Advanced Digital Signal Processing – Wavelets and Multirate
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Module No. # 01
Lecture No. # 21
Short Time Fourier Transform and Wavelet Transform In General

A warm welcome to the twenty first lecture on the subject of wavelets and multirate digital signal processing. Let us recall in a few words what we discussed in the previous lecture and put the lecture today in perspective.

In the previous lecture, we had brought in the idea of the time frequency plane. We could think as we saw of the time frequency plane as a floor if you please a two-dimensional surface and you used functions to put tiles on that surface analyzing tiles or synthesizing tiles. What the uncertainty principle told you was the smallest size that a tile could have. When I say size, I mean the smallest area that a tile could have.

Of course, depending on the units that one uses, one would have different values for that smallest area, and all those limits ultimately come from the fact that the product of the time variance and the frequency variance must be greater than or equal to 0.25 or one-fourth.

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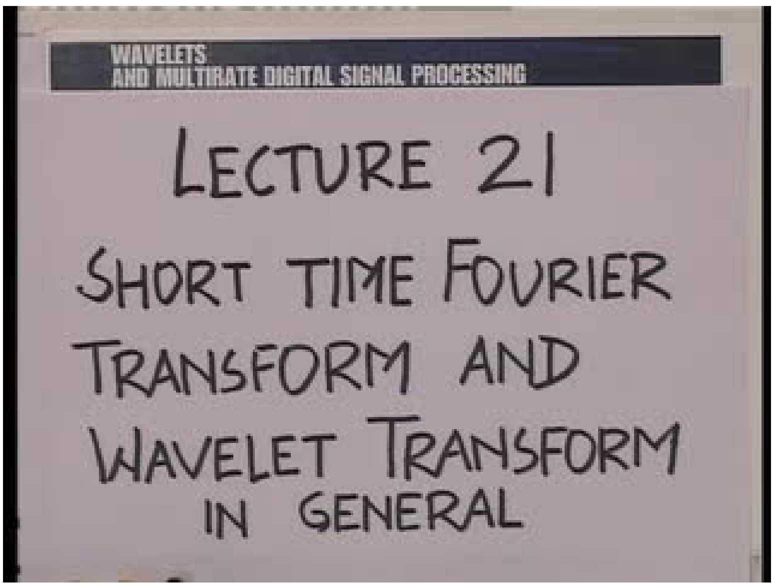

Now, what we intend to do today and that it is captured in the title that I have given in the lecture today, is to build two kinds of transforms in a more generalized way based on this idea of tiling, namely - the short time Fourier transform and wavelet transform in general. So, we wish to look at both of these from the perspective of tiling the time frequency plane.

With that background, let us go straight away to the first of these two transforms, namely - the short time Fourier transform. In fact in some sense to replicate how things proceeded historically, you know, when people realized in analyzing signals or in dealing with two domains simultaneously as we are trying to do that there is a basic uncertainty that hits you when you try to do something like that. The first thing that they thought of was the simplest, namely…

If you cannot find out frequency components at a particular time, you could probably find them out over an interval of time, and the obvious way to do that is chop to break the signal into parts into pieces. Take the Fourier transform of each piece. That is exactly what the short time Fourier transform proposed. Let us explain this mathematically.

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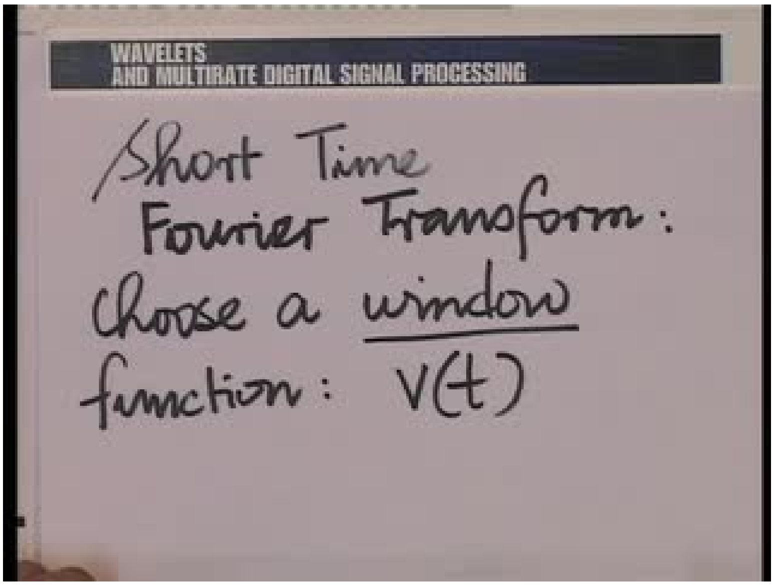

So, when people talked about the short time Fourier transform said they said, well, choose first a window function. Let us call that window function V t, and we shall define the characteristics that we desired out of the window function. So, essentially what we would want out of the window function is a finite time variance and a finite frequency variance. So, let us write that down in mathematical terms.

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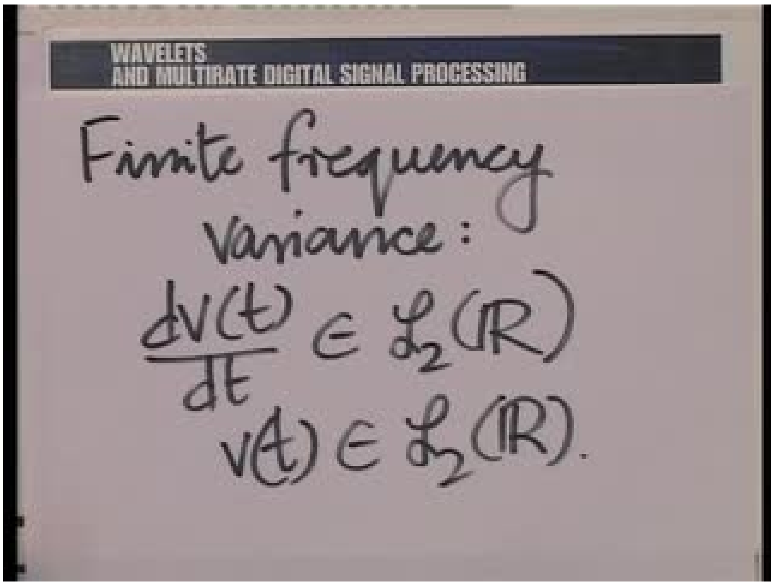

Finite time variance: which essentially means you want t times v t to belong to L 2 R, needless to say v t belongs to L 2 R, that is taken anyway, and we also want finite frequency variance and that means we want the derivative of v t to belong to L 2 R. Let me just write that down for completeness.

Now, if we use this definition of a window function, the most common window or the simplest window that we have not counted so far disqualifies. So, the so called rectangular window, the rectangular pulse in place of v t which we have been using as the scaling function in the Haar multi resolution analysis is disqualified.

However, we shall take it as an extreme case. An extreme case where one of these is not satisfied the other is. So, in the Haar multi resolution analysis, we are using a so called window function for the scaling function of that matter even for the wavelet function, which has an infinite frequency variance. So, it disqualifies from the point of view of frequency variance, but it definitely qualifies from the point of view of time variance.

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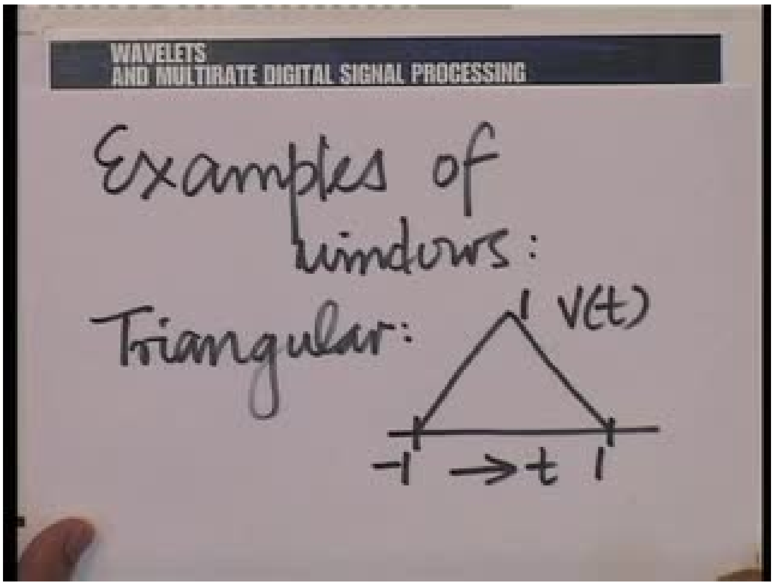

Now, without little background we could also take other window. Let us take a couple of examples just to give an idea. So, last time we looked at the time frequency product of the triangular function that could be a possible window. So, examples of windows could be the triangular window. I will just sketch it the Gaussian window.

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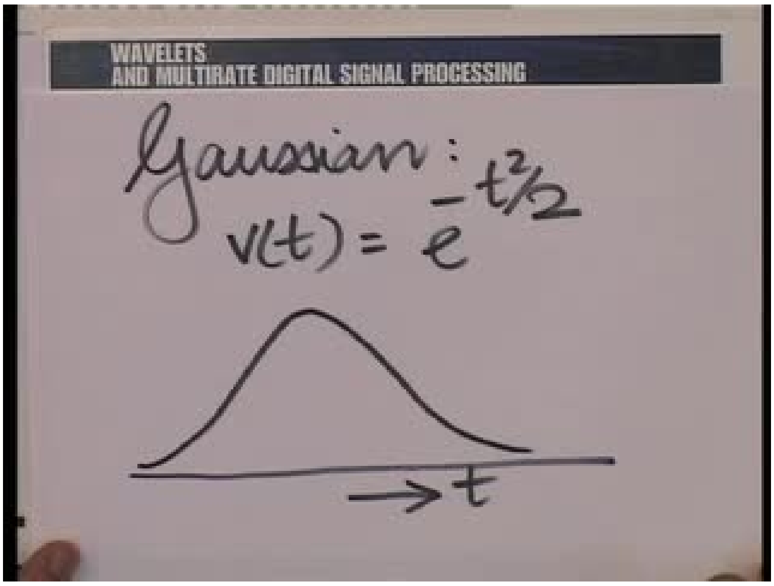

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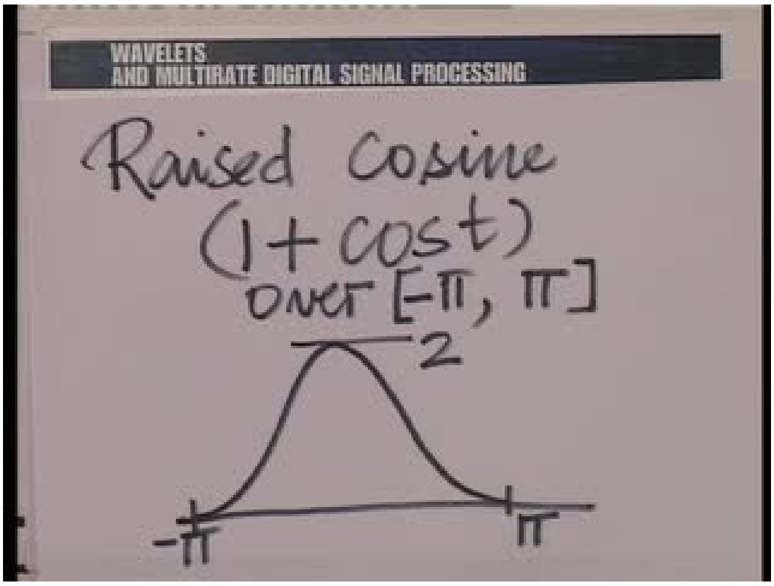

Again, let me sketch it. A so called raised cosine window which look something like 1 plus cos t over a limited interval that would have an appearance like this, a raised cosine mind you at the two ends it would be 0 it would have a maximum of 2 in between anyway, what you do notice commonly among all these windows is that there is a limit in time in spread and a limit in frequency in spread.

Now, for the raised cosine case I have not proved explicitly about this limit in frequency, but in fact I leave it you as an exercise to calculate the sigma t square sigma omega square product for the raised cosine window. So, with that we have chosen a window and we as I said allow that extreme poor case of the rectangular window with which we built our concept of multi resolution analysis as well.

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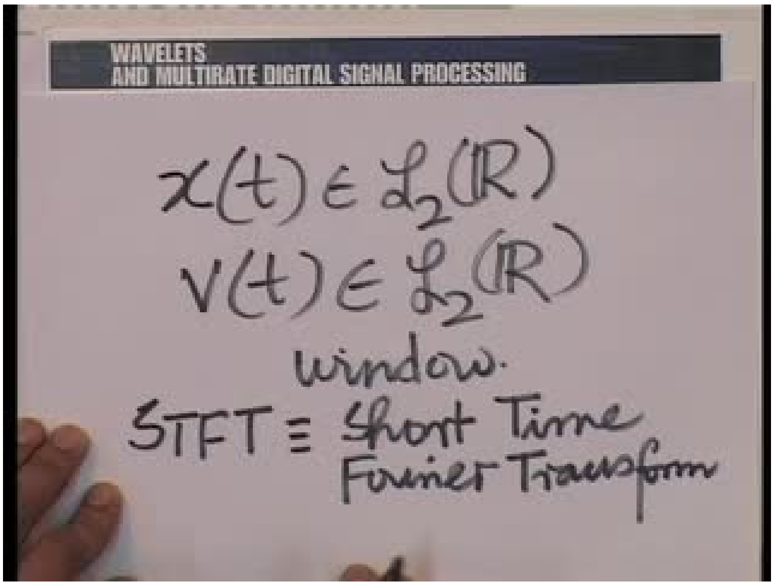

Anyway, with that background, let us now use as tiles translates and modulates of this window. So, what we do is to construct of a continuum of the following kinds of dot products. We have a function x t belonging to L 2 R and we have chosen a window v t. The short time Fourier transform of x t, so, the short time Fourier transform is abbreviated often by STFT. So, let us use this abbreviation now onwards.

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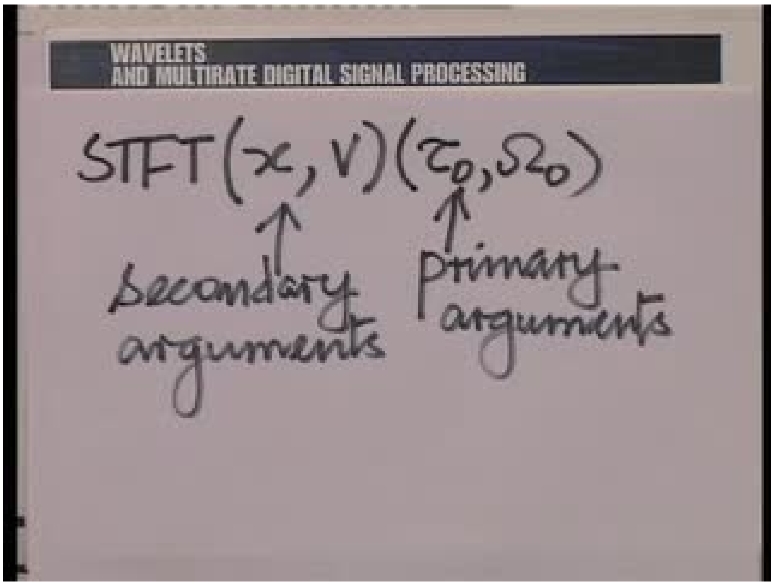

The short time Fourier transform or STFT of x with respect to v as we shall call it STFT of x with respect to v, so we have two arguments. The function whose short time Fourier transform is been constructed and the window with respect to which it is being constructed.

Now, this is essentially the set of arguments which determine what short time Fourier transforms we are constructing, but there is also a pair of arguments for this transform which are the translation and the modulation arguments. So, these are I would call the so called primary arguments of the short time Fourier transform, and these one should call the secondary arguments to explain this idea of primary and secondary arguments. Let us take for example, the Fourier transform. In the Fourier transform function, the primary argument is the triangular frequency - capital omega. The secondary argument is the function whose Fourier transform is being calculated.

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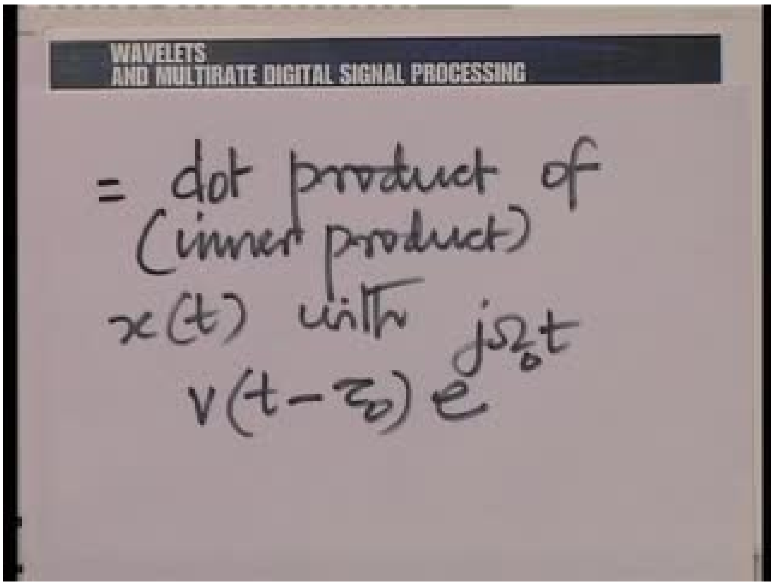

So, when you calculate the Fourier transform of x t, the secondary argument is the function x. The transform is the whole operator. It takes this function x as a secondary argument, and the primary argument is the angular frequency. So, with that little explanation, let us go back to the short time Fourier transform of x with respect to the window v evaluated, this is how we would read it, the short time Fourier transform of x with respect to the window v evaluated at the translation tau 0 and the modulation omega 0. Let us write that down; it would be essentially a dot product of the following.

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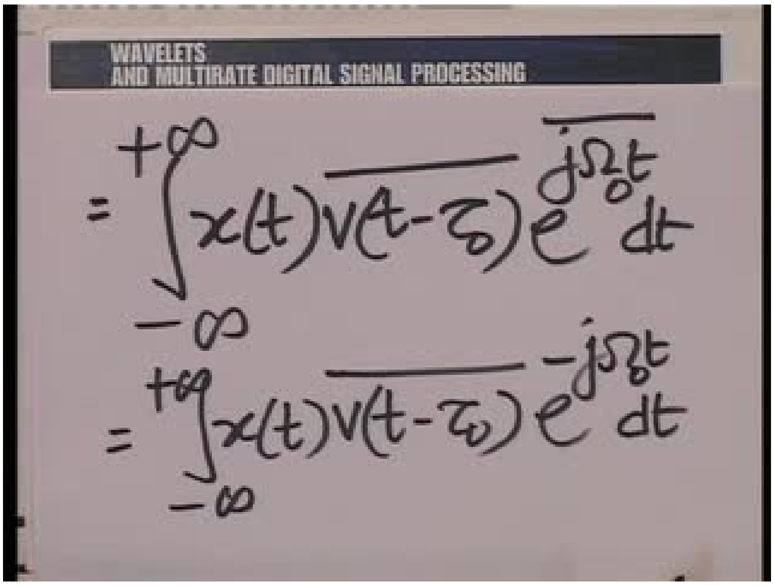

A dot product or inner product of x t with v translated by tau naught and modulated with e raise the power j omega naught t and how this dot product would look, Let us write it down x t multiplied by v t minus tau 0 bar e raise the power j omega naught t bar dt, and if you please, we could simplify this.

Now, a couple of words here - if v t is a real function as we been considering all this while, whether it is the Gaussian or whether it is the raised cosine or the triangular function or if you like the extreme case of the rectangular window, this complex conjugation is redundant, it is not required. If you look at it, what we are doing here in an alternate sense is to first multiply x t by a window appropriately translated.

So, tau 0 is the location of the window. Essentially what we are trying is to extract information of x t around the point t equal to tau 0, and then, we are taking a Fourier transform; that is another way to interpret this. So, short term Fourier transforms is essentially a process of piecing or breaking the function into pieces followed by a Fourier transformation. Here, we are assuming a continuous tau 0 and a continuous capital omega 0, and therefore, we are talking about the continuous short term Fourier transforms.

Now, let us invoke Parseval’s theorem to get a different perspective on the same expression here. So, let us go back to the expression here. We have taken a dot product of x t with a translated version of the window and then a Fourier transformation on this product. So, for example, if v t were to be a triangular window, essentially it would extract the information of x t around the point tau 0 with some weighting done by the rectangular or by the triangular function or the rectangular function.

In the rectangular function, there is no weighting; in the raised cosine or the triangular or the Gaussian function, there is a weighting. Different weights given to different points around tau 0 followed by a Fourier transformation, so, looking at the frequency content in a certain region.

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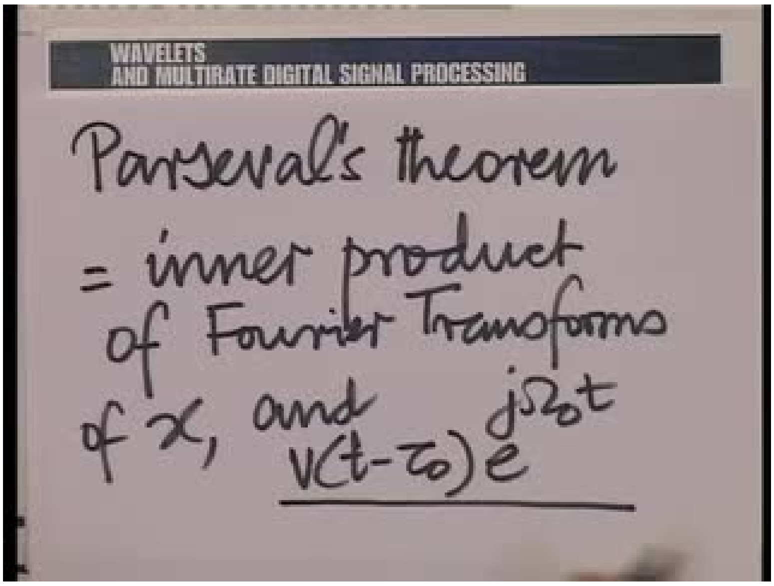

Now, if we invoke Parseval’s theorem on this, what is it give us. Parseval’s theorem says this is also the inner product of the Fourier transforms of x and v translated by tau naught and modulated by capital omega naught. Therefore, we need to calculate the Fourier transform of this quantity; that is the task that we need to do.

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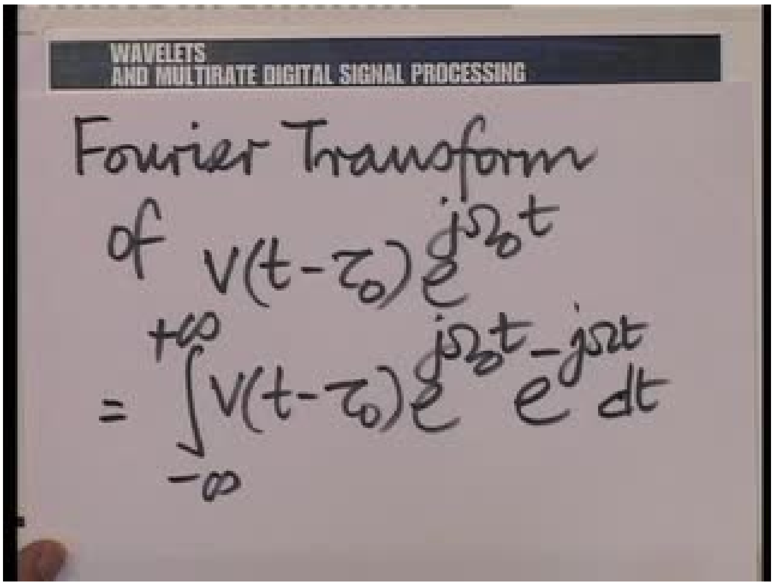

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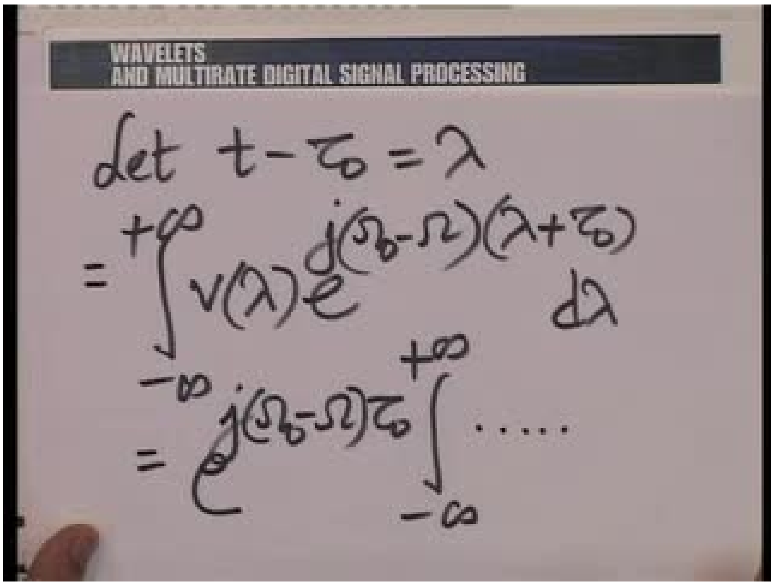

The Fourier transform of v t minus tau 0 e raise the power j omega naught t can easily be seen to be the following. So, it is minus to plus infinitive v t minus tau 0 e raise the power minus, well, j omega naught t multiplied by e raise the power minus j omega t integrated with respect to t. Now, let us simplify this. So, of course we employee the standard process of replacement of argument; so, let us replace the argument t minus tau naught.

I am noting that for a fixed tau naught, when t runs from minus to plus infinity, lambda also runs from minus to plus infinity; this would became integral from minus to plus infinity v lambda e raise the power j. Now, of course collecting terms, this is omega naught minus omega times t which is lambda plus tau naught d lambda.

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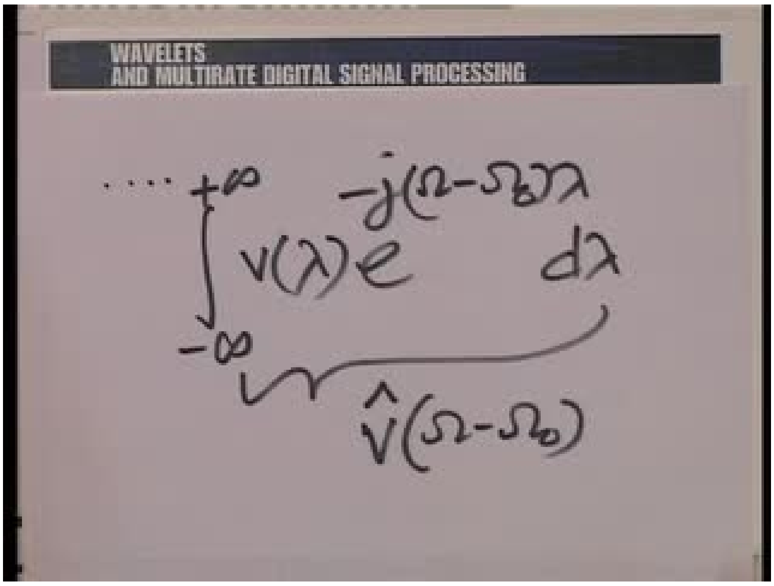

$$\int_{-\infty}^{+\infty} v(\lambda) e^{-j(\Omega-\Omega_0)\lambda} d\lambda$$

$$\hat{v}(\Omega-\Omega_0)$$

And now, we have a simple answer there. We can rewrite this little bit taking out common terms. So, you may notice that when we expand this product, we have the term e raise the power j omega naught minus omega times tau naught and e raise the power j omega naught minus omega times tau naught is independent of lambda, so, I could bring it outside the integral. Some say this becomes e raise the power j omega naught minus omega times tau naught and the rest of it inside, and now, let me see what remains inside. What remains inside is v lambda e raise the power j omega naught minus omega times lambda d lambda. So, let me write that part down.

Now, here, I should write only the integral, I have left out the other term. That integral is essentially v lambda e raise the power, I will rewrite it as minus j omega minus omega naught times lambda d lambda, which is easily seen to be the Fourier transform of v, but evaluated at omega minus omega naught instead of at omega. So, all in all, we have the following.

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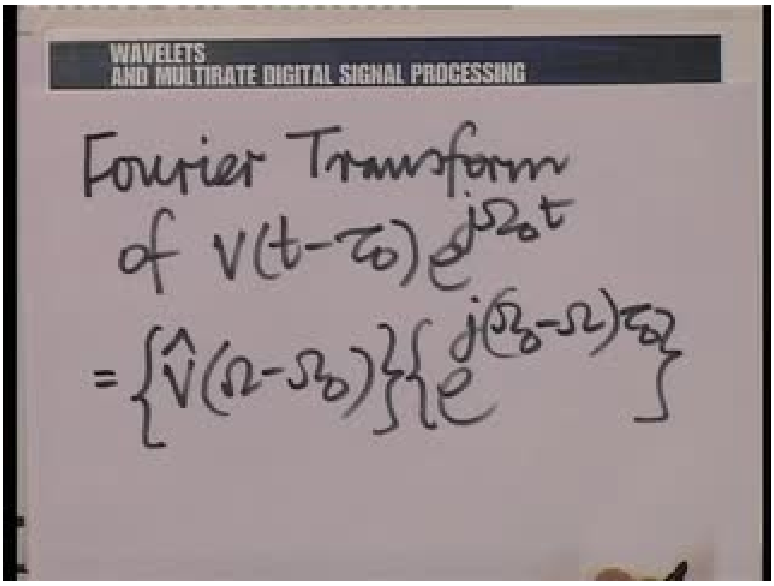

We have the Fourier transform of v t minus tau 0 e raise the power j omega naught t is the Fourier transform of v evaluated at omega minus omega naught multiplied by e raise the power j omega naught minus omega times tau naught and we shall substitute this in the Parseval's theorem expression.

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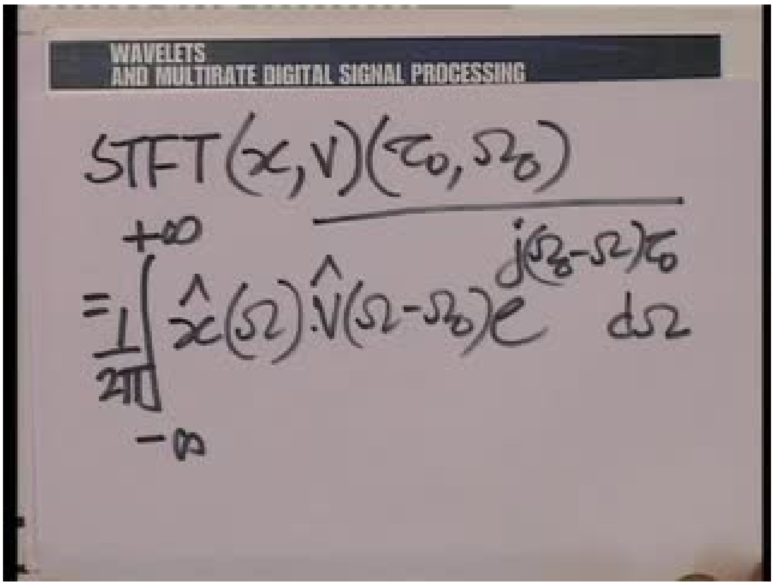

So, we have the short time Fourier transform, secondary arguments of x with respect to the window v, evaluated at tau naught and capital omega naught is essentially, the following inner product with a factor of 1 by 2 pi remember. So, it is x cap omega v cap omega minus omega naught e raise the power j omega naught minus omega times tau naught d omega with this all complex conjugated.

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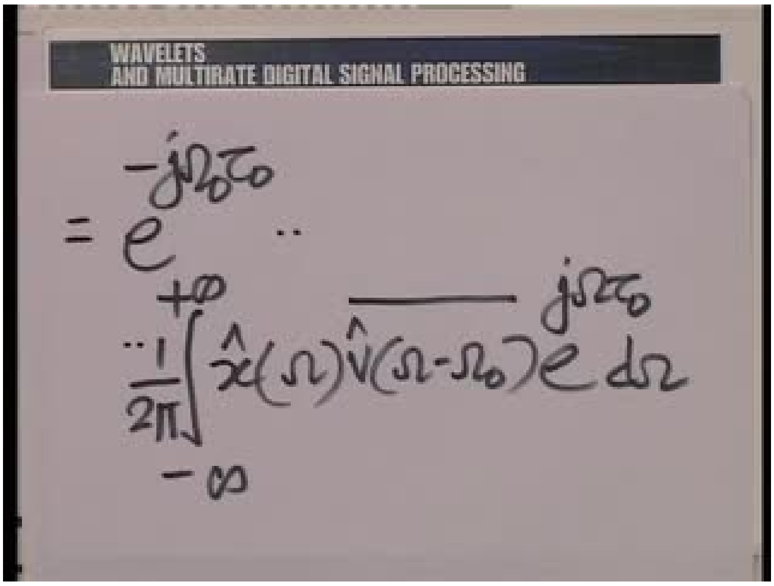

And let us simplify that in once again, let us note that when we expand this, we get two terms - e raise the power j omega naught tau naught and e raise the power minus j omega tau naught out of which only the second depends on capital omega, the first does not; the first can be brought outside the integral. So, what we have in effect is e raise the power minus j omega naught tau naught times the following, 1 by 2 pi just a constant x cap omega v cap omega minus omega naught e raise the power j omega tau naught d omega.

This is complex conjugated here. Here the complex conjugate has been taken care of as it is, and now, we have a very beautiful interpretation for this. This looks very much like an inverse Fourier transform if you think about it. It is the inverse Fourier transform evaluated at the point tau naught. Inverse Fourier transform of what? Of the Fourier transform of x multiplied by the Fourier transform of the window shifted to lie around omega naught. So, now this makes a lot of sense.

The short time Fourier transforms as we expected has an interpretation, a very similar interpretation both in time and frequency. Let me put back before you; the frequency interpretation first and then, we will see how it is similar to time interpretation that we had a few minutes ago.

So, the frequency interpretation is like this. You know, if you look at it, this is essentially a constant term it is magnitude is 1. This is again a constant 1 by 2 pi. So, one need not pay too much of attention to these two terms.

Essentially, it is this integration which is important. In this integral what are we doing? We are multiplying the Fourier transform of the function which is being analyzed that is x. The Fourier transform is x cap of capital omega by the Fourier transform of the window shifted to lie around capital omega naught. So, we are trying to analyze the content of the Fourier transform around omega equal to omega naught and we are doing so, at the point t equal to tau naught if you please because we are taking inverse Fourier transform here.

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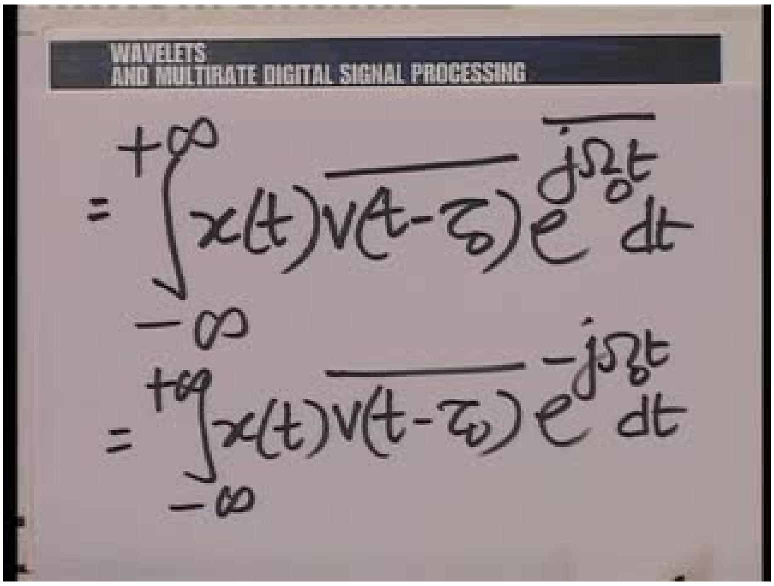

If you will just go back a few steps, we had the expression for the short time Fourier transforms in time; let me put back that expression for you, that expression is here. What did we do there? We need exactly a dual thing in time. So, we had this is, you see, if you before we calculate it, we have this essentially as a time expression, and with this time expression, what we did was the following.

We multiplied x t by translate. So, we extract the information of x t around the point t equal to tau naught and then, took a Fourier transform at the point omega equal to omega naught. This completes the beauty of duality in the interpretation. We are doing exactly the dual of what we have done in time in frequency that is what we have said.

The operation is very similar in time and frequency. It extracts a region of time and it also simultaneously extracts a region of frequency. So, in a certain sense, the short time Fourier transform was a very good idea when it came. Although, when they use the rectangular window, it was a bad idea, and the rectangular window was not sufficient for that reason. You had to taper off the window at both ends to make the function continuous.

In fact, all this is also the basis of the idea of windowed FIR filter design in discrete time signal processing. People talk about designing finite impulse response filters using windows. Now, the ideas that we have talked about here are also replicated there in a slightly different context, but let me not go into that for the moment. Come back to this point. So, short time Fourier transforms is one way of tiling the time frequency plane. What is the tile is doing in short time Fourier transforms? You must try in visualize them.

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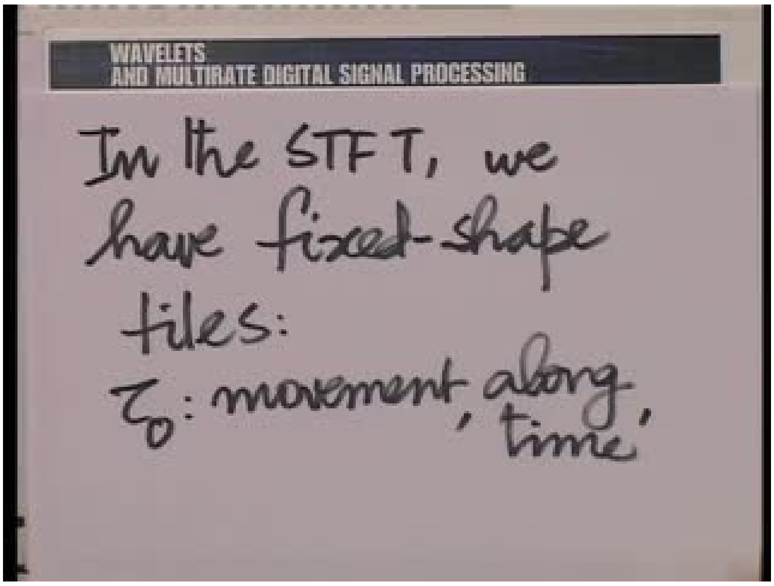

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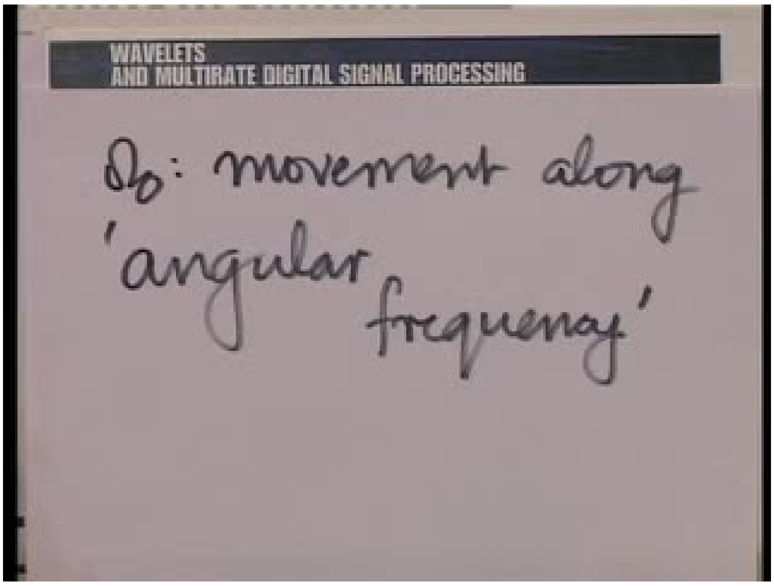

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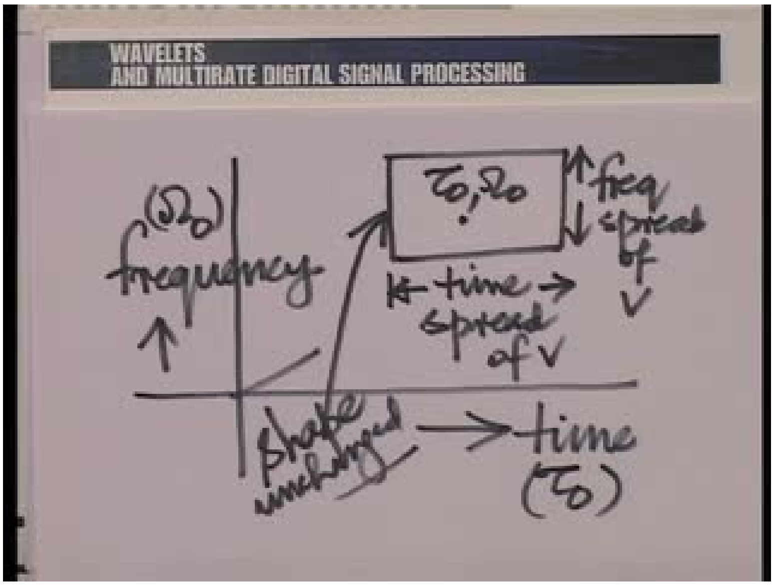

So, in the STFT, we have fixed shape tiles. You may call them that fixed shape tiles. Tau naught is a movement along time, and simultaneously, omega naught is a movement along frequency or angular frequency if you please.

So, in fact I think it will be best to understand this graphic. What we saying is, we have this time frequency plane as we constructed the last time, and in fact, the time frame is indexed by tau naught, and the frequency frame, the frequency region is indexed by capital omega naught different values. So, at a particular tau naught and omega naught, what the short time Fourier transform does is to extract information in a region of the time frequency plane spread around tau naught and omega naught, where this is indicative of the time spread or twice the time spread of v and this is indicative of twice the frequency spread of v.

And what we saying is, we can visualize this, think of this as a tile now and visualize this tile as being moved along this teo dimensional plane. So, forget about this axis for the moment. Just visualize that tile, just visualize the plane underneath as a time frequency plane and this tile moving around for different tau naught and omega naught that is what we are doing. The shape is unchanged that is important. We should mark it here shape unchanged.

Now, in the same spirit, let us look at the continuous wavelet transform. So, we now going to introduce a more general version of the wavelet transform. So far we have being seen the very specific kind of wavelet transforms. What we call the dyadic discrete wavelet transform. Then, the scale parameter is changed in powers of 2. The translation parameter is changed by uniform steps; the step depends on the scale.

So, if you look at the Haar multi resolution analysis, of course the scale is changed dyadically; that means in powers of two, and the translation is change in steps of unity when you take the basic or the middle so called sub space v 0, and as you go towards higher sub spaces in that ladder, the step size becomes smaller in powers of 2 steps; as you go lower in that ladder, the step size changes again by factors of 2 becomes bigger and bigger.

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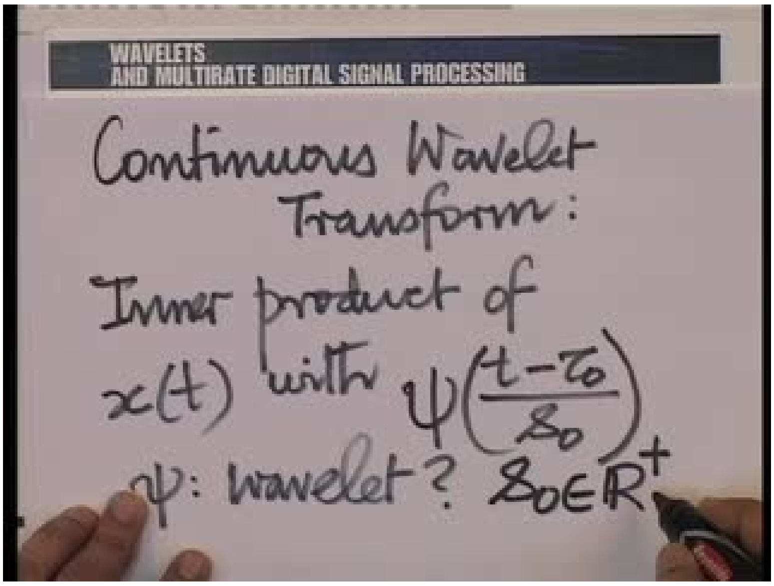

Anyway, so we need to understand this from a time frequency plane perspective. So, in general, what is the continuous version of the wavelet transform? The continuous version of the wavelet transform is essentially, is essentially a dot product again it is an inner product of x t with. Now, this time, instead of a window, we have a wavelet, a translate and a dilate of a wavelet. So, we take the wavelet psi; we translate it by tau naught and we dilate it by a factor s naught.

Psi is a wavelet. Now, what an earth to be mean by that? What in general is a wavelet? In fact, we shall indirectly postpone the answer to that question for a while until we complete this discussion on the continuous wavelet transform. So, what qualifies as a wavelet? Is a question that we need to answer?

Well, we know examples of functions that qualify as wavelets with tongue in cheek, we will say the Haar function qualifies as a wavelet. I said tongue in cheek because; actually it has an infinite frequency variance. So, we have trouble there, but anyway, as I said tongue in cheek.

But we know better examples of wavelets. We know the Daubechies wavelets for example, may be more difficult to construct, but nevertheless, there for us, and we have a whole family there of Daubechies wavelet. So, we have examples of wavelets, you know, which are restricted in time and restricted in frequency. So, one thing that we very clearly understand is that the wavelet function needs to be restricted in time and restricted in frequency. It needs to be a window function in some sense, but just any over window function that we shall take some time to answer anyway.

So, for the moment leaving open the question of what qualifies as a general wavelet, let us come back to this inner product. We are taking to construct the continuous wavelet transform and inner product of x t with this wavelet function dilated by s 0 and translated by tau 0. Of course, s 0 must be non-negative; so, the way to write it is, s 0 belongs to R plus; this plus means that s 0 is a positive real number excluding 0 and excluding all negative real numbers.

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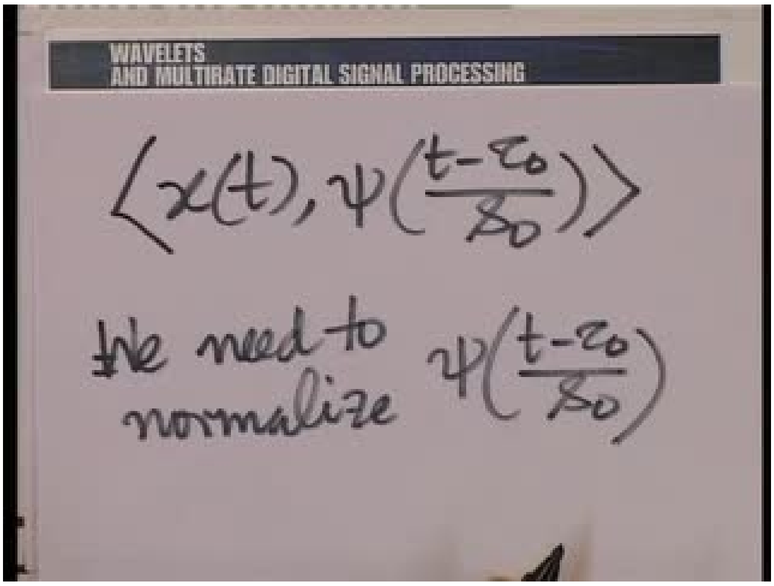

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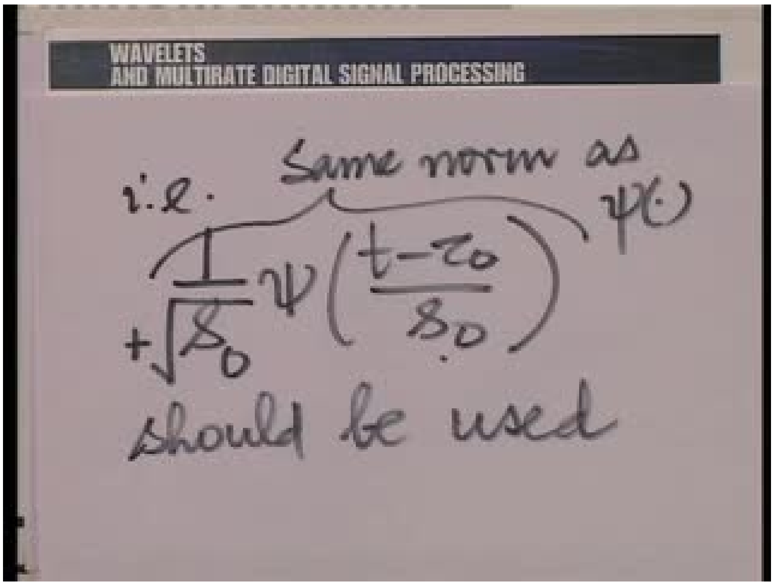

Let us write that inner product down for ourselves. So, we are saying this. Now, again here, we need to exercise a little bit of caution. You see, if we use this as it is, then we have what is called the problem of normalization. This problem of normalization did not come in the short time Fourier transforms, because, when we modulated and when we translated in time, the norm of the function was unchanged, but here when we dilate, then the norm changes and we need to take care of that. So, we need to normalize and that I leave to you to prove can be done by multiplying by a factor of 1 by square root of s 0. So, if we take 1 by s 0 positive square root time psi t minus tau 0 by s 0, then it is normalized.

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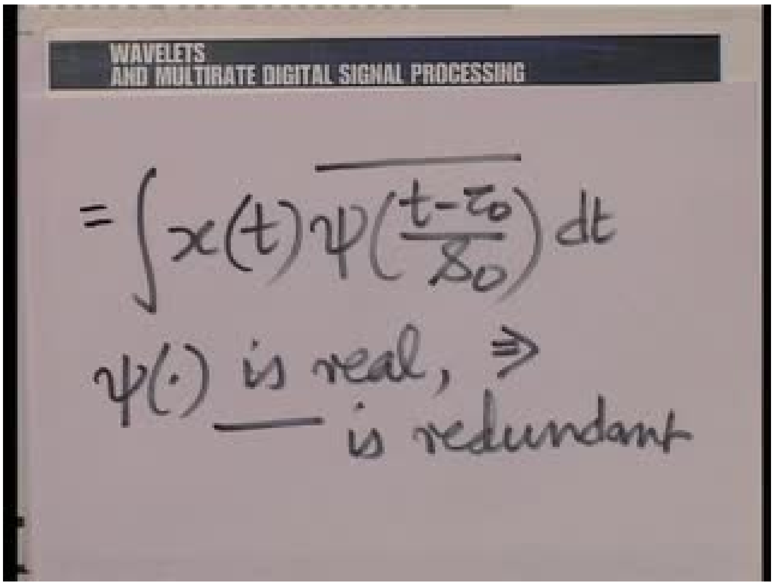

It has a norm equal to the norm of psi t. So, this has the same norm as psi t without the translation and dilation, anyway. So, let us construct this dot product. The dot product becomes integral x t psi t minus tau 0 by s 0 dt with a complex conjugate on this. Now, of course if psi t is real, then the complex conjugate is redundant and that is what we have been doing in all the real wavelet that we have been using ignoring that complex conjugate.

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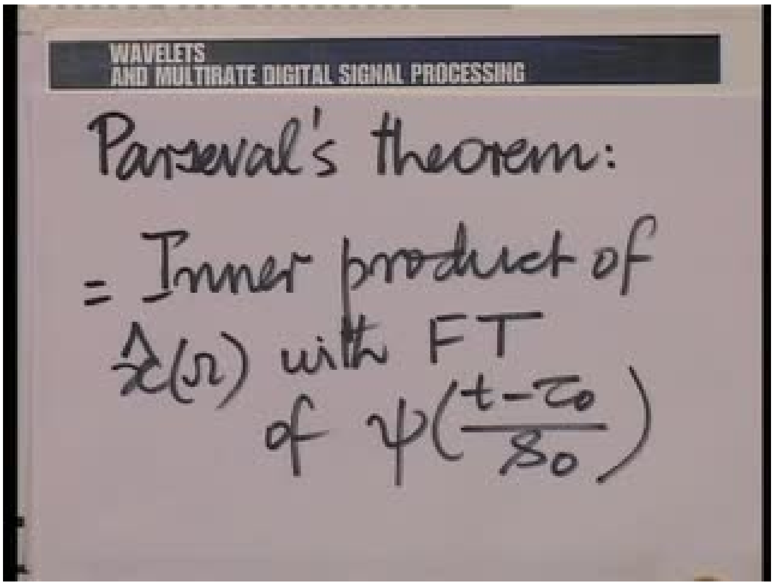

Now, let us interpret this also in the frequency domain sets what are we doing there. Let us use Parseval's theorem again. This will also equal then to be inner product of x cap omega with the Fourier transform of psi t minus tau naught by s naught.

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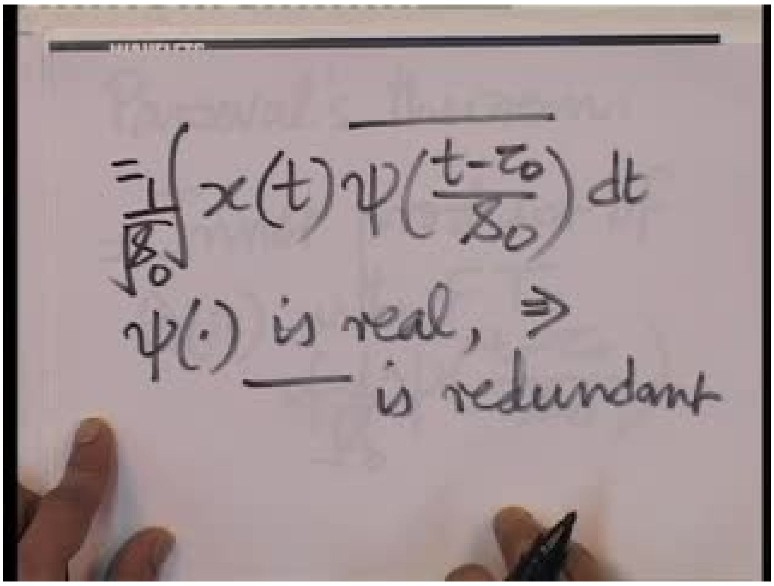

And let us evaluate the Fourier transform of t minus tau naught by s naught, of course normalized with this, and here again, although I did not need it inside the integral sign, I should keep the 1 by s naught to the power of half outside the integral sign. So, I must bring in the factor here for completeness. Let us evaluate this.

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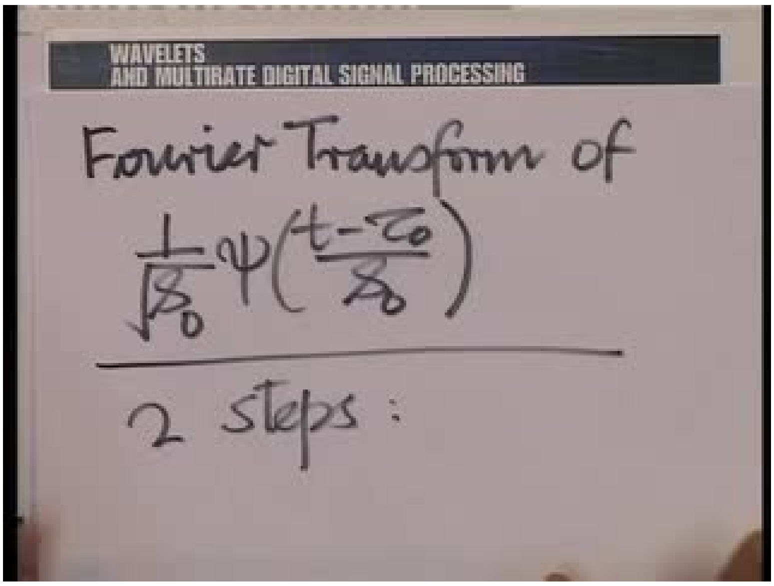

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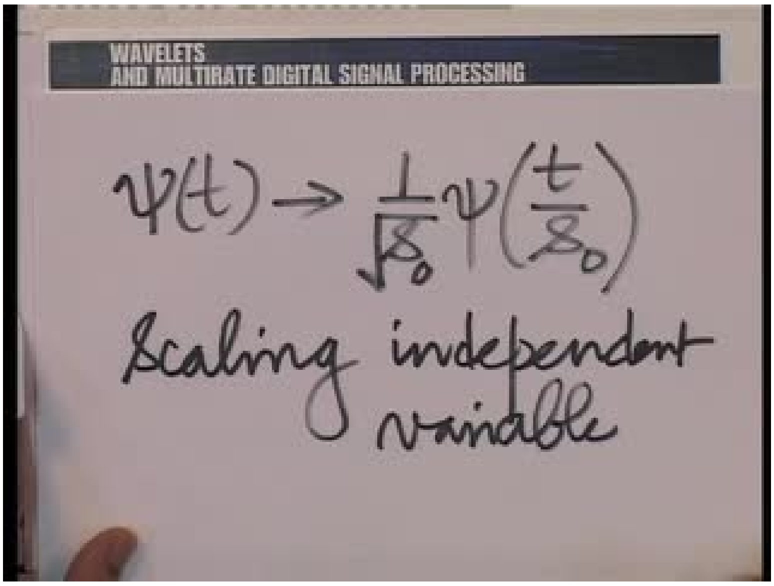

Now, the Fourier transform of 1 by s naught square root psi t minus tau naught by s naught can be calculated as follows. We will do it in two steps - we will first go from psi t to 1 by square root s was square root s naught psi t by s naught and we make use of the property of the Fourier transform pertaining to scaling the independent variable. So, make use of that property first.

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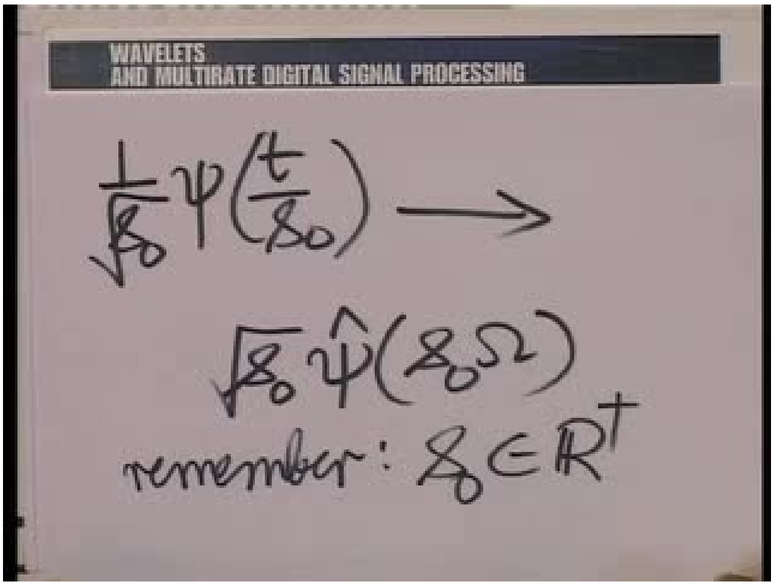

So, it is very easy to say that the Fourier transform of 1 by s naught to the power half psi t by s naught becomes s naught to the power half psi cap s naught omega. So, because of the normalization, the 1 by square root of s naught here and the 1 by s naught becomes the square root of s naught here and s naught there, and of course, remember s naught is positive in real. So, we do not need to worry about the sign a modulus is not required.

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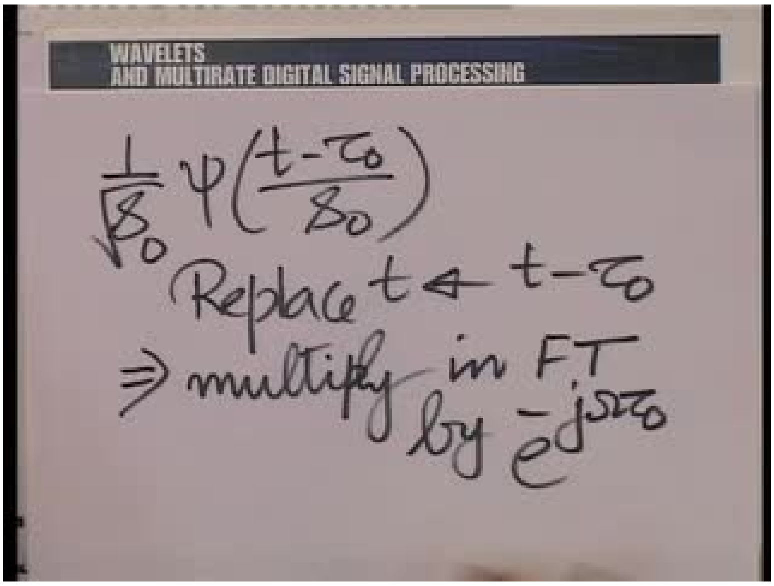

Now, if you wish to find the Fourier transform of 1 by s naught square root psi t minus tau naught by s naught all that we are doing is to replace t by t minus tau naught and that amounts to multiplying in the Fourier domain by e raise the power minus j omega tau naught.

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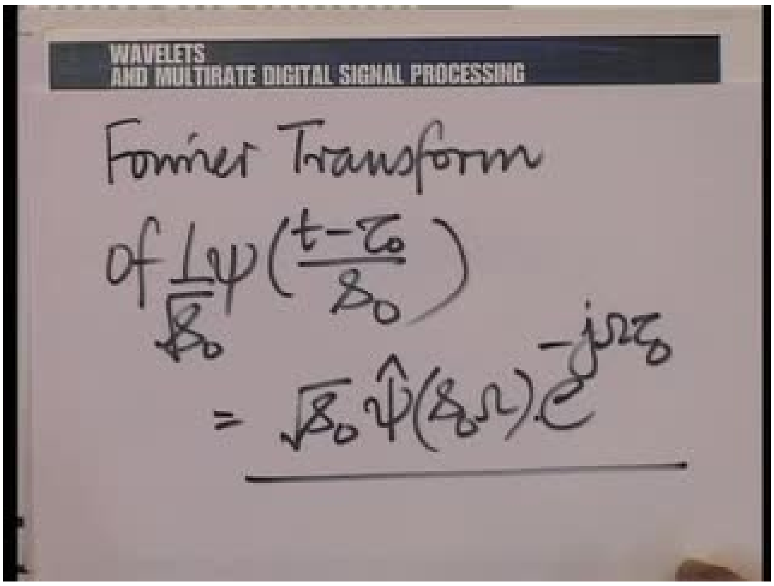

So, we will do exactly that. Therefore, we have the Fourier transform of psi minus tau naught by s naught into 1 by s naught square root is square root of s naught psi cap s naught omega multiplied by e raise the power minus j omega tau naught, and we put this back in the expression that we had for the continuous wavelet transform. Remember, the continuous wavelet transform is a continuous function of the translation tau naught and the scaling s naught; s naught is only positive real. Tau naught is both negative and positive real. Let us emphasize that.

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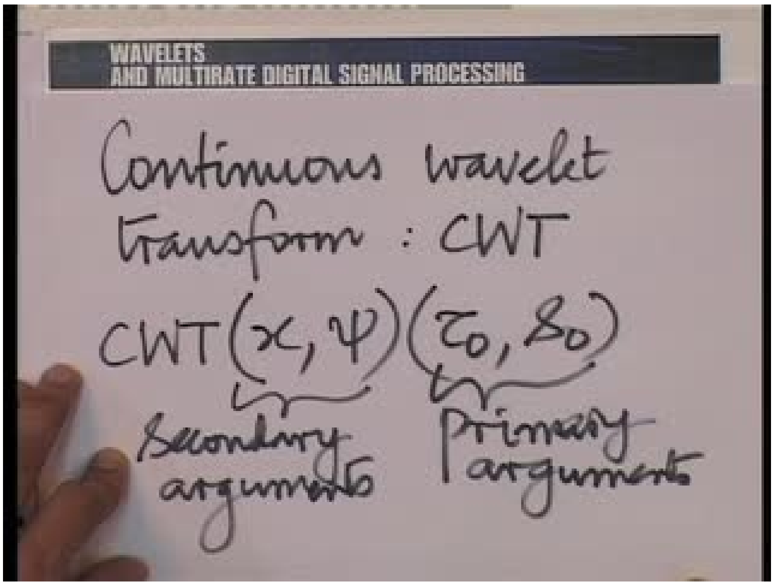

So, let us write both of these down and let us introduce notation here. The continuous wavelet transform, which we shall abbreviate by CWT, and here, again it has primary and secondary argument. So, CWT will have the secondary argument x and psi, and the primary argument tau naught and s naught and this reads as the continuous wavelet transform of the function x which presumably belongs L 2R with respect to the wavelet psi evaluated at the translation tau naught and the scale s naught.

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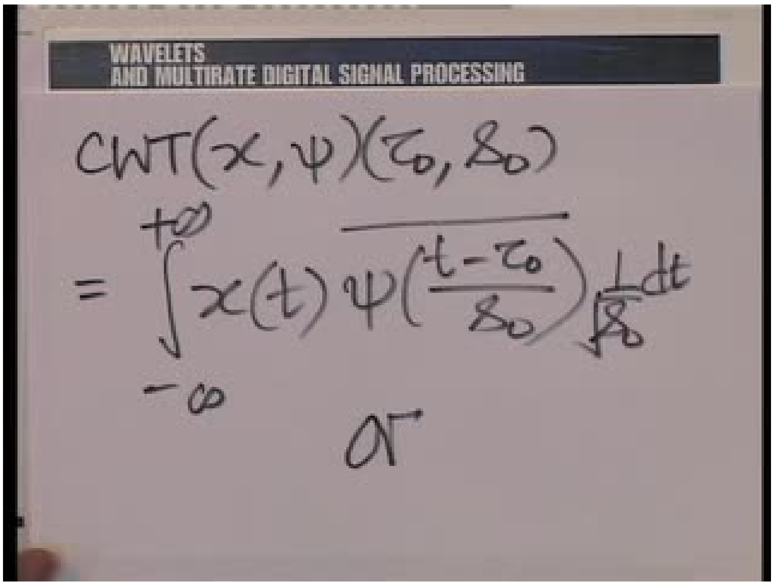

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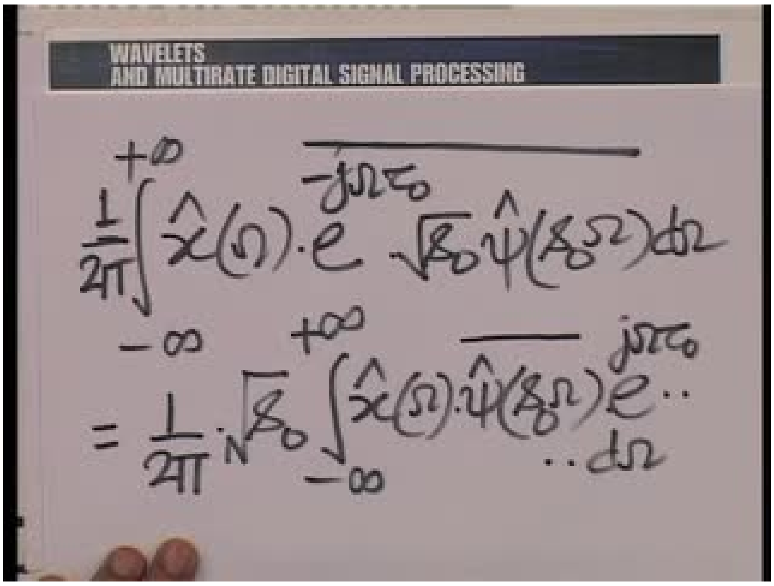

So, this quantity CWT of x with respect to psi evaluated at tau naught and s naught is either if you wish to look at it that way, the dot product of x t with psi t minus tau naught by s naught normalized with 1 by s naught either this or as we have just seen using Parseval’s theorem the following, of course with a factor of 1 by 2 pi. So, let me keep that factor of 1 by 2 pi here. This is of course complex conjugated. So, I need to rewrite this little bit.

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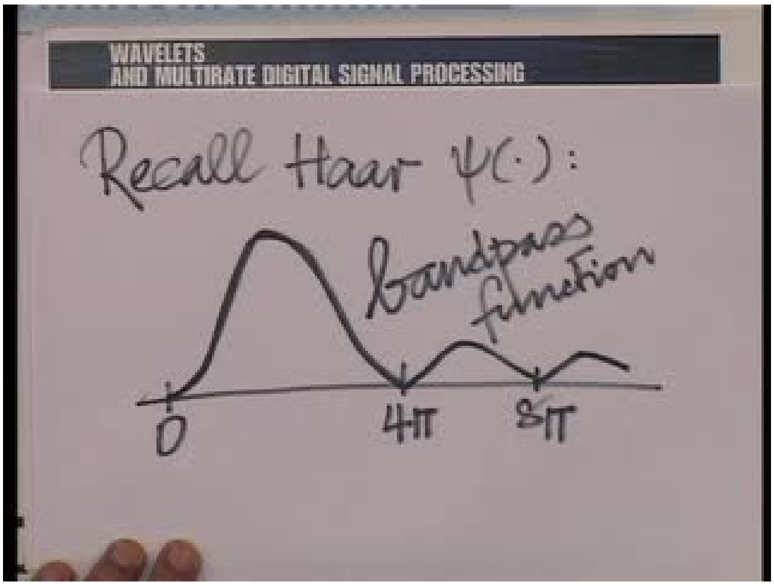

There is a complex conjugate here in general. I have taken care of the complex conjugate here by replacing the minus by a plus. Now, this has a very interesting interpretation provided. We require the nature of the Fourier transform of psi from the example we seen so far. You will recall that if we consider the haul wavelet for example, psi t was a band pass function. In fact, just to recall, let me put down the magnitude pattern of the Fourier transform of psi t in the Haar case, recall the Haar case.

It had magnitude Fourier transform which look something like this, which had the first null at 4 pi and subsequent nulls of side loops that all multiplies of 4 pi beyond, and the main loop essentially was a band between 0 and 4 pi. So, it was a band pass function. By a band pass function, I mean it did not have a non-null Fourier transform at 0. As omega tends to infinity, again the Fourier transform decays towards 0. So, the Fourier transform magnitude is 0 at 0, 0 at infinity and maximum somewhere in between. It emphasizes a band of frequencies.

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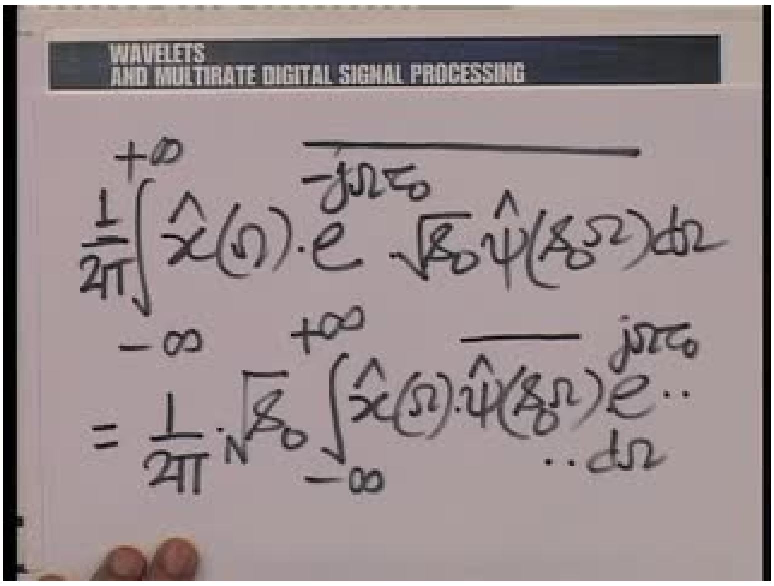

We saw that explicitly in the case of the Haar function. I encourage all of you to explicitly calculate the Fourier transform of the Daubechies 4 wavelet for example, to be interested to do. It has to be done numerically and verify that would also have this band pass character. So, we see the trend in these so called wavelet functions.

They have a band pass character, and if we allow that interpretation, then what we have written here has a beautiful meaning. It means that we are multiplying the Fourier transform of x with the band pass function scaled in the Fourier domain by the factor s naught and we are calculating the inverse Fourier transform of the same, of course this factor of square root of s naught is here the normalize.

Now, if you accept that psi is a band pass function, then, what you are doing here is essentially to extract a region of frequencies in the Fourier transform of x which lies around the appropriate dilate of the Fourier transform of psi and you are calculating the inverse Fourier transform. The inverse Fourier transform, this integral after multiplication by e raise the power j omega tau naught with respect to omega essentially means the output after doing this work in the frequency domain. So, what we are saying in effect is the following.

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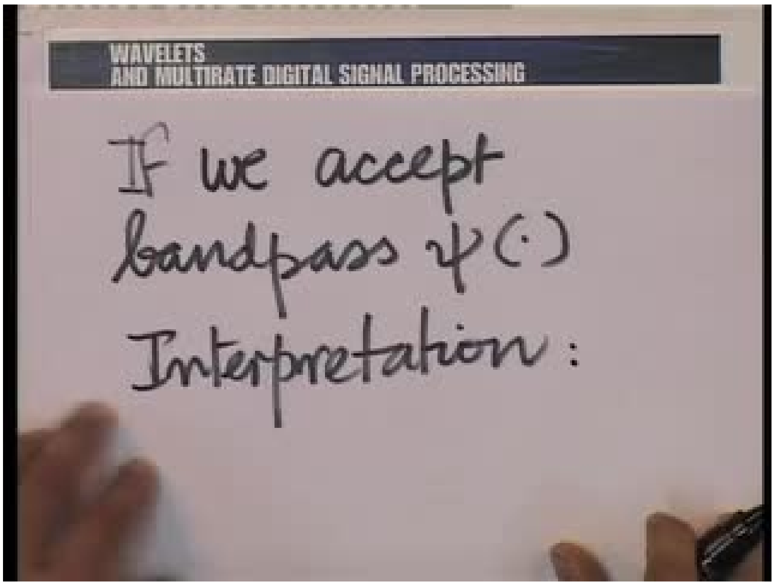

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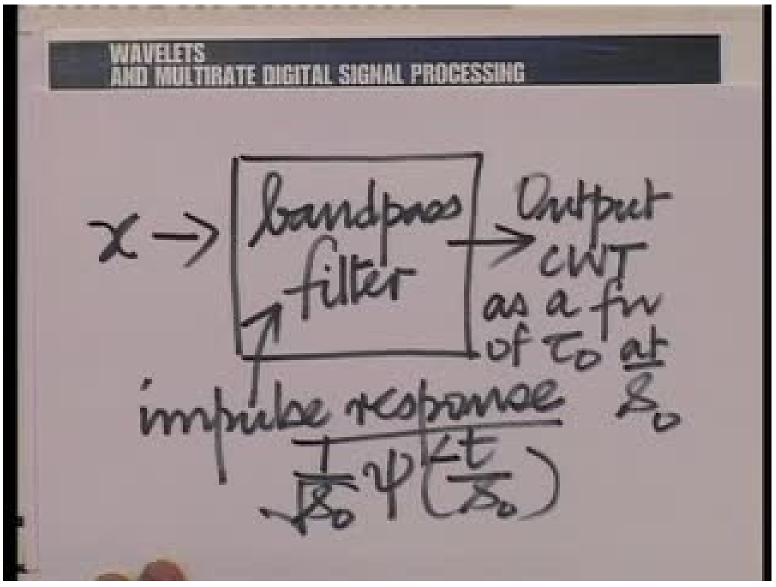

We are saying that in effect if we accept that psi is a band pass function, then the interpretation is as follows. In the continuous wavelet transform, we are taking x, we are passing it through a band pass filter whose impulse response is essentially 1 by s naught square root psi t by s naught.

Well, if you like one should complex conjugate this, because you are doing a complex conjugation there as well, for your complex conjugate this and strictly you should also put a minus sign here, because this is, this would be the inverse Fourier transform when you complex conjugate in frequency and then scale by s naught.

The output is the CWT as the function of tau naught at the scale s naught. So, at every scale, there is different filter. You have a continuum of filters indexed by s 0; for every scale s 0, there is a different filter. It extracts information in x cap around the center frequency appropriately scaled by s 0 and the band is also scaled, remember, when you scale the center frequency, also scale the band, recall all this discussion in the Haar.

Now, we are doing it for the general band pass function and that inverse Fourier transform is operated by tau naught. So, you are calculating the output at each point tau naught. This is the interpretation of the continuous wavelet transform.

Now, based on this interpretation and based on what we have understood as the interpretation of the short term Fourier transform, we shall go to the inversion of these two transform in the next lecture.

So, with that then we come to the end this lecture where what we have seen is essentially, the definition and the interpretation of the short term Fourier transform and the continuous wavelet transform.

Thank you