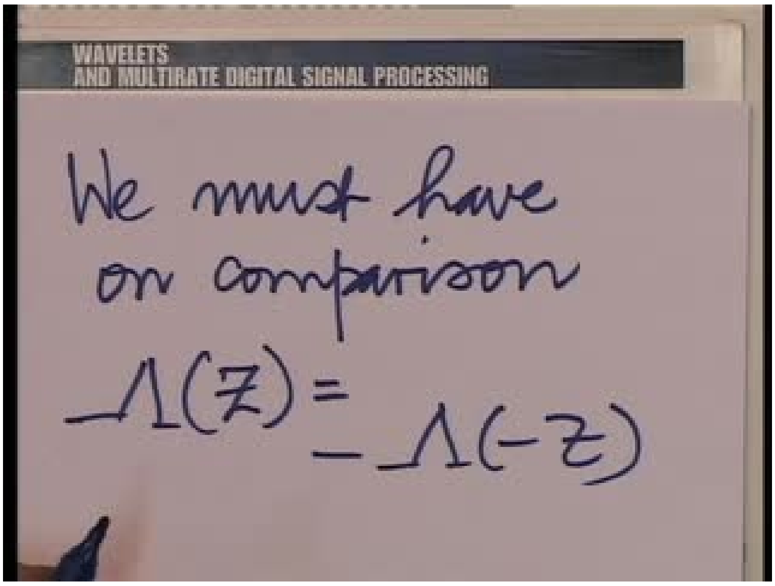

We f of z is minus lambda z times h of z inverse minus z inverse and f of minus z is this, but then, here, if we substitute z by minus z, we get f of minus z from here would be minus lambda of minus z h of z inverse, and now, if we compare this and this, we note that since h of z inverse is a common factor and is not identically zero, these must be identical. So, you must have lambda of z is equal to minus lambda of minus z. We must have and this amount to saying that lambda z plus lambda minus z must be zero.

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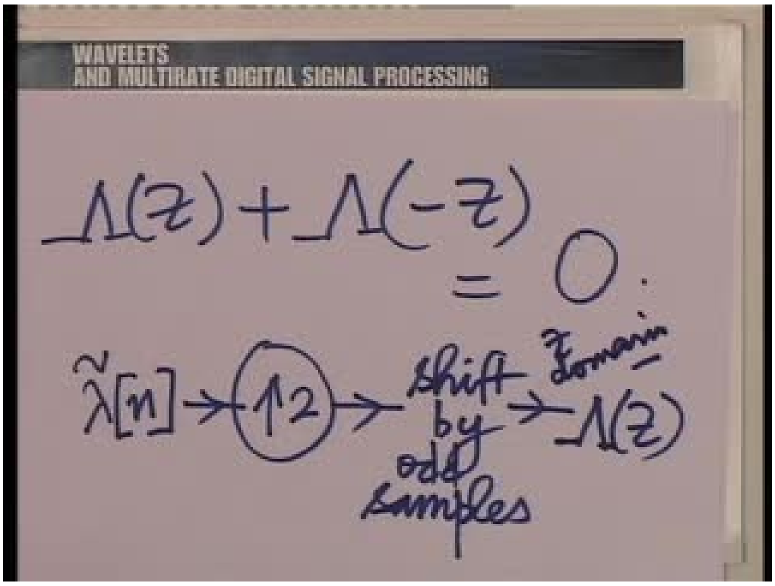

Now, what is this mean in terms of sequences? You see, it means that if lambda z is the z transform of a sequence, then that sequence should be zero at all the even locations; which means that the sequence should have been obtained by, you could think, you see, when a sequence is zero at all the even locations, we are effectively saying that the sequence could have been obtained by up sampling another sequence, and then, shifting by one place. You see, when you up sampling a sequence, you introduce zero's at all the odd locations up sample by a factor of two.

So, when you up sample a sequence by a factor of two, you introduce zero's at all the odd locations. Now, if you wish all those zeros to shift to the even locations, all that we need to do is to shift this up sample sequence by an odd number of samples.

So, in other words, it is a sequence; let us call it lambda tilde, a lambda tilde n, which on up sampling by a factor of two, and then shifting by an odd number of samples gives us the z transform lambda z. This is of course, the z domain representation.

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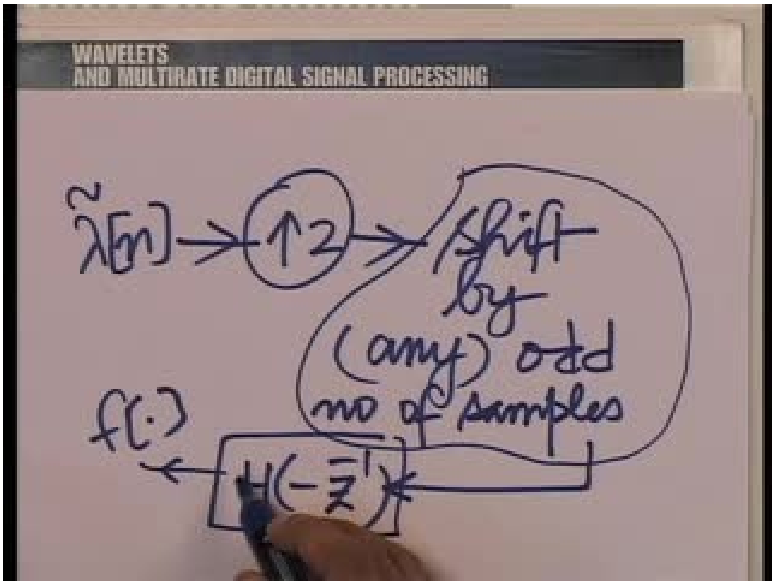

And what we are saying is an edition is that to get f said, we effectively have to cascade this. You have lambda tilde n, and then you have an up sampler by a factor of two. You could shift by any odd number of samples, and then subject this to the action of h of minus z inverse and what we get here is essentially the sequence f. This is what we essentially said.

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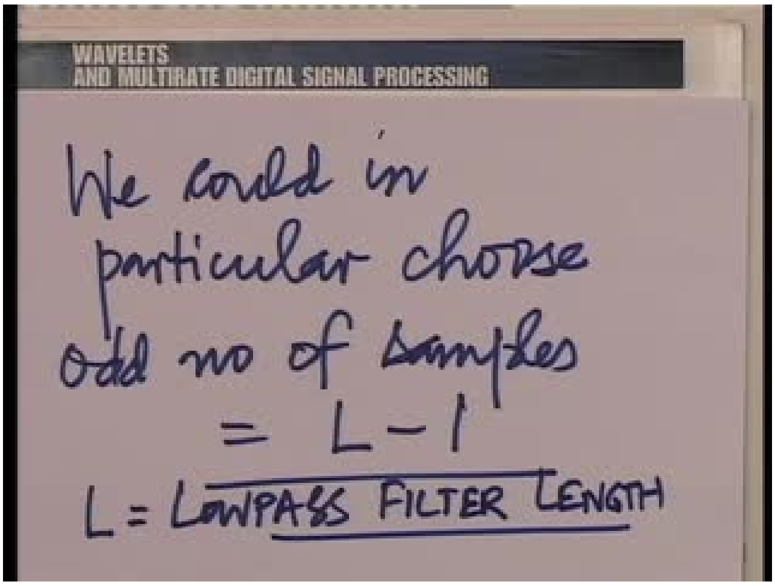

Now, we could choose that odd number of samples strategically. You know h of minus z inverse in this discussion. If you look at this sequence of steps here, this h of minus z inverse looks very much like the analysis high pass filter. The only difference between this and the analysis high pass filter is that you need to put an odd delay here. So, we could in particular choose the odd number of samples to be equal to l minus one, where l is essentially the low pass filter lengths. So, for example, for Daubechies four or in other words, Daubechies filter bank whether length of the filter is four, l is equal to four, and if you recall, l minus one is the delay that we need to introduce once we replace h z by h of z inverse to get the analysis high pass filter.

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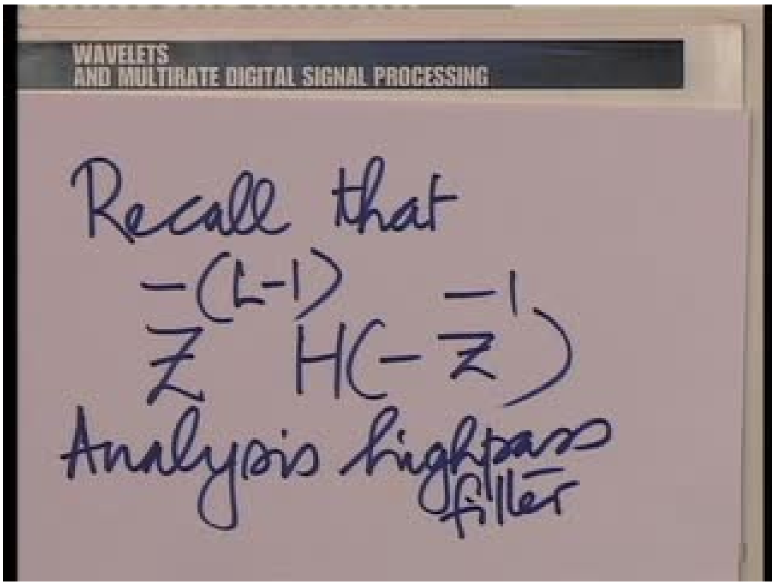

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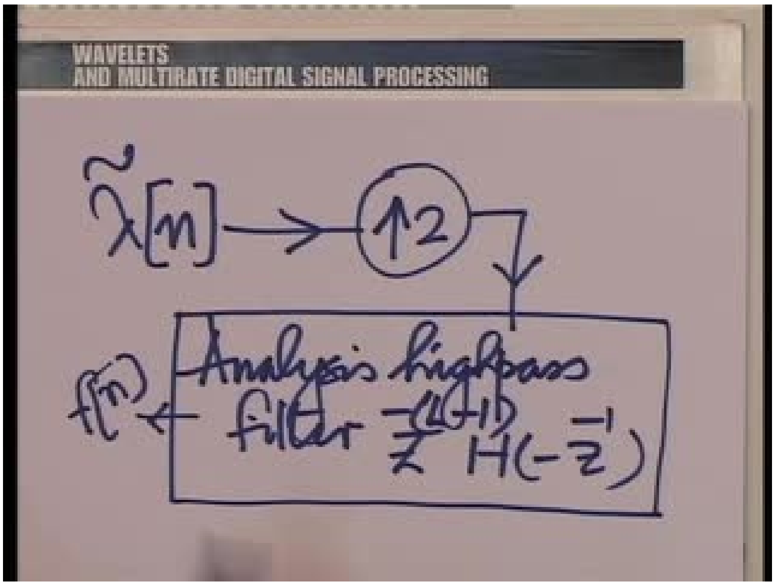

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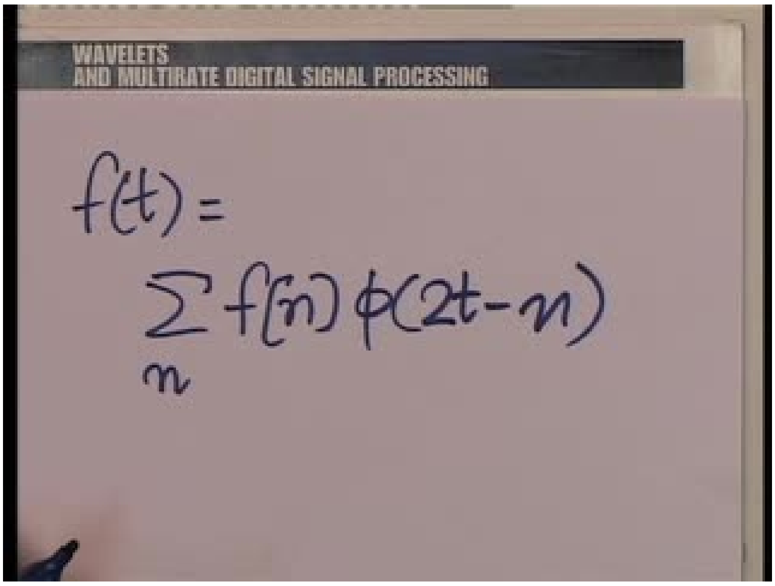

What I am saying in effect is recall that z raise the power minus l minus one times h of minus z inverse is essentially the analysis high pass filter, and therefore, what we are saying in a effect is that we could obtain f n as follows. We could take this hypothetical lambda tilde which has been up sample by two. We could subjected to the action of the analysis high pass filter z raise the power minus l minus one h minus z inverse, and this would produce f for us, and once we have f n f of n the sequence f, we can reconstruct the function f of t, and lowed be halt f of t is therefore, summation on n f n of course, when not stated, we name the summation runs over all the integers. So, summation f n phi two t minus n.

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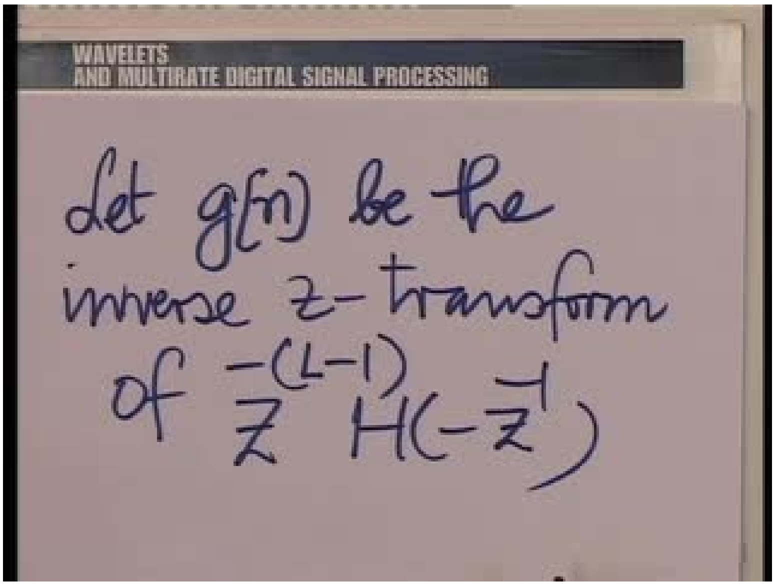

Now, let us denote the impulse response of the analysis high pass filter by g n. So, let g n be the inverse z transform of z race the power minus l minus one h of minus z inverse.

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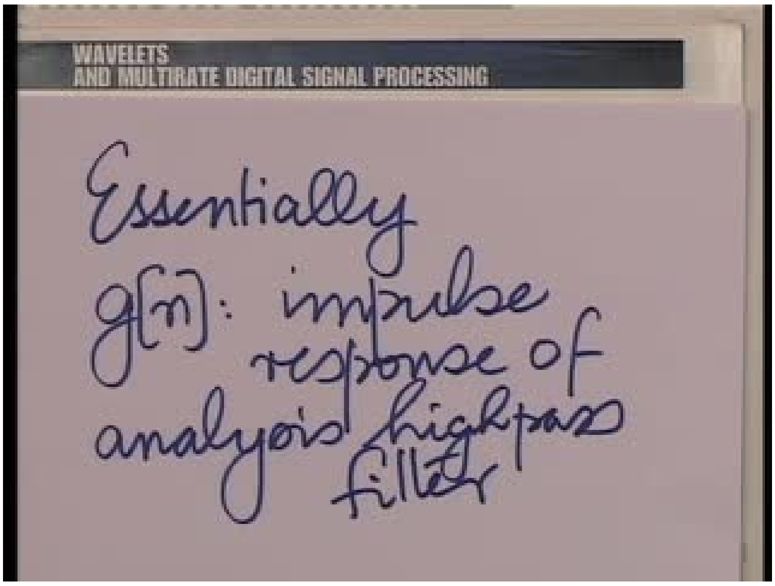

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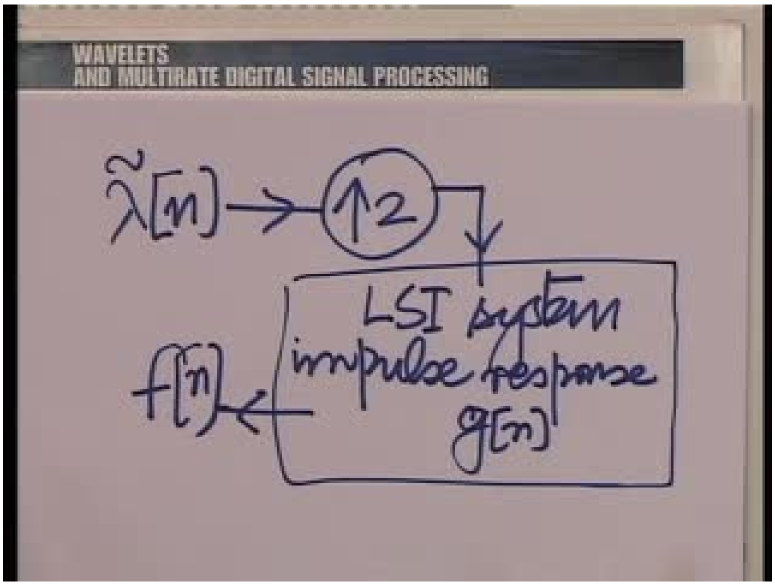

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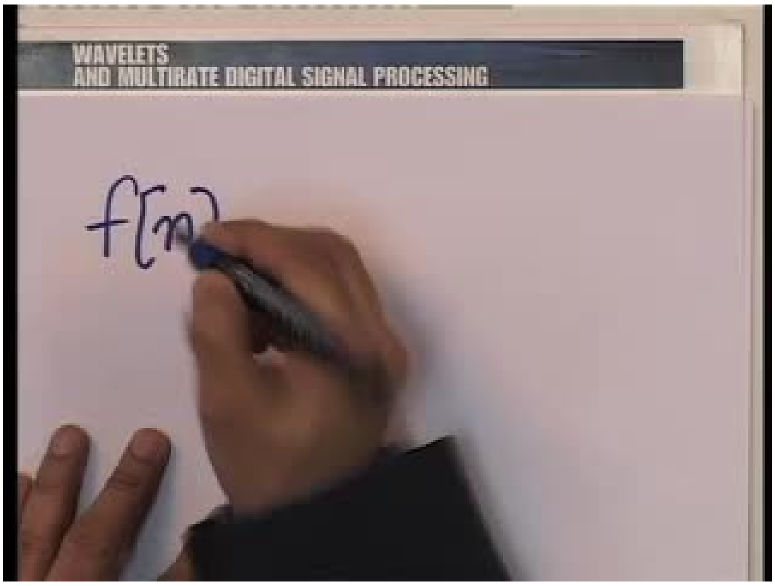

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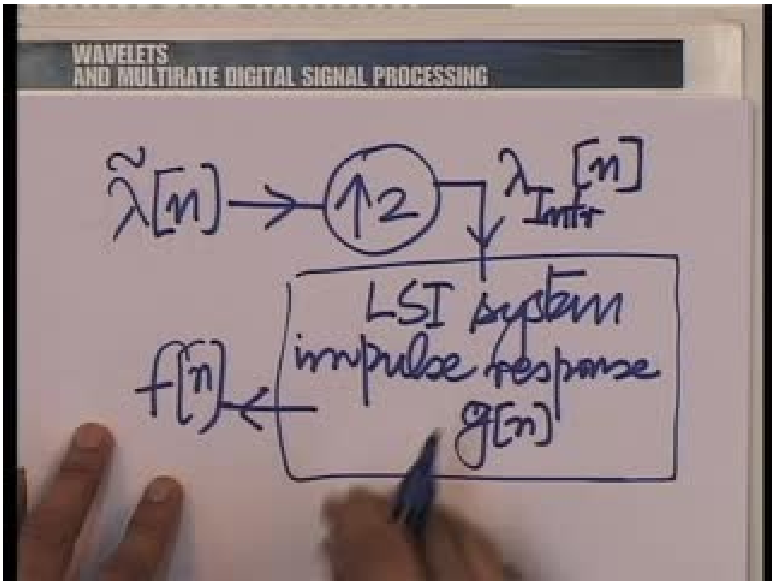

Essentially, g n is the impulse response of the analysis high pass filter. Where upon, what we are saying is - lambda tilde a up sample by two subjected to the action of an l s i system linear shift and variance system with impulse response g n results in f f of n to wish to be very precise, and f of n must therefore be, and for that we shall first introduce an intermediate sequence here. Let us call it lambda intermediate, lambda intermediate of n.

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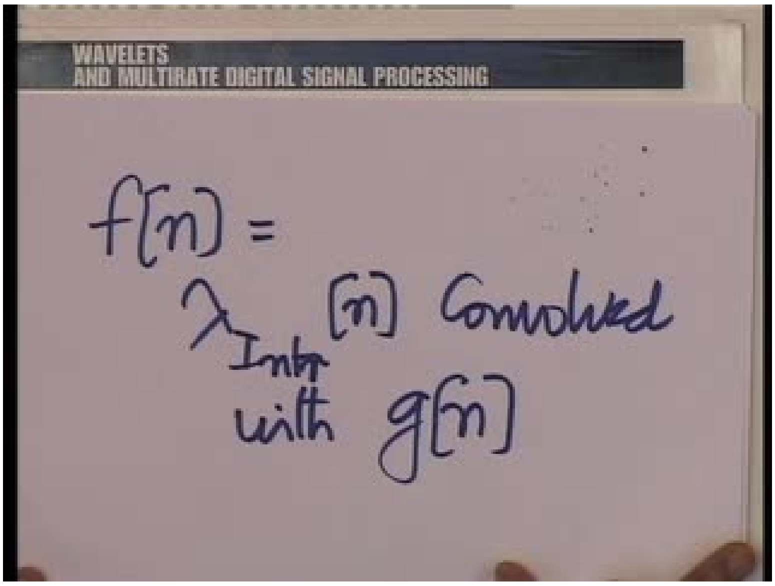

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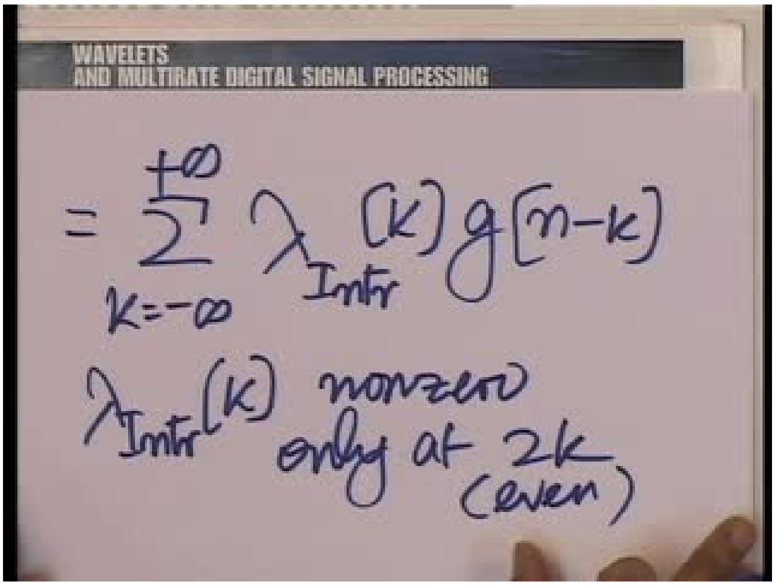

This lambda intermediate is of course, convolve with g n to get f n. f n is lambda intermediate convolved with g n, and that can be written as summation k running from minus to plus infinity lambda intermediate k g n minus k, but then lambda intermediate k is non zero only at in at even k, and therefore, we need to replace this summation only with summation on k and k replaced by two k here, only even locations.

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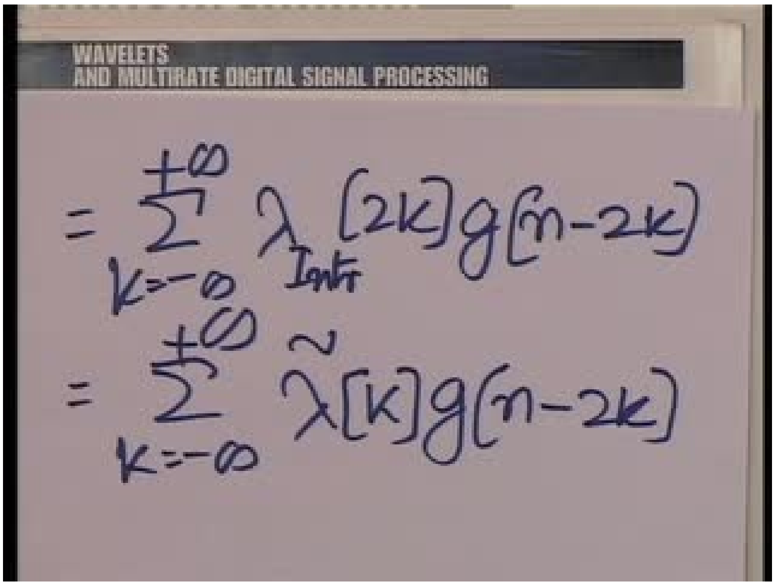

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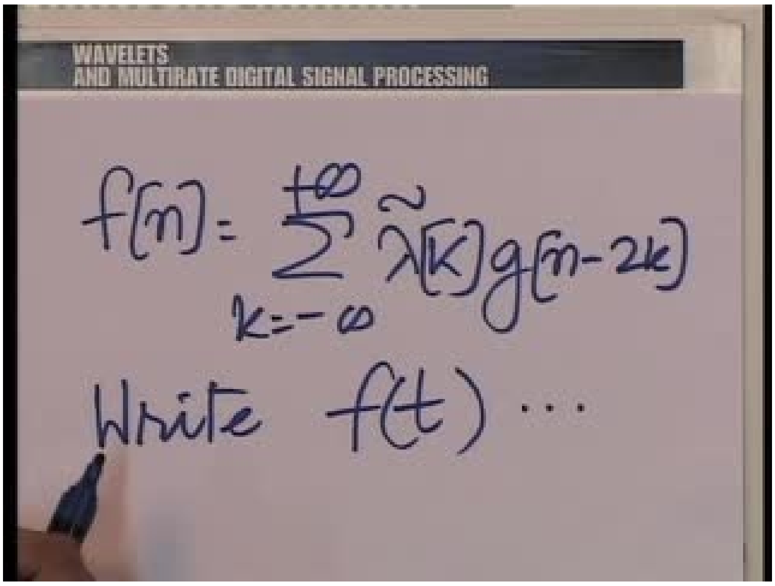

So, this summation can we return as summation k again going from minus to plus infinity lambda intermediate that two k, all other points are zero, remember, because it is up sample g n minus two k, and lambda intermediate at two k is simply lambda tilde a at k, an interesting conclusion, and this is f n for you. So, we will just get a barring once again what we concluded is that f n has this form, f n is summation over all integer k lambda tilde a k g n minus two k.

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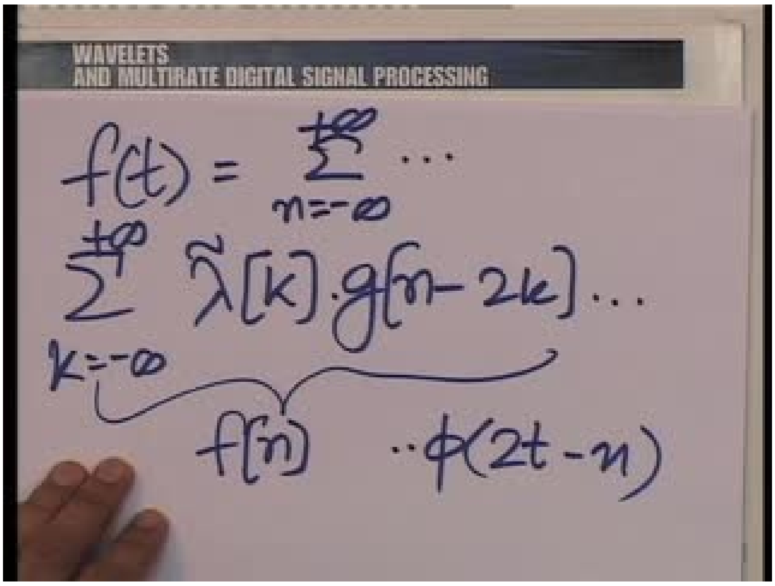

And therefore, we shall now write down f t, and f t the typical the total typical function in w zero has the following form: f t is summation on all k lambda tilde k g n minus two k times. Now, this is, you see, there is a summation, this is f of n, the whole thing is f of n, and this things to be multiplied by phi two t minus n and some over all n to some over all n times this.

Now, here, we have an external summation on n, on an internal summation on k. Let us interchange the order. So, let us do the summation k outside and the summation on n inside.

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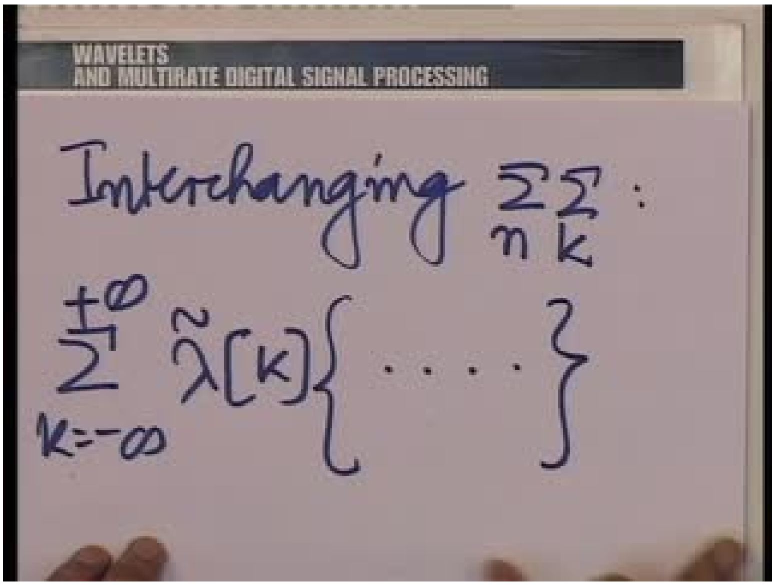

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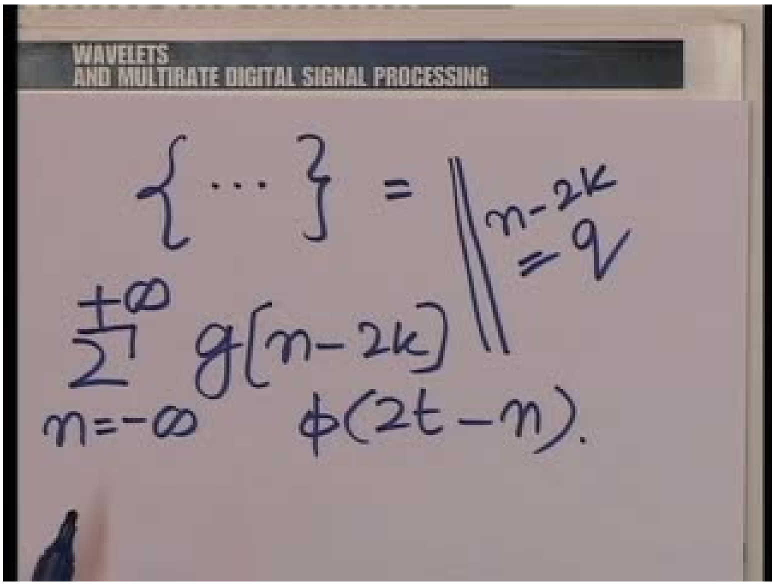

So, interchanging the summation, we would get summation on k outside lambda tilde a k, and then, I will write the summation on n inside as follows: summation on n over all n g n minus two k five two t minus n, and now, we play the standard trick - k is fixed here. So, let us put n minus two k equal to another variable. Let us call it q.

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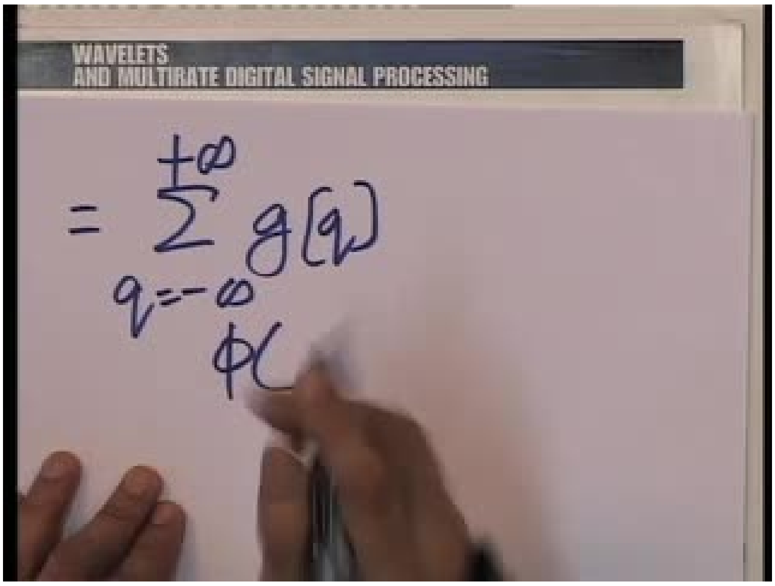

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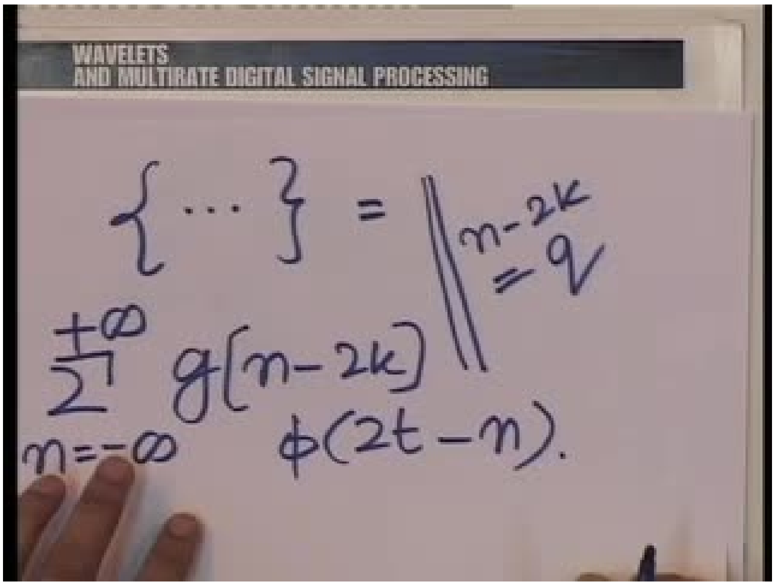

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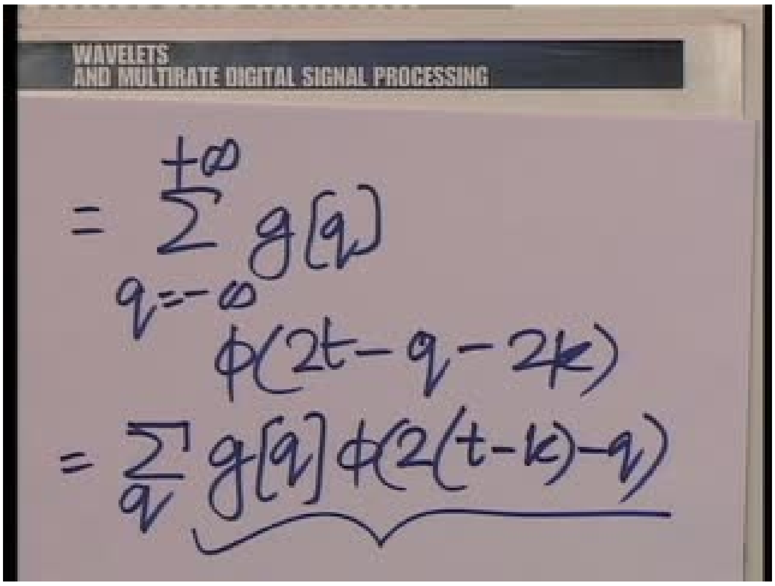

Since k is fixed, when n runs over all the integers, so does q, and therefore, we can re-write this as summation q running over all the integers g of q phi of now, well, n is clearly q plus two k, and therefore, we have two t minus q minus two k here, and now, we can take two column from this and write summation on q again interpreting it over all the integers g of q times phi two t minus k minus q, very interesting.

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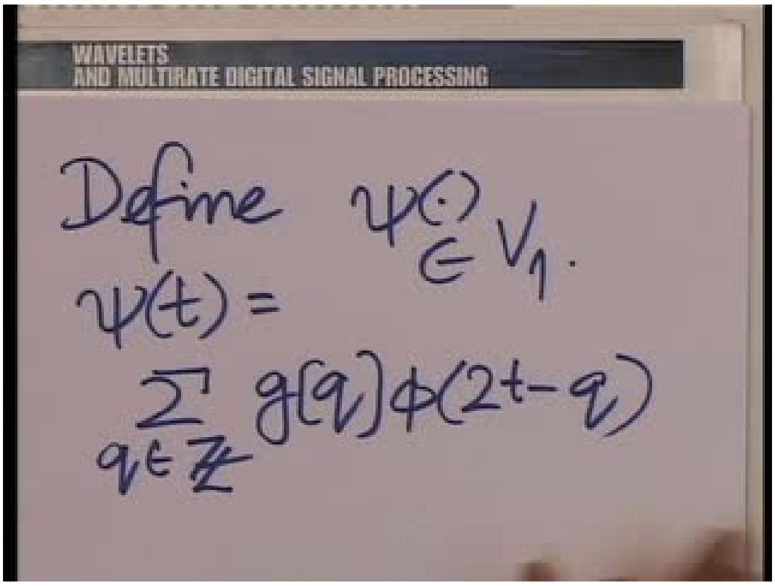

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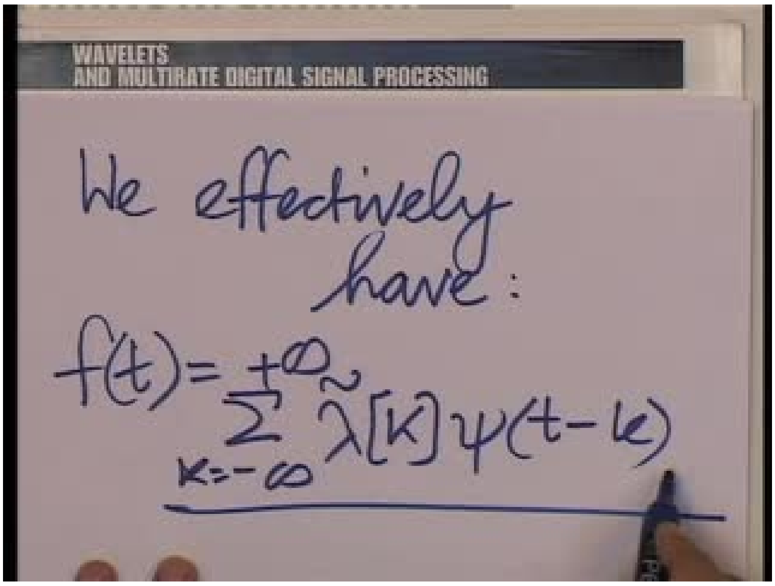

Now, here we are almost at very wish to be. You know, if you look back at the expression that you got here, this t minus k is a shift on the continuous variable. So, suppose for the moment you forget t minus k and replace it by t, and you notice that you are essentially making a linear combination of phi two t minus q with the coefficients g q. You realize that you are trying to create a function in v one here. Define: therefore, the function psi t to be summation on oval integer q g q phi two t minus q. Essentially psi belongs to v one, and then, what we have is effectively, this proto typical function f t is summation on all k lambda tilde k psi t minus k. That is what we have.

So, effectively, what we are proved is that this proto typical function f t in the orthogonal complement of v zero in v one, that is w zero, is expansible in terms of integer translates of psi t, and that is exactly where we wanted to go. If you could capture the single function psi t, and all its integer translates could form a basis. Essentially could span the space w zero or job is more or less done, and therefore, we more or less prove what we wanted to.

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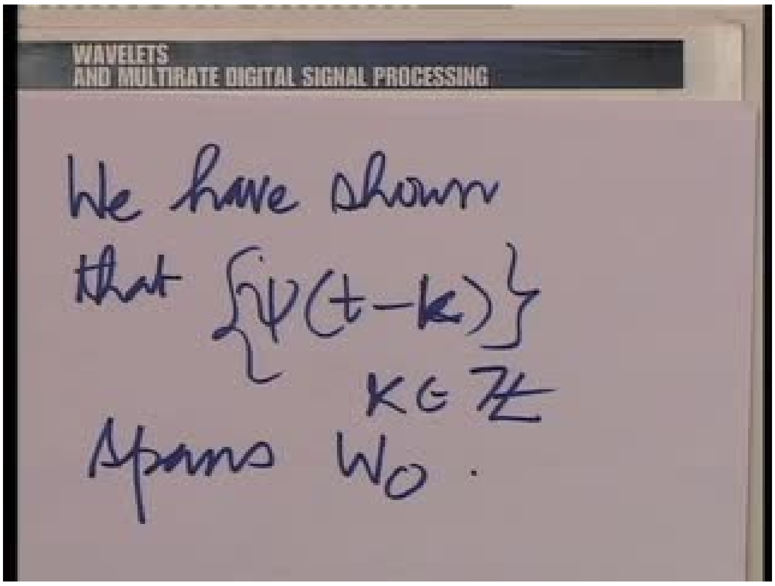

This was the psi t for which we are looking. We have shown that psi t minus k k over all the integers expands w zero, because it can represent any function in this orthogonal complement by linear combinations.

Now, some details need to be completed. We need to show that psi t is orthogonal to its integer translates. So, we of course, of shown that psi t minus k expands w zero. What we have not shown is that the psi t minus k form on orthogonal basis. We also technically need to show that the psi t minus k are orthogonal to all integer translates of phi phi t minus m if you please for integer m, and these are the detail that we need now to take up and complete.

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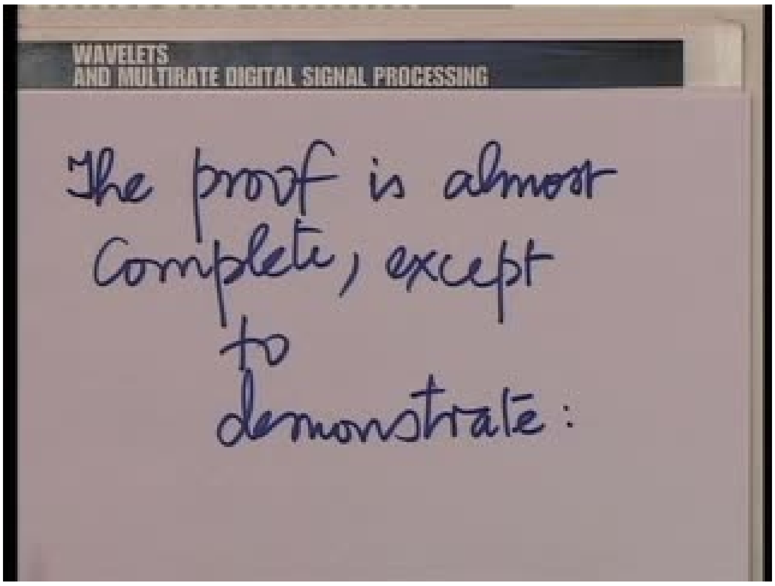

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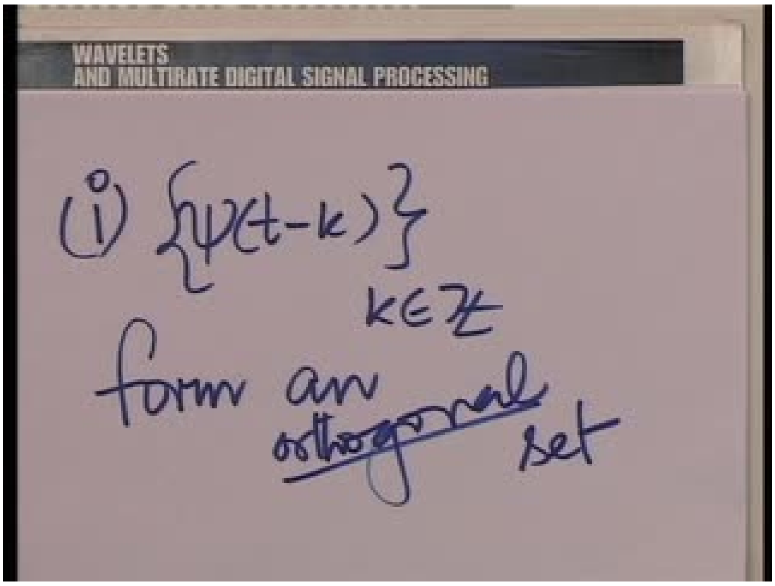

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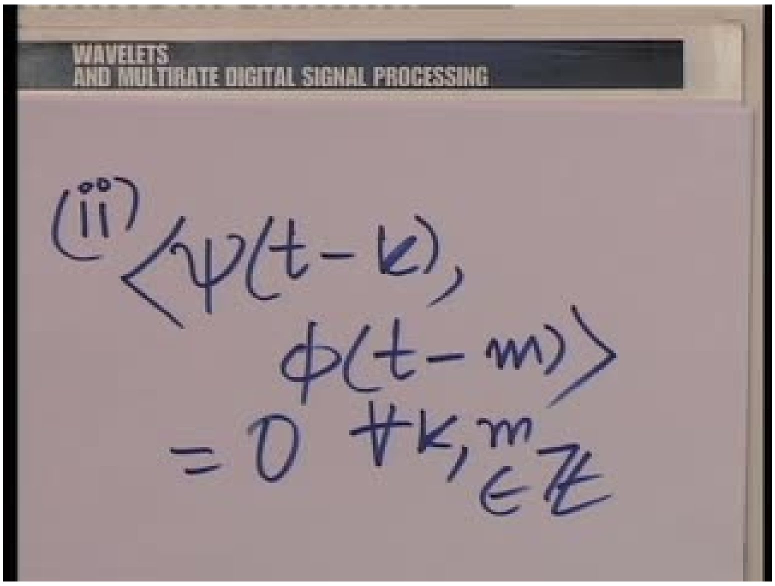

Let us put down what we need to do. The proof is almost complete expect to demonstrate the following things: psi t minus k over all integer k form an orthogonal set. For, the, the inner product of psi t minus k and phi t minus m is zero for all k and m belonging to the set of integers. So, essentially that the integer translates of psi and the integer translates of phi are orthogonal.

Now, how would be go about proving this? We again use the same strategy of cross correlation. We would expand psi t in terms of its basis phi two t minus n and phi t also in terms of the same basis, and we shall use the z domain properties of the low pass filter. That we have to prove this two results. We shall do this in the next lecture before we proceed to discuss variations of dyadic multi resolution analysis that emerge from here. Thank you