Advanced Digital Signal Processing-Wavelets and Multirate
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Module No. # 01
Lecture No. # 25
The Theorem of (DYADIC) Multiresolution Analysis

A warm welcome to the 25 th lecture on the subject of Wavelets and Multirate Digital Signal Processing. Let us put in perspective what we are going to do in the lecture today. In the previous two to three lectures we have been moving from continuous translation scale to discrete scale were the discretization of scale is logarithmic in nature and then specifically to dyadic scale, where this scale is discretized in powers of 2. However, the translation parameter is still continuous and today, we intend to discretize the translation parameter with dyadic scale leading to the specific situation of dyadic multi resolution analysis that began this course with.

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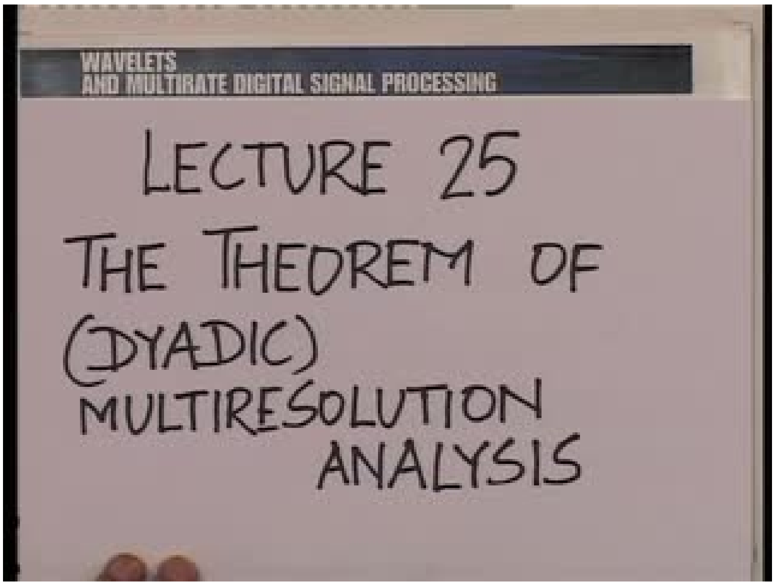

Before we proceed to prove a very important principle which essentially is the principle of discretization of translation in the contest of dyadic scale, call the theorem of multi resolution analysis and that is how I title the lecture today.

I have focused today on the theorem of dyadic multi resolution analysis. Essentially, an effort to discretize the translation parameter knowing that the scale has been discretized in a dyadic manner in powers of 2. Before we proceed to prove that theorem and to establish the results of that theorem, we would like to bring out the link between what we did in the previous lecture and were this theorem stands in the next few minutes.

Yesterday, we had indirectly talked about orthogonal filter banks. We had said that if the - so called - wavelet and the corresponding synthesis wavelet, remember we have talked about a psi and a psi tilde; psi on the analysis side and on a psi tilde and the synthesis side. We said in principle the more general situation is that you have different wavelets on the analysis side and the synthesis side when you discretize the scale parameter in a logarithmic manner.

What it really means is that the most general situation is where the analysis wavelet and scaling function is not the same as the synthesis scaling and wavelet function and this is a discovery for us beyond what we been doing all the while before we started talking about uncertainty.

Now, we would like to (( )) give a name to those kinds of filter banks which lead to different wavelets on the analysis and the synthesis side. Then, we would like to make specific the case whether they are same.

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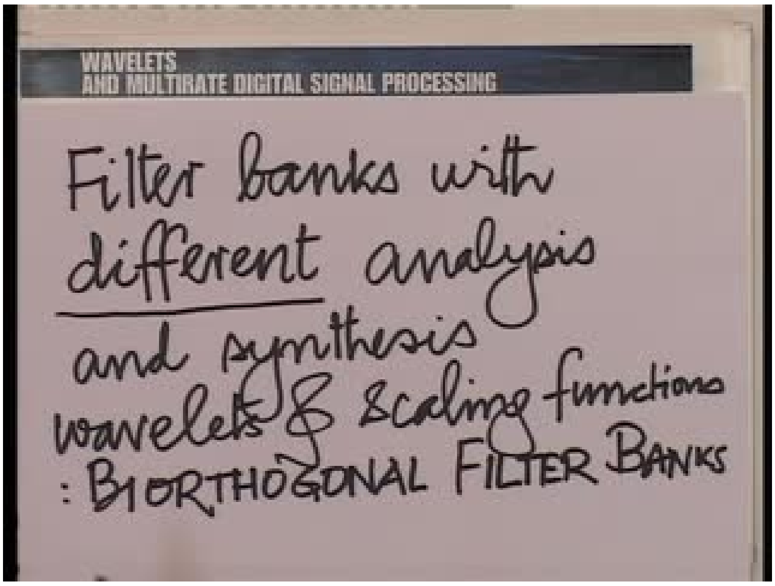

Therefore, let us give these names, filter banks with different analysis and synthesis wavelets and with scaling functions are called bi-orthogonal filter banks, so these are more general. Now, please remember that when we talk about filter banks here we are always preferring to filter banks with perfect reconstruction and I think that point must be written down very clearly for emphasis.

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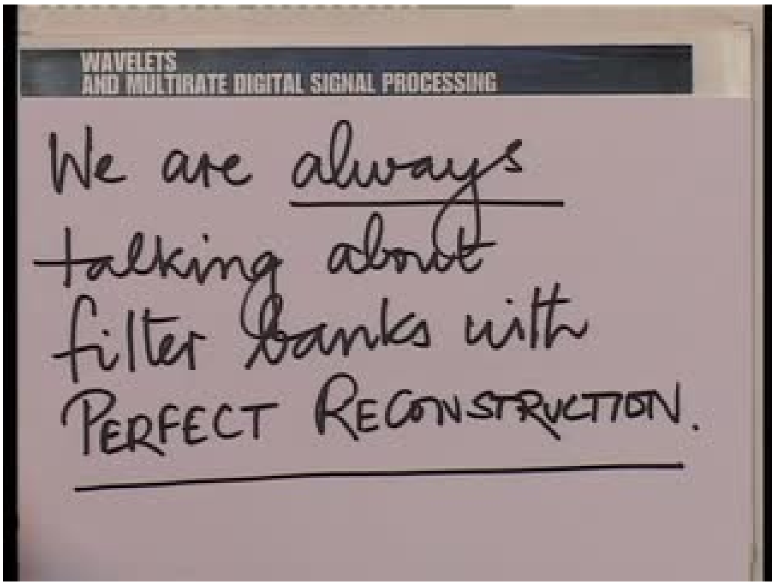

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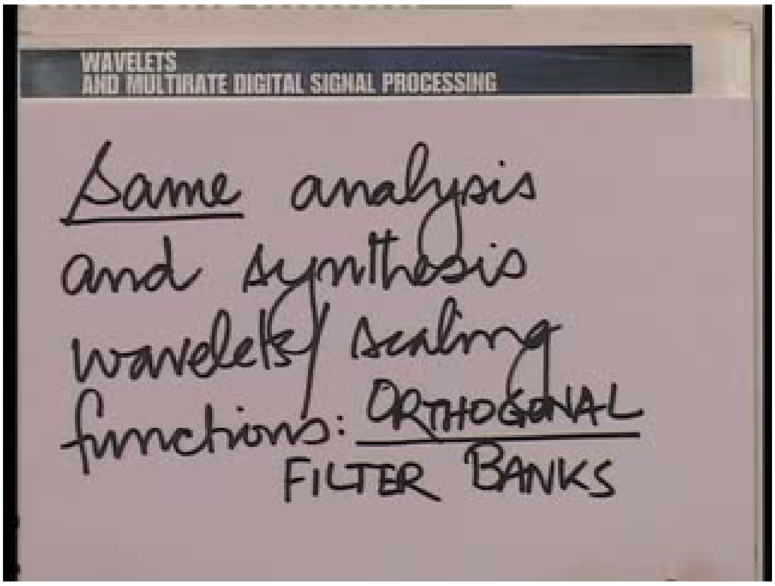

We are always talking about filter banks with perfect reconstruction. In particular, when the analysis and the synthesis wavelets and scaling functions are the same then those filter banks are called orthogonal.

We shall be largely focusing on orthogonal filter banks today. Before we proceed to the filter bank where we have essentially discretized - of course last time when we talked about filter banks we were talking about continuous time filters, remember. Now, we have moved one step ahead from the kinds of filter banks that we refer to earlier on in this course, where we only spoke of discrete time filters. Now, we have continuous time filters in the filter bank and we will just recall a couple of important steps that let us to a discussion towards the end of the previous lecture.

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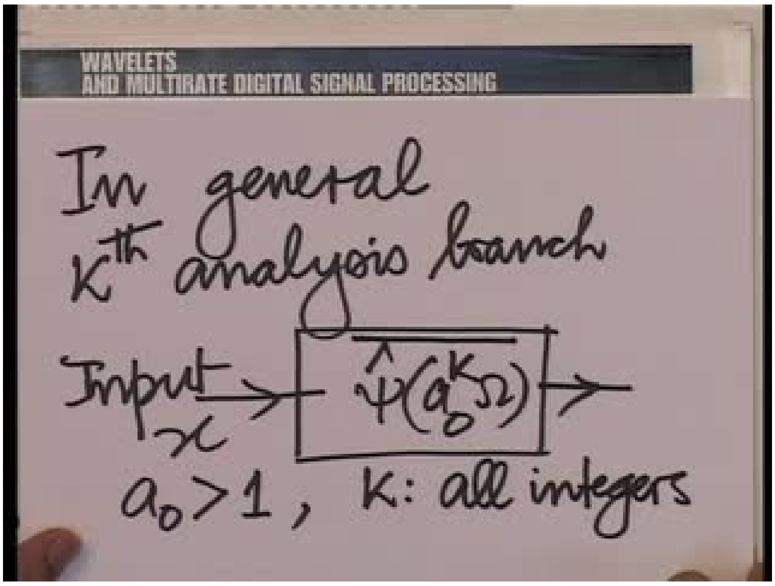

Namely, we said that the K th analysis branch who take the input x and subject it to a filter with a frequency response psi cap a naught raise the power of K omega complex conjugate a naught - as you recall - was any number greater than 1 and of course, K ran overall the integers.

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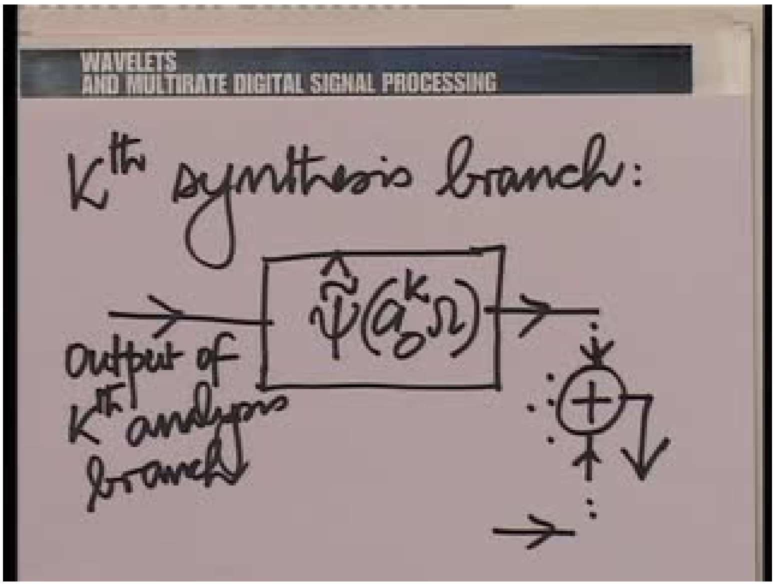

The K th synthesis branch essentially took the output of the K th analysis branch as it input and subjected it to the action of a filter whose frequency response was psi tilde cap a naught raise the power of K omega and all this K th synthesis branches were added together on the synthesis psi to produce the output.

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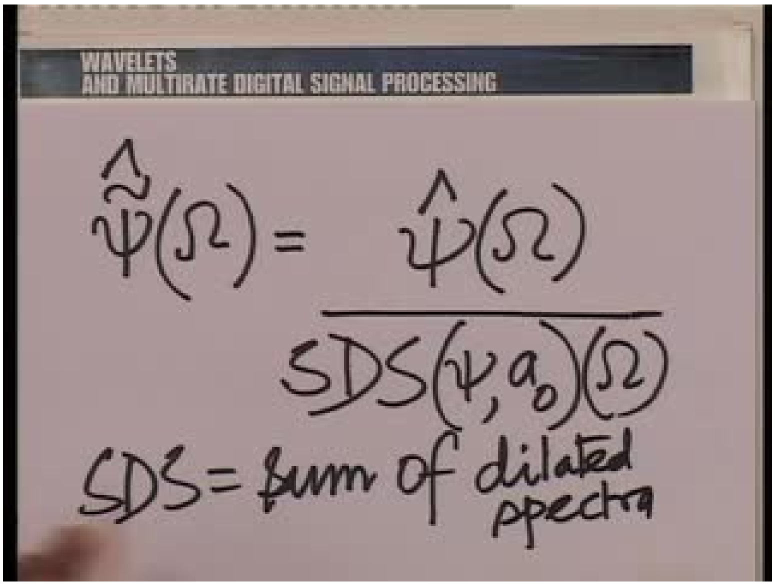

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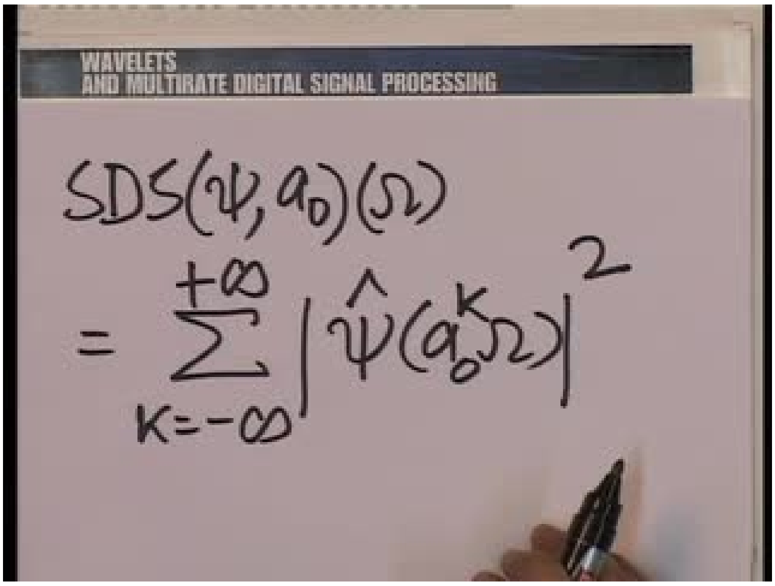

We are also specified what psi tilde was, we began by specifying psi tilde in terms of Fourier transform, in fact it is very difficult to specifying terms of the time domain expression. So, its specify psi tilde cap omega to be psi cap omega divided by the sum of dilated spectra of psi with the parameter a naught evaluated at omega. SDS as you recall was an abbreviation for the Sum of Dilated Spectra and we had defined the sum of dilated spectra as follows.

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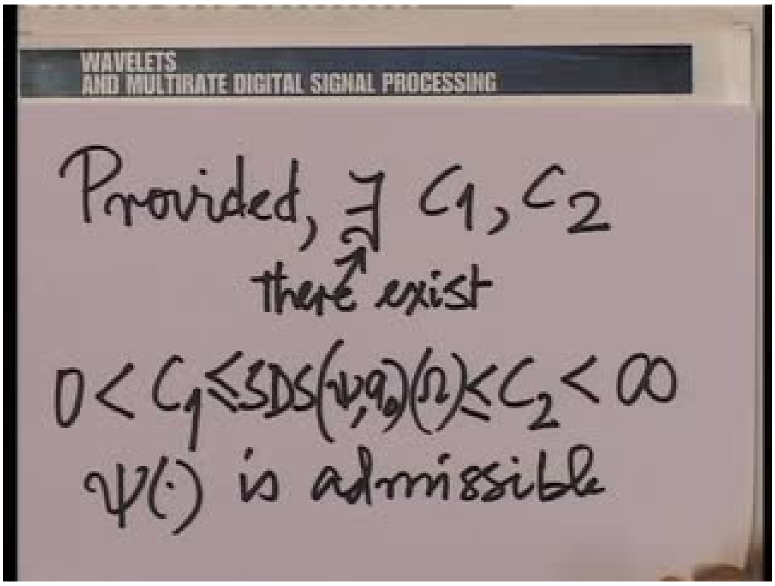

Dilated spectra because of this term the frequency variable is replaced by a dilated version of the frequency variable, a naught raise the K power times omega, that is why a sum of dilated spectra. We noticed that, provided there exits - this means there exits or their exits - C1 and C2, so C1 is strictly greater than 0, C2 is strictly finite, strictly less than infinity and this SDS of psi and a naught for all omega is between C1 and C 2. You may guaranteed that psi is admissible and this followed from the upper bound here.

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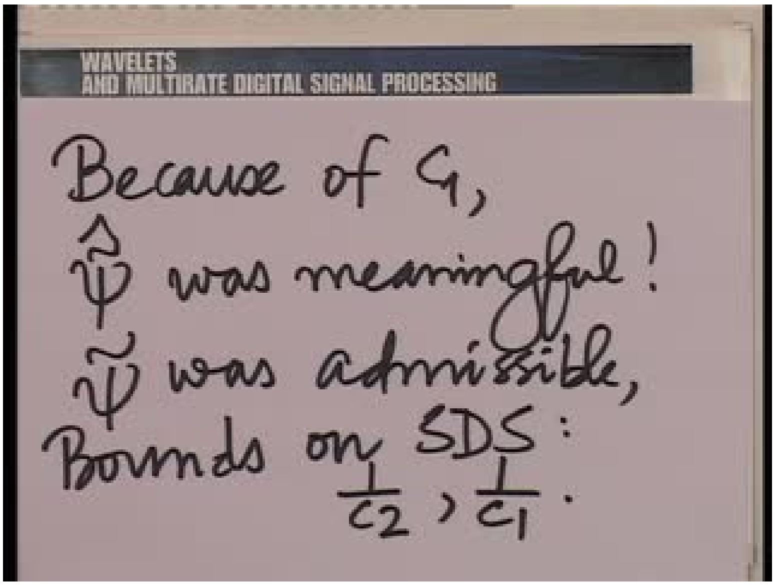

Further we guaranteed because of C1, psi tilde cap was meaningful. Because of this condition on upper and lower bounds it was always true that psi tilde was admissible and the balance on SDS for psi tilde were essentially 1 by C2 and 1 by C1. Now, with all this recapitulation we also said that in case the sum of dilated spectra SDS psi a naught was constant for all omega you had an orthogonal filter bank there.

In fact, the sum of dilated spectra being constant for all omega essentially means that there is no difference in the way different frequencies are treated and we could see that if had ideal filter so called an coat, an coat ideal frequency responds for psi for example, a frequency responds which was 1 between pi and 2 pi then, you automatically get perfect reconstruction and consequently a constant some of dilated spectra.

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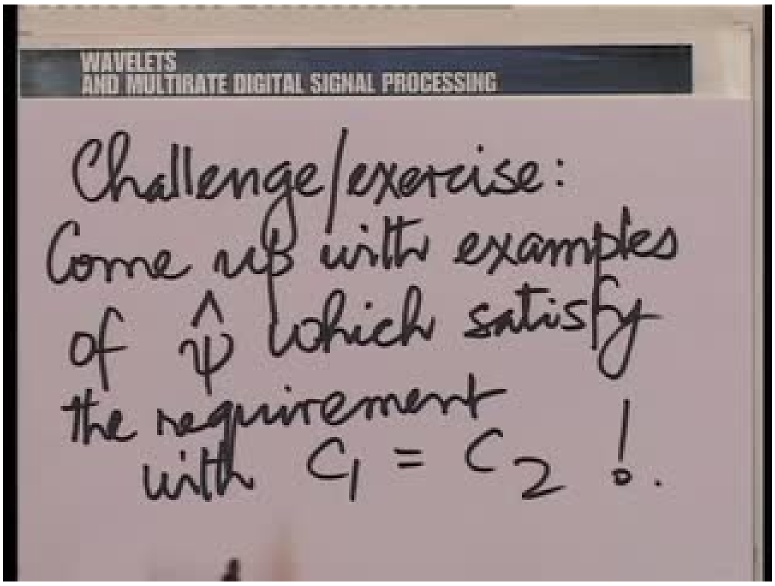

So, we begin to see the inter relationships of course, the ideal case is not the only case. At this point I would like the class to reflect to another possible cases that they might be, I answer the question right the way because some curiosity should be evoked. So, I put this exercise or this challenge before the class - partial challenge, partial exercise. Come up with examples of psi cap which satisfy the requirement with C1 equal to C2.

Essentially, the sum of dilated spectra being a constant, anyway not just the ideal case. Well, what I wish before we proceed to the dyadic case is to note one more variation that can be introduced. You see here we are admitting the idea of biorthogonal filter banks, where the analysis and the synthesis side are different in terms of the wavelets and scaling function involved. Now, we can see in a minute that, once we have this conditions satisfied by psi we can also construct an equivalent orthogonal filter bank from that psi, were the analysis and the synthesis wavelets are the same but different from psi and that is done as follows.

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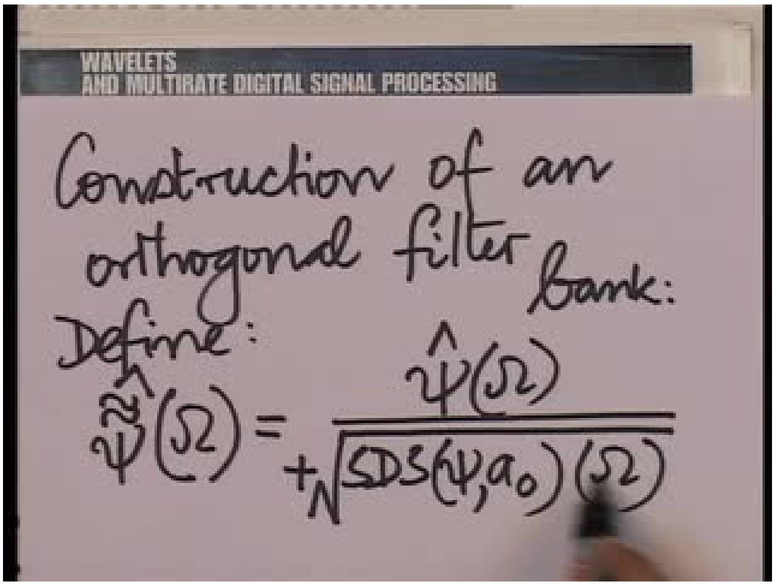

So, construction on orthogonal filter bank, let us define psi double tilde in terms of Fourier transformer, psi double tilde cap omega is defined to be psi cap omega divided by not the sum of dilated spectra, but the positive square root of the sum of dilated spectra.

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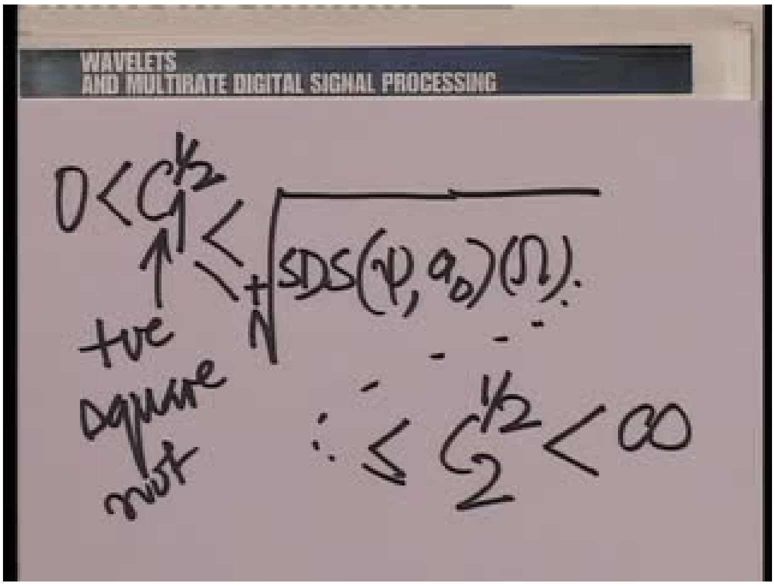

Now, again please note that because of the upper and lower bound on SDS it is meaningful to put this in the denominator after all we are guaranteed because of the positivity that 0 is strictly less than C1 to the power half - I mean - I am talking about the positive square root less than or equal to the positive square root of SDS psi a naught omega continued this way, less than equal to C2 to the power half positive square root again and this is of course strictly less than infinity.

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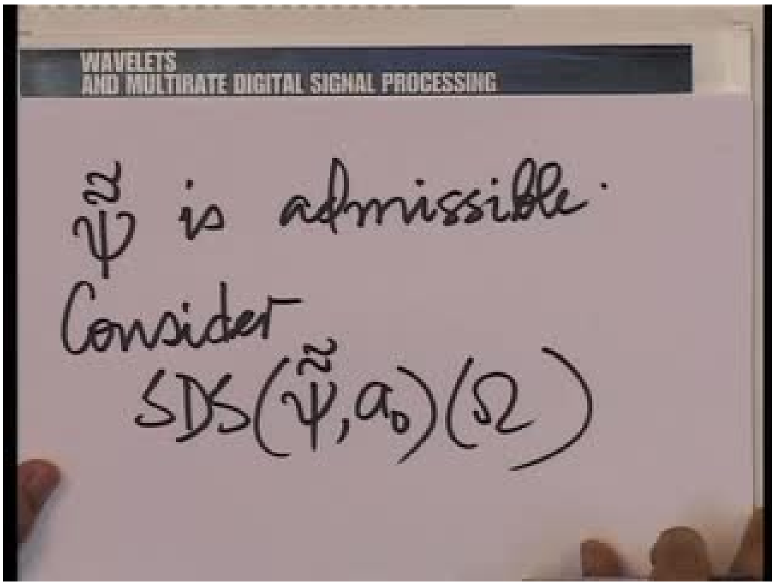

Therefore, it is meaningful to divide in the denominator that is a meaningful operation. With that observation, we shall first proof that psi double tilde cap for other psi double tilde is admissible and to proof that admissible we only need to establish the sum of dilated spectra. So, we need to consider the sum of dilated spectra of psi double tilde with a naught evaluated at omega, if you can say something about this we have done our job.

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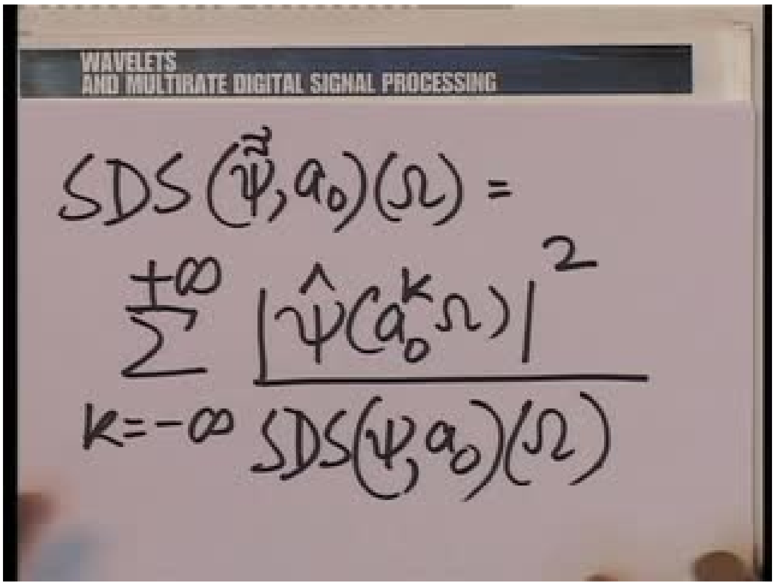

In fact that quantity as you can see with great ease is simply summation, K going from minus to plus infinity psi cap a naught raise the power K omega mod squared divided by - well please remember the sum of dilated spectra is independent of multiplication of the independent variable by a naught to the power of K. Anyway, what I will do is still write down, here square we get the denominated, psi as it is, a naught into omega and what you saying is that this is naught affected when we replace omega by a naught to the power of K, I will just recall that reasoning.

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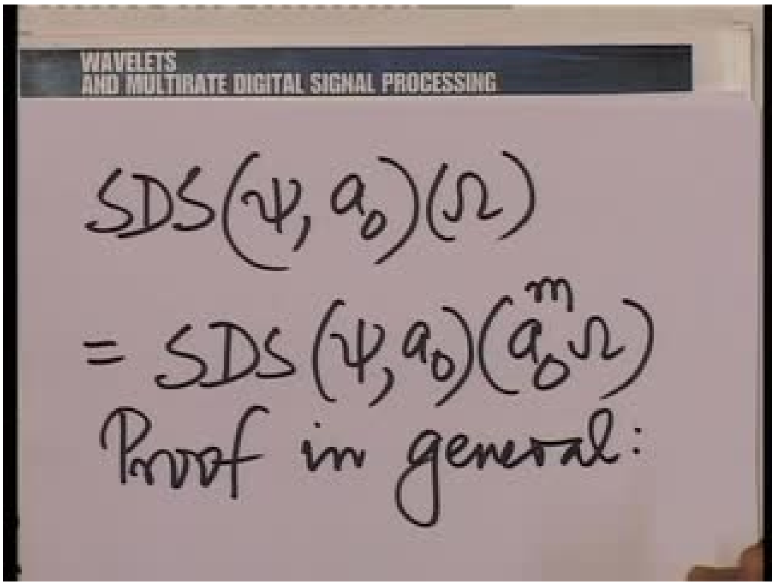

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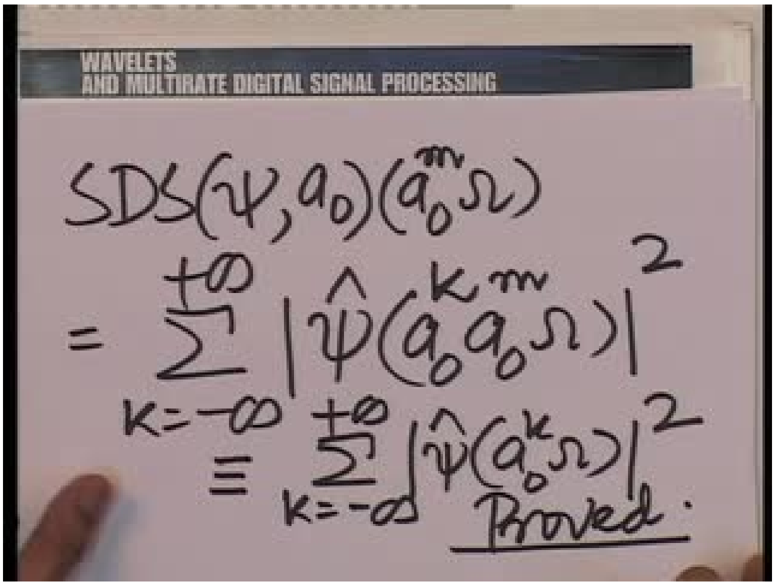

You see what we are saying is SDS of any psi a naught of omega is equal to SDS psi a naught a naught to the power of m omega, we proof this yesterday but let us just quickly recall the proof in general. Indeed SDS psi a naught a naught raise to the power of m omega is essentially summation K running from minus to plus infinity psi cap a naught raise to the power of K a naught raise the power of m omega mod squared.

Please note that this is a naught raised the power K plus m times omega, the spectrum evaluated that frequency modulus square, m is fixed. So, when K runs over all the integers K plus m also runs over all the integers, so this is equivalent to summation K going from minus to plus infinity psi cap a naught to the power K omega the whole square and that proves what we wanted to.

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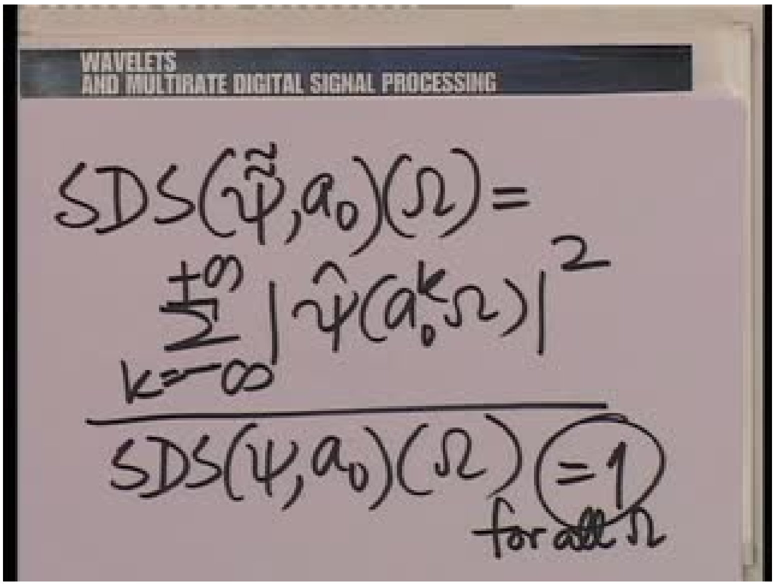

So, with this observation we have essentially SDS psi double tilde a naught evaluated at omega is summation K running from plus to minus infinity psi cap a naught raised the power K omega mod squared divided by SDS psi a naught at omega. Notice that this is essentially SDS psi a naught at omega and therefore, the numerator and denominator cancels on account of the bounds - the lower and upper bound - and this become is equal to 1 for all omega.

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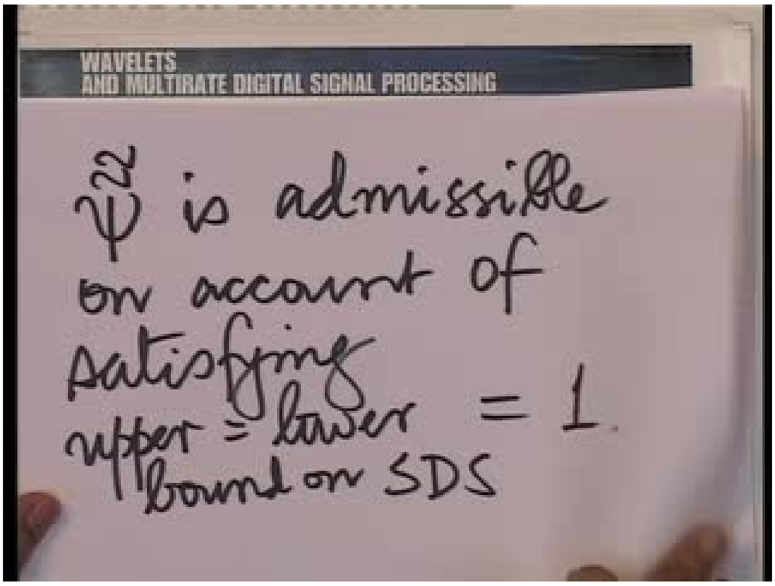

Therefore, we have a constant - something even more than bounds we have a constant - sum of dilated spectra in the context of psi double tilde. Therefore, psi double tilde can be used both on the analysis and the synthesis side the fact that is some of the dilated spectra is a constant means that it is admissible and therefore, psi double tilde is a wavelet and an orthogonal wavelet at that.

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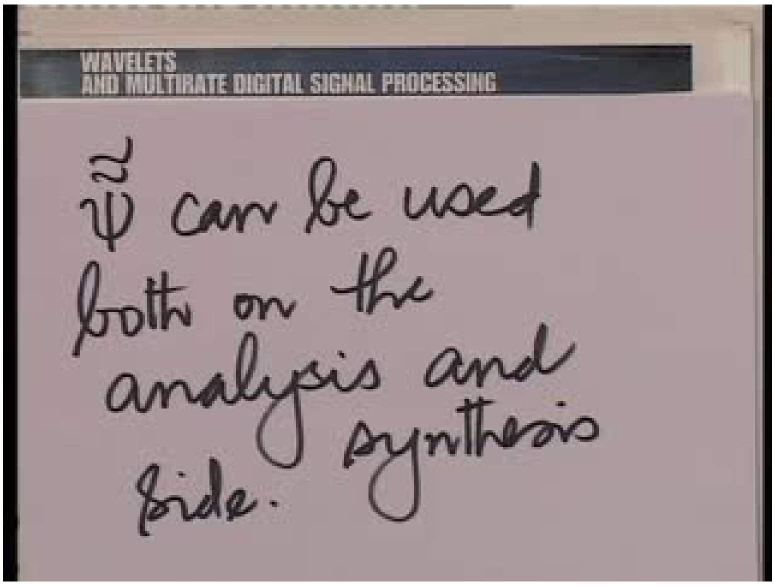

So, psi double tilde is admissible on account of satisfying upper equal to lower bound on SDS equal to 1 and psi double tilde is an orthogonal wavelets, psi double tilde can be used both on the analysis and the synthesis side.

So, what we have done is to complete the process of constructing an equivalent orthogonal wavelet given by orthogonal situation. A wavelets psi which gives you upper and lower bounds on its sum of dilated spectra. Now, a few words before we proceed to the next step the discretization of translation, one must told that the in the all discussion one has not discretize the translation. One is still talking about continuous translation parameters tau and when we talk about orthogonal here we are talking about orthogonality keeping the continuous translation parameter working in continuous time here not in discrete time.

For example, if you were to take the hour wavelets you could certainly show that it has these bounds, but one must not confuse the fact that the hour wavelets gives orthogonality with those discrete shifts, with the concept of the orthogonality as just talk about minute ago where the translation is continuous.

For example, the hour wavelet will not satisfy the upper and the lower bounds being equal to the same, being equal to 1 for example or any constant for that matter because their the orthogonality is with respect to discrete shifts and that is a weaker requirement in fact even though it is a weaker requirement that is what we strive towards in implementation because we do not want retain again the whole continuous translation parameter here, though notionally it is good to know.

Now, what we are going to do is to accept a wavelet psi which has this property of admissibility and reconstructability because of the lower bound, so it has a C1and a C2 and we are going to ask can be discretize it given a naught equal to 2 specifically, so will concentrate on a naught equal to 2 will accept a wavelet which gives us some of dilated spectra being between 2 positive bounds. We will discretize the translation parameters strategically to construct dyadic multi resolution analysis and that is where we once again recall the axioms of the dyadic multi resolution analysis.

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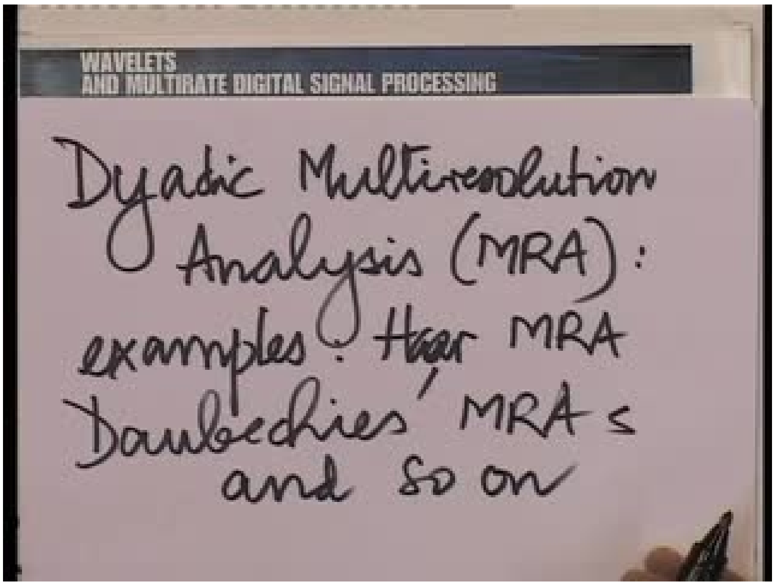

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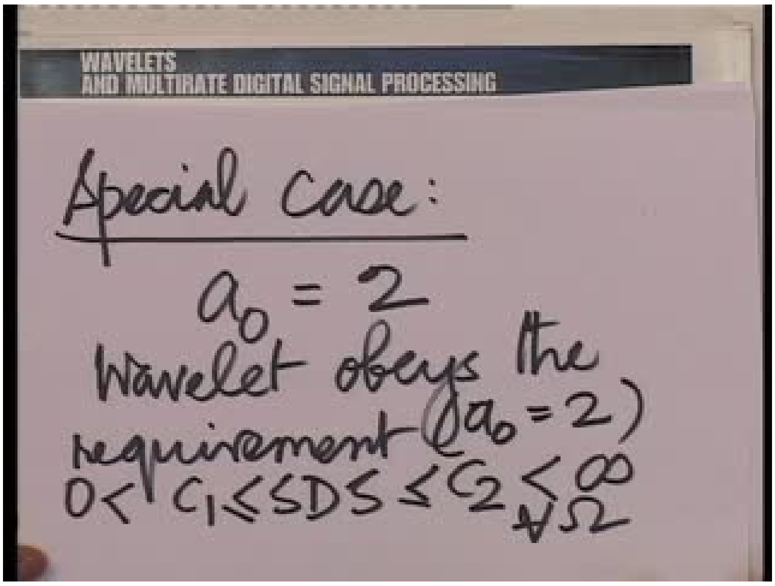

Now, we put all our discussion in perspective what we did earlier on this course the special case of the most general wavelet transform. The Dyadic Multi Resolution Analysis or MRA whose examples constitute Haar MRA, Daubechies MRA and so on. Essentially, the special case where a naught is equal 2 the wavelet obeys the requirement SDS between 2 positive bounds for all frequencies. Essentially, the requirement for discretization and of course, here it obeys the requirement with a naught equal to 2 that is what we need, it may not obey the requirement for all values of a naught greater than 1 but obeys the requirement with a naught equal to 2 for this.

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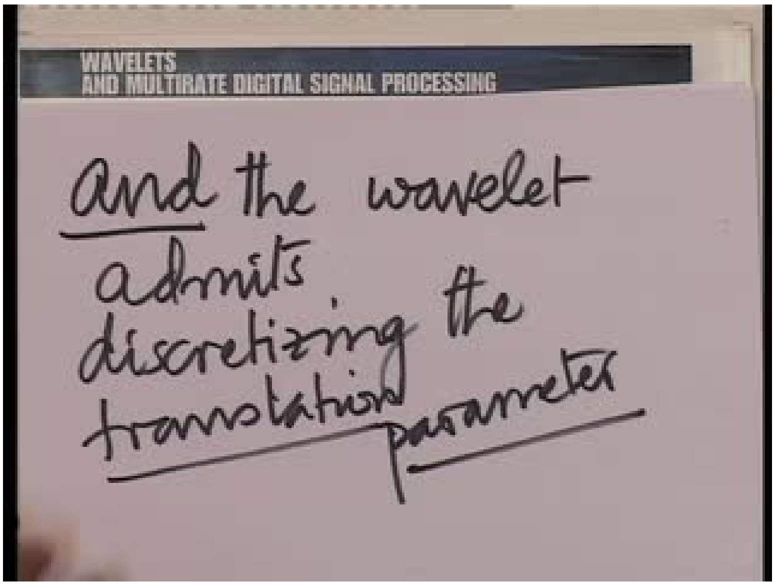

So, as you can see these bounds will of course depends on a naught in general, were the wavelets has these two properties and the wavelet admits discretizing the translation parameter.

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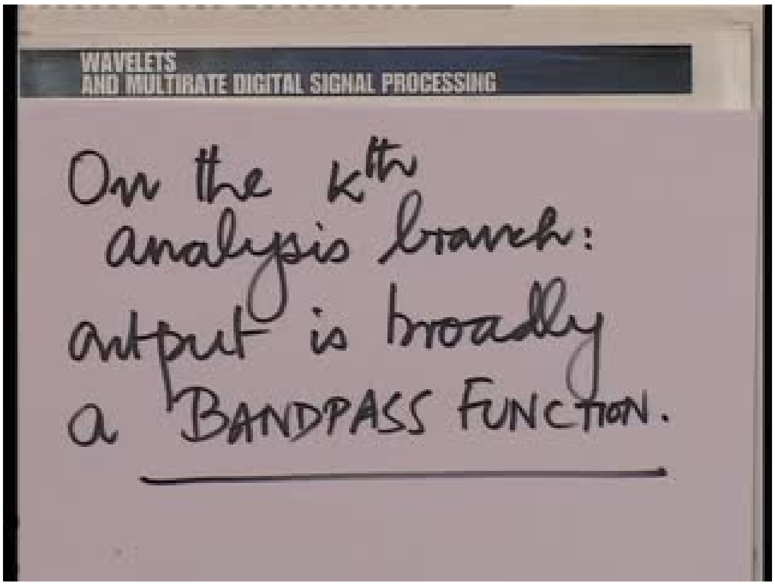

How do we discretize the translation parameter can be understood by a straightly different consideration and how should we go about discretizing the translation parameter, should we do it in the same way for all the branches or should we do it differently, the answer is very simple. You see, if you look at what you are doing on the k th analysis branch the output is broadly a band pass function. It is a function which is significant in a particular band of frequencies not around 0 - band pass remember - and for different values of k there is a logarithmic variation of this band.

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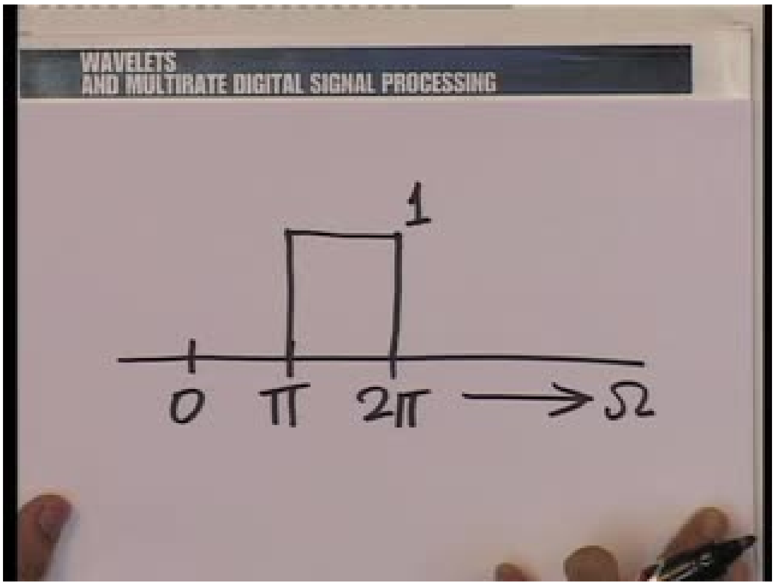

Now, we shall invoke a generalization of the sampling theorem, as we know it to the case of band pass functions. I shall give a generalized statement illustrated with an example suppose, we consider a band pass function were the band on omega lies between pi and 2 pi, suppose we consider band pass function like this.

Now, what could be the sampling rate? You see, after all when we talk about discretizing the translation parameter we are essentially talking about sampling the output of this filter on the k th analysis branch and feeding those samples instead of the continuous function to the input k th synthesis branch.

So, how do we sample in such a way do not lose something that is an equivalent question to how do we discretize the translation parameter. So, if you wish to sample this you have two alternatives, you could sample it obeying the Nyquist criterion thinking of the highest frequency as 2 pi or you could sample it remembering that this band is blank here and you have an occupancy only of pi.

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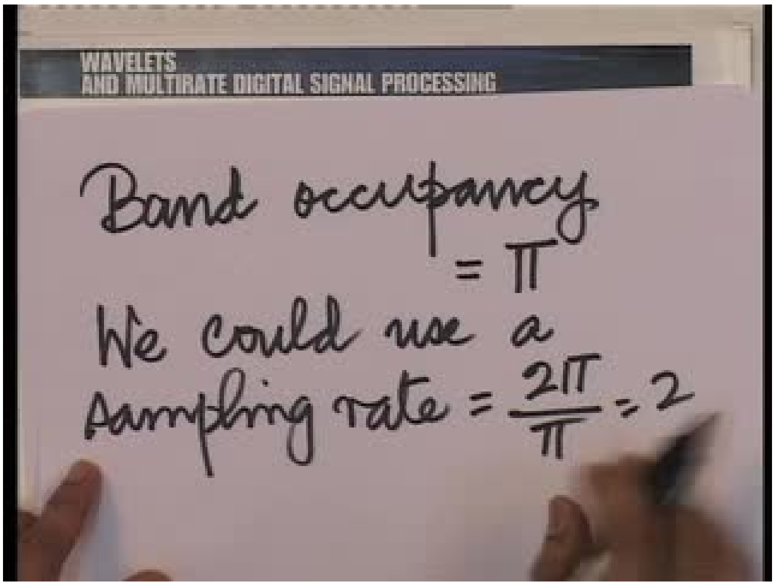

Now, this second approach where you sample noting that the occupancy is only pi can be used in specific circumstances and this is one of them. In fact, you can in principle sample this at a rate equal to twice or more not more, again more has to be taken with a pinch of salt not any value more but definitely with a sampling rate twice of the band. In other words, the band occupancy is pi and therefore, we could use a sampling rate equal to 2 pi divided by this band occupancy pi which is 2.

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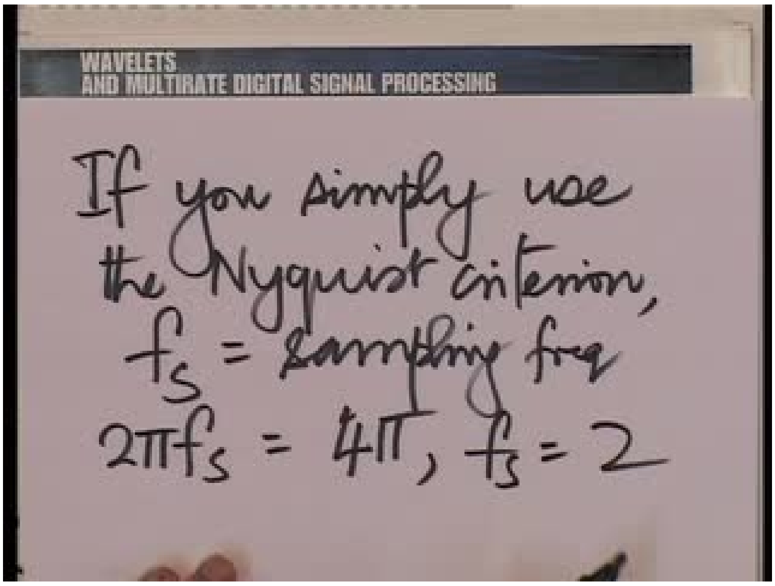

What I am trying to say is, you see when you use a sampling rate of 2 effectively your saying, what does if it really mean, let me explain it in a slightly different way. If you simply use the Nyquist criteria, we should have f s as the sampling frequency in such a way that 2 pi f s is equal to 4 pi, so f s is equal to 2.

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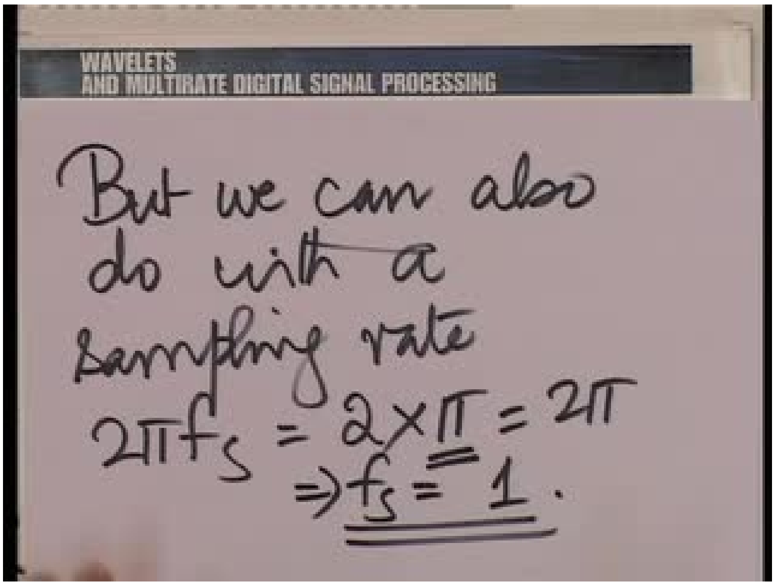

In that sense we could use a sampling rate of value 2, but we can also do with the sampling rate 2 pi f s is equal to only 2 times the band which is pi, this is the band occupancy pi and therefore, f s is equal to 1, this is what I am trying to say. This is the band pass sampling theorem where you can look at the band occupancy and decide your sampling rate not the maximum frequency and suppose we do that, suppose we use a sampling rate of 1.

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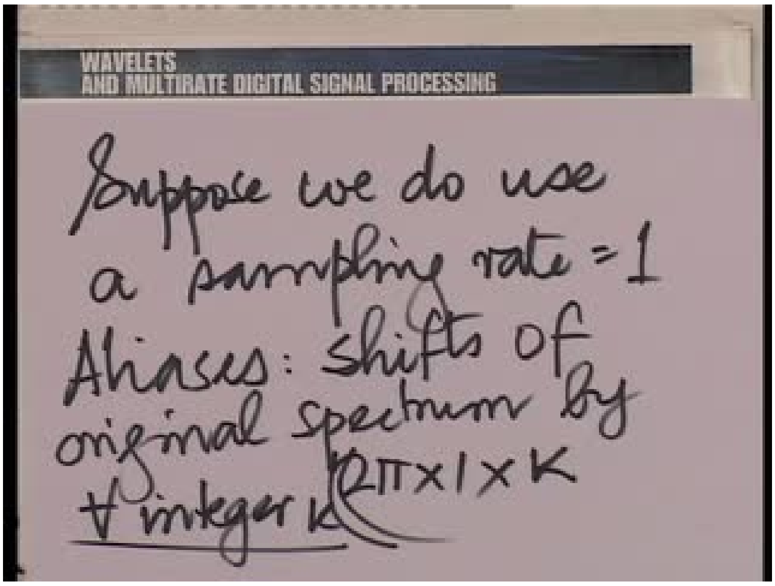

What are we doing? We are essentially adding whole aliases which are shifts of the original spectrum by 2 pi multiplied by 1 times K for all integer K. So, take the original spectrum shift it by whole multiples of 2 pi times this sampling rate 1 essentially, 2 pi K for all integer K and add up these translates let us do that.

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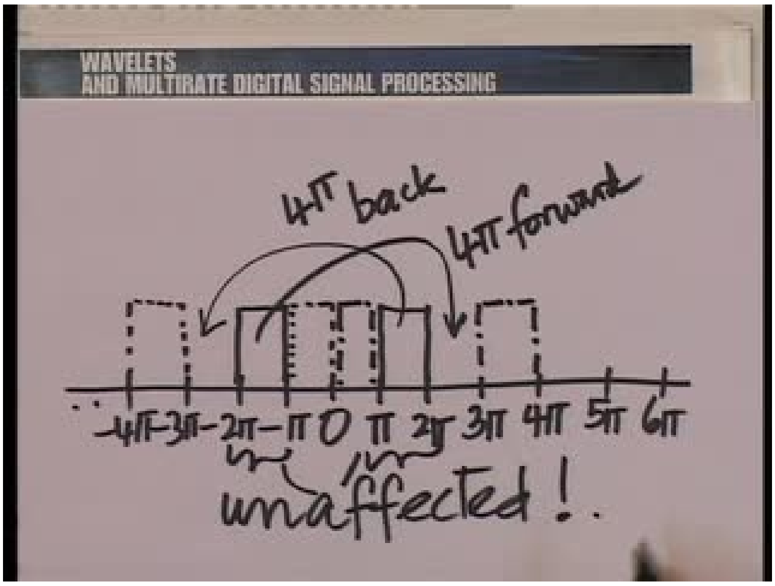

So, let us take the original spectrum, now I will show both the positive and the negative sides of the frequency axis for completeness. I will contract the drawing to make it clear and so on. My original band is here, when I translate pi 2 pi backwards I get this alias, so I will just show that this would go back here.

When I translate 2 pi forward this comes here, so I show that with dot and dash and this goes there, of course I could keep doing this. Now, I must translate in all multiples of 2 pi, so I must take a translation by 4 pi, a translation by 6 pi and so on.

When I translate this band by 4 pi backward and forward, when I translate this by 4 pi backward, I would have this coming here. Of course, when I translate this 4 pi forward and going to 5 pi, I don't need to bother there. When I translate this by multiples, 6 pi and so on and keeps going further back, I don't need to bother about that. This too, I need to worry about what will happen when I translate it by 4 pi and 6 pi, so once again when I translate this by 4 pi it comes here and then, 6 pi and 8 pi and so on.

Now, you can see that other translates I am not going to interfere with this, this original part of the spectrum, this and this is unpolluted or unaffected. So, I could retrieve the original band pass signal by putting a band pass filter between pi and 2 pi, this is the band pass sampling theorem.

Now, you know one should not generalize this too quickly, this does not mean that wherever I put that band of pi I could use the sampling rate of 2 pi - I mean - essentially you know twice that band and not twice the highest frequency that is not true in general. It is true depending on the location of the band as well and that is why the band pass sampling theorem a little more complicated than the low pass sampling theorem - the conventional Nyquist theorem. As we know it but certainly more economical and in fact, what we are doing in dyadic multi resolution analysis is essentially, very implicitly, very covertly invoking the band pass sampling theorem. If we just take a minute to understand this much of the rest of the discussion could be far more clear.

What I have shown you for between pi and 2 pi can now be repeated for between 2 pi and 4 pi and then, between 4 pi and 8 pi and also between pi by 2 and pi and so on, the same principle would hold. For all those cases one could use the sampling rate which is twice the band and not twice the highest frequency using the same kind of an argument.

So, what am trying to say is that for different branches on the analysis side I would have to use different sampling frequencies now and in fact those sampling frequencies are also going to be related logarithmic, they going to be in powers of 2. Recall that in the dyadic multi resolution analysis as we know that is exactly what happens, when you go from V 0 to V 1 you have twice the number of point, twice the number of function in certain interval. When you go from V 1 to V 2 again the number of function is doubled; when you go from V 0 to V minus 1 the number of function is halved, so all that is essentially a manifestation of this band pass sampling theorem.

Now, we must ask the question which we indented to all the while when we discretized the translation parameter with a naught equal to 2.

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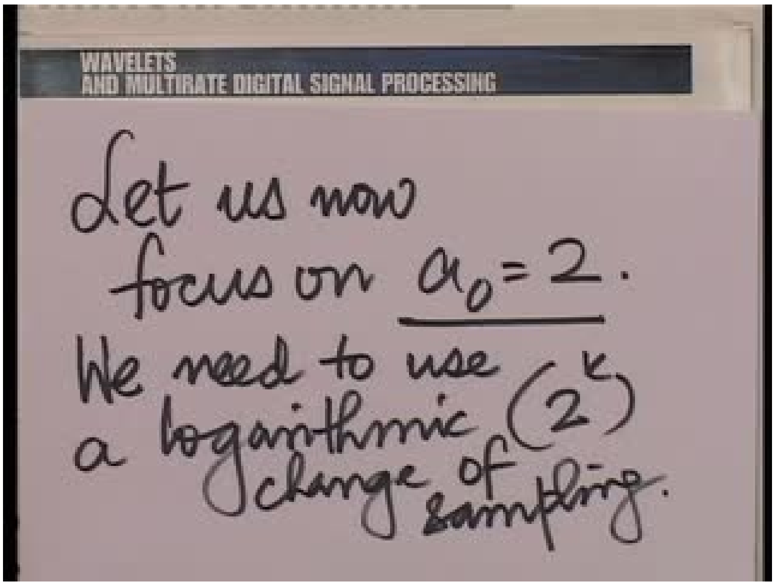

Let us now focus on a naught equal to 2, we need to use a logarithmic change of the form 2 to the power of K of sampling.

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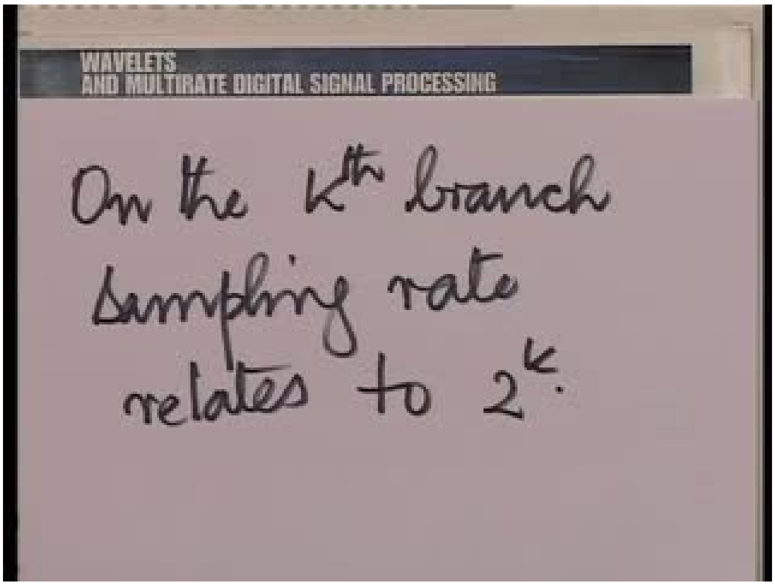

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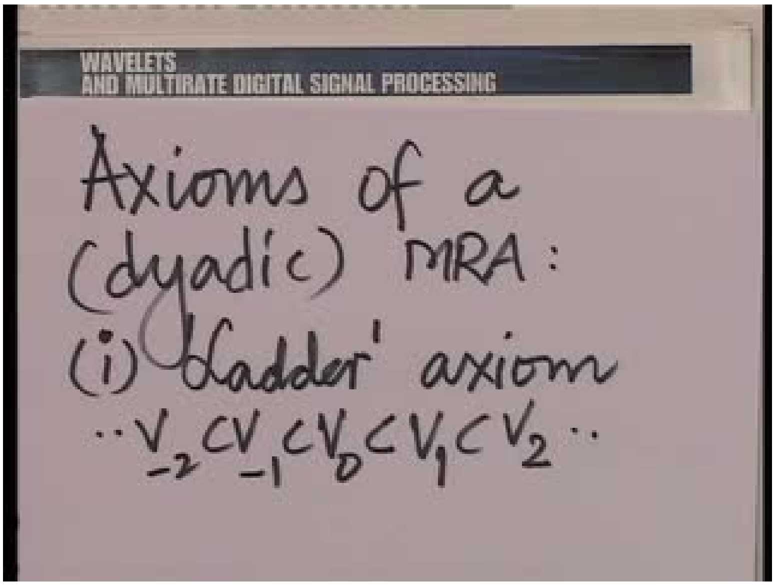

So, what we are saying in effect is that on the k th branch the sampling rate relates to 2 raise the power k. We do explicitly need to ensure it because, that is what the dyadic MRA axioms do, it is now put down the theorem of dyadic multi resolution analysis, as we know it and as we are going to prove formally step by step.

The axioms of a dyadic MRA are as follows, the first axiom - you will recall what we are going to do is to put these axioms in the light of the discussion that we had. The first axiom is you have a ladder of sub space V 0, contain in V 1, contain in V2 and so on. I should not go over the details of these V0 s and V1 but I shall bring out the connection between this subspaces and the discussion that we been having essentially each of these subspaces, for example, V0 is the subspace were the functions are band pass in a certain band. V1 is the subspace whether functions of band pass in the next higher band, V2 in the next higher band and each time the frequency occupancy is doubled, as you go downwards the frequency occupancy is halved. Therefore, as you go upwards the sampling rate is doubled as you come downwards the sampling rate is halved.

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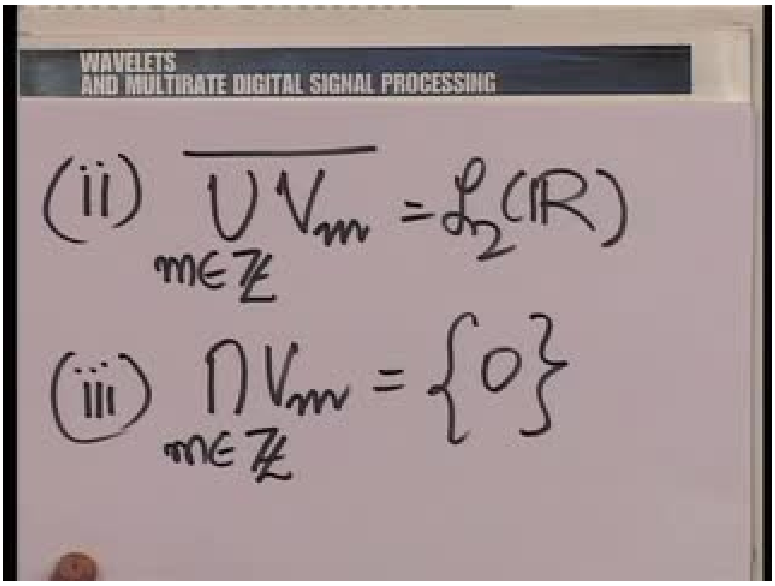

So, that was the first axiom the second axiom is that of course the union axiom, so union V m m over all Z with the closure is L 2 R which essentially as an axiom of perfect reconstruction or when you collect all those incremental subspaces together you go back to the whole input x that is what we are saying, so it is an axiom of reconstruction.

Now, the third one is of course an axiom where we are saying that we always remain in L 2 R, so as we go downwards. If we are going towards smaller and smaller bands finally, we are going to reach a constant function with 0 part that is what we are saying.

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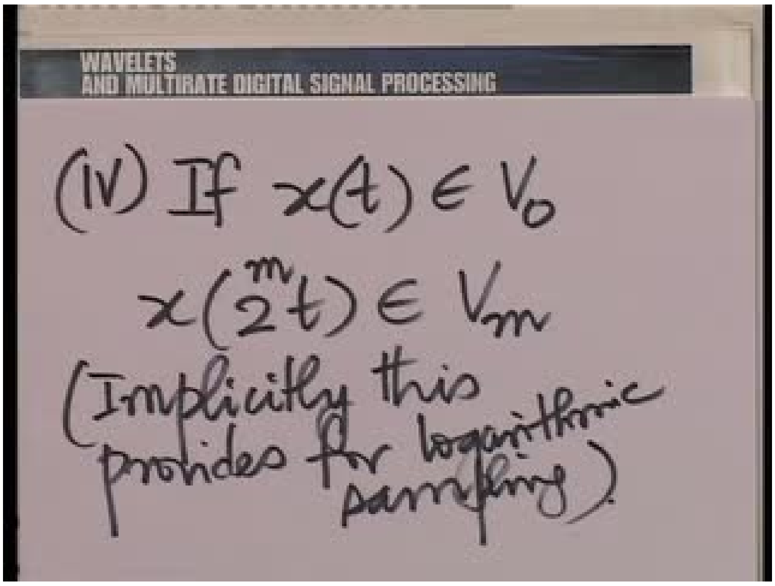

The 4 th axiom is important, the 4th axiom says that if x of t belongs to V 0 then, x of 2 raise the power of m t belongs to V m and implicitly this provides for logarithmic sampling. If we just take a minute on this, it is essentially a statement of logarithmic sampling; in fact logarithmic sampling with the logarithm of 2.

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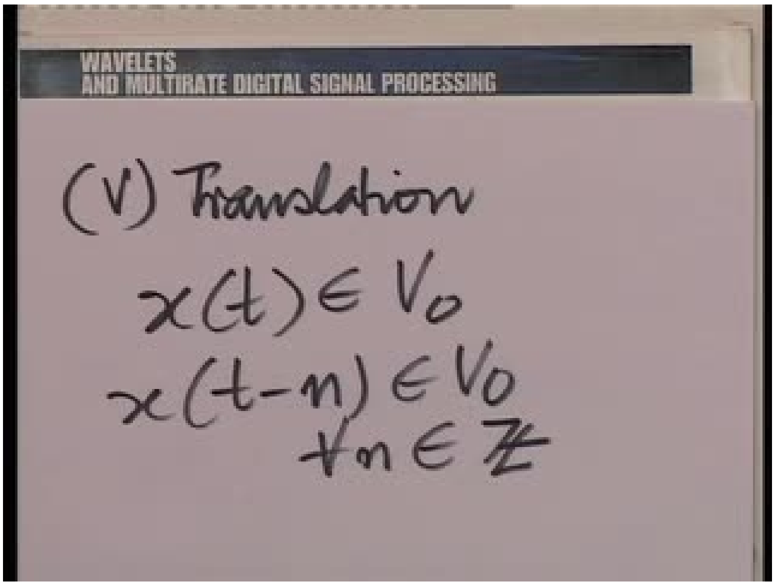

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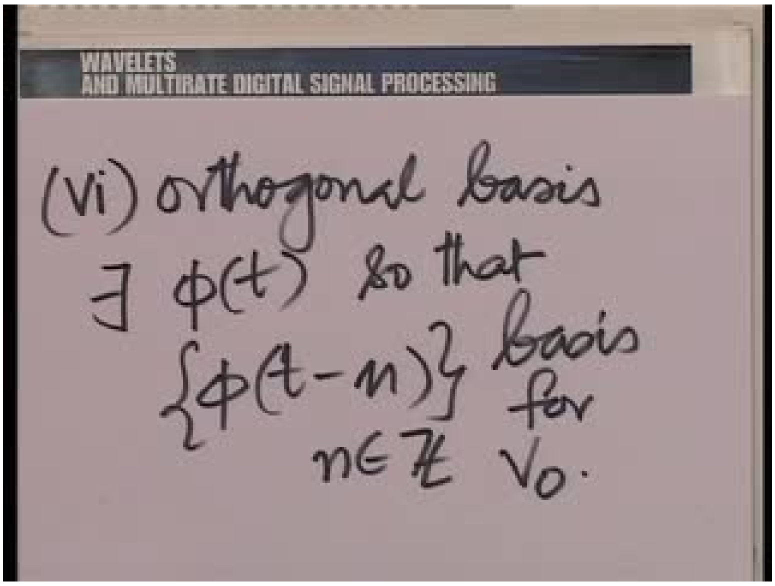

Of course, the axiom of translation that is, if x t belongs to V 0 then x t minus n belongs to V 0 for all integer n, essentially says that we have a uniform sampling. Now, we have interpret this axioms in the context of the discussion that we had and finally, we have the axiom of an orthogonal basis. The orthogonal basis axiom says that we have the exists of phi t, so that phi of t minus n with all integer n is a basis for V 0. Given the 4th and the 5th axiom, we have a corresponding orthogonal basis for each of the V n's in the ladder.

Now, what is the meaning of this axiom in the context of our discussion? This axiom essentially gives us a way to reconstruct from samples, after all the coefficients in the expansion of the function with respect to phi t are essentially like the generalized samples of the output after filtering.

Now, you know actually V0 is the collective subspace but what we are trying to do here although here we are sampling, we are sampling a collective subspace not an incremental subspace. The theorem of multi resolution analysis is going to give us an incremental subspace, what we are saying here is a collective subspace. We are saying that if you go all the way up to 2 pi for example, you need to sample using the conventional Nyquist theorem at 4 pi. If you go all the way to 4 pi you need to sample at 8 pi that is what we are saying here. When we write down this orthogonal basis we are essentially giving a scheme for generalize sampling and reconstruction from the sample but with the conventional Nyquist theorem.

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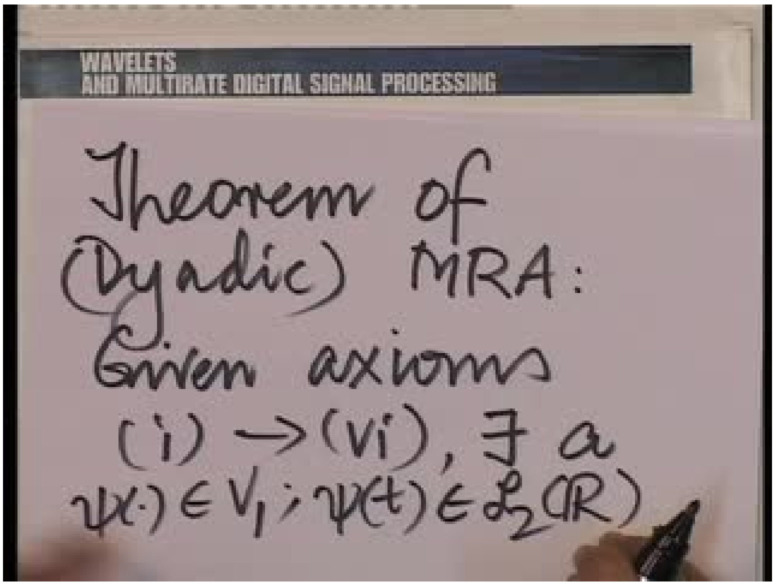

Now, we put down finally the theorem of multi resolution analysis which says that given axioms 1 to 6 there exists a function psi, of course this psi t is a function in L 2 R and in fact psi t is also in V1.

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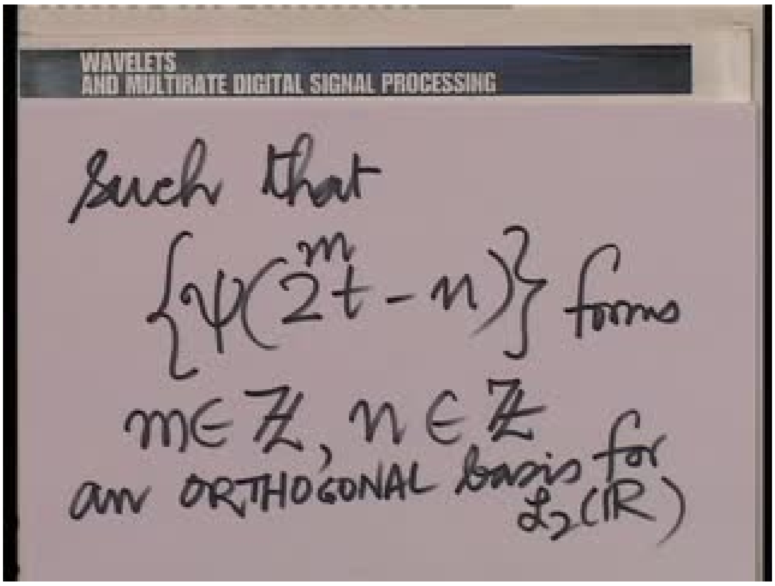

Well, psi 2 raise to the power of m t minus n for all integer m and all integer n forms an orthogonal basis for L 2 R, this is the theorem of dyadic multi resolution analysis.

Now, let us put down the statement of the theorem in the language of our discussion. In this statement we are saying that we have this psi t which covers the incremental subspace between V 0 and V 1.

We can use translates of that pis t to cover the incremental subspace what are we saying they are effective we can band pass sample that band which is covered by the incremental difference between V0 and V1. So, if you think about it this wavelet its spectrum would cover a band and not a band starting from 0. We are saying we can move from low pass sampling to band pass sampling. We can discretize the axis with band pass sampling.

For example, in the light of what we discussed a few minutes ago, if psi essentially covers the band between pi and 2 pi; we are saying you could do by sampling that band at 2 pi and not at 4 pi. You could go from V 0 to V 1, when you represent functions in V1 you need to sample at twice the rate as compared to when you represent them in V0 if you use the conventional theorem. What we saying is you could represent functions in V1 by a low pass sampling and up to V0 and a band pass sampling for the increment.

When you go from V1 to V2, you could either use a low sampling from the V1 and then band pass sampling for W1or you could use a low pass sampling for V0 a band pass sampling for W0 and a band pass sampling for W1. If you take this argument to the extreme, the entire representation of a function in L 2 R can be done by band pass sampling each of these incremental bands, that is what the theorem of multi resolution analysis essentially sets.

In the next lecture, our endeavor shall be to prove this theorem formally, starting from these axioms that we know where this theorem fixes in the entire scheme of things, thank you.