Advanced Digital Signal Processing - Wavelets and Multirate.
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Lecture No. # 20
The Time Frequency Plane And Its Tilings.

A warm welcome to the twentieth lecture, on the subject of wavelets and multirate digital signal processing. Recall, what we had done in the previous lecture, we at build up the uncertainty principle to completion, we found that nature imposed a fundamental limit. If we look at the time bandwidth product, you cannot go below a certain number that is for pre-finely in fort; in fact, we also in fort which function could give us that minimum product. Let us therefore, put the theme of the lecture today, and some of the important conclusion, such we had drawn in the previous lecture, before ourselves to put our discussion in perspective.

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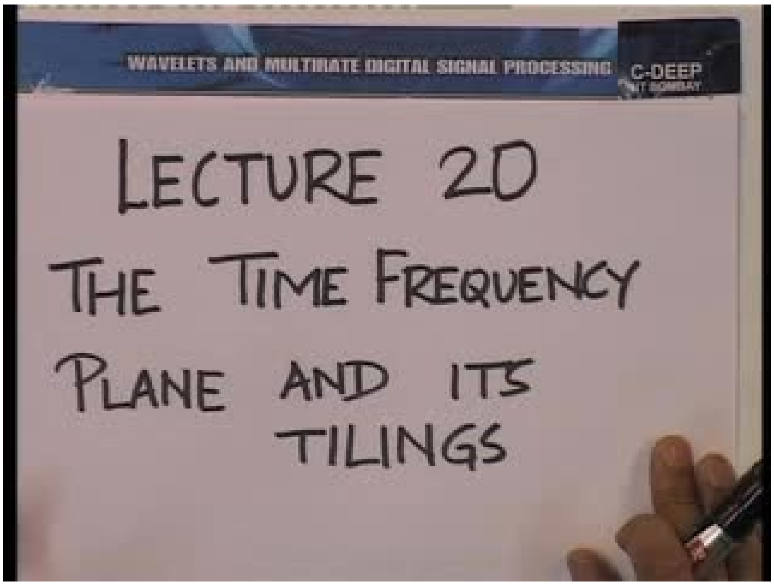

So, what we intend to today is to talk about what is called the time frequency plane and the idea of tiling; the time frequency plane, just like you would tile a flower for tile a surface, we would tile the time frequency plane. And to recall what we had done in the previous lecture, we had drawn the following conclusions. Conclusion number one, what is the minimum time bandwidth product that you can get.

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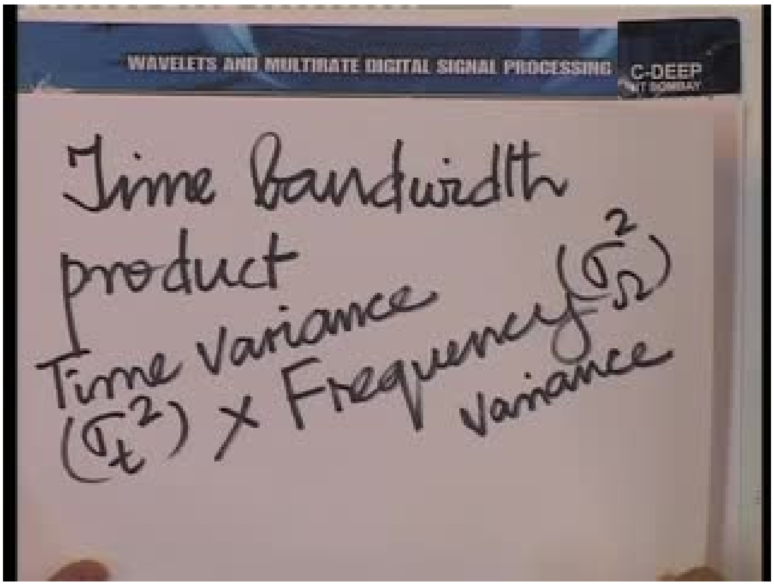

So, recall that we had defined the time bandwidth product, to be the time variance multiplied by the frequency variance. And we said that this quantity which we also described as sigma t squared times, sigma omega squared cannot fall short of 0.25.

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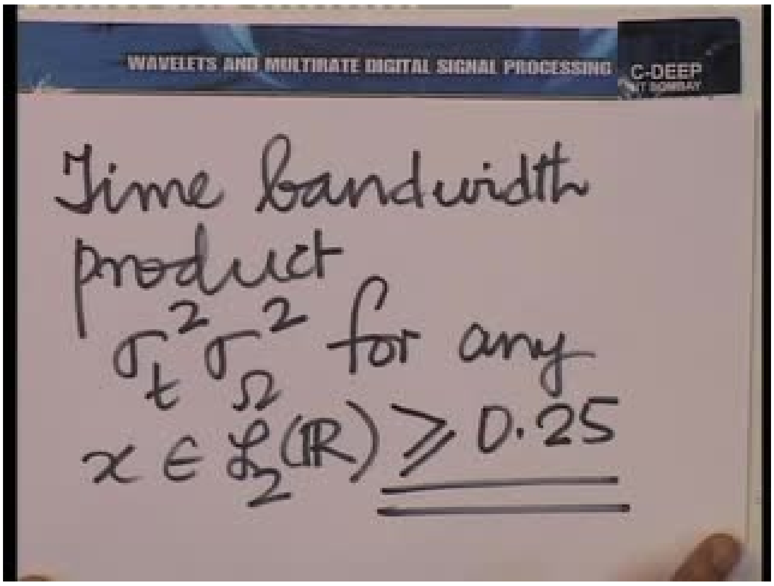

So, we said the time bandwidth product sigma t squared sigma omega squared, for any function x belonging to l 2 r is always greater than or equal to 0.25, we had prove this last time. And we also conclude it which function x t in l 2 r gives as this time bandwidth product.

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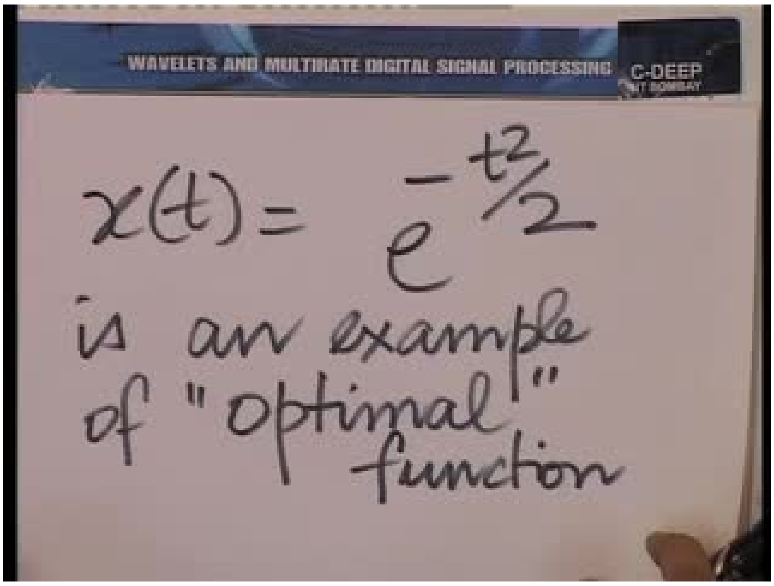

So, we showed that x t is the Gaussian namely, e race the power minus t squared by 2, for example, is an example of a, so called optimal function; optimal, in the sense of time bandwidth product, all right. Now, other optimal functions can be obtained by modulating this with a term of the form, e race the power j alpha t squared.

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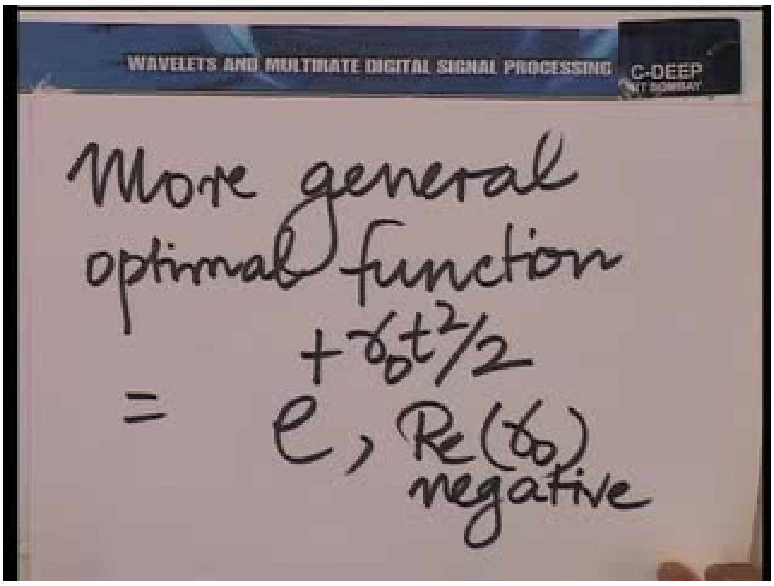

So, we saw the more general optimal function; more general optimal function is of the form, e race the power minus gamma 0 t squared by 2, are you could say, plus gamma 0 t squared by 2 with real the part of gamma 0 negative; if you wrote minus gamma zero, of course, you could say the real part is positive, other way whatever you wish to write.

So, gamma 0 could be complex in general that is what I mean, anyway. Essentially, it is a Gaussian that is optimal, in a way that is good news, in a way it is bad news; the good news is that we know what the optimal functions, we known that Gaussian is optimal; the bad news is that the Gaussian is unrealizable, in the exact sense and physical systems; you know this may seem like a puzzling statement. One of the favorite probability density functions of most scientist and engineers is the, so called normal of the Gaussian density. And in fact we go to the extent of saying that when we put together a number of independent, identically distributed random variables, in most situations, the some random variables goes towards Gaussian; so, that is what is called the central limit theorem.

And therefore, we also justify the use of the Gaussian density, in most statistical situations; so much, so that, we are abscessed with the use of normal distribution, and we use the term variance, to even denotes the spread around the mean. We say, well in a Gaussian, the spread around the mean, tells us is indicated by the variance, and tells us more or less in what range that variable likes; the variance mean plus variance to mean minus variance.

Well, so that is the Gaussian for you, then why are we saying that this is physically unrealizable? I am talking about a Gaussian time wave form, take for example, the exponential time wave form or the exponential time wave form modulated by a sinusoid; these are easily realizable. Circuits which comprise of resistances, inductances and capacitances when exited say with a step or even for the matter with a sinusoid, give as either exponentially decaying sinusoids or exponentially decaying transience; and therefore, those are easy to generate with physical systems; unfortunately, there is no meaningful physical system which can generate a Gaussian in the same way.

So, that is one of the fundamental reasons, why Gaussian is good news in a statistical density, but Gaussian is bad news as far as function scope. You know I must mention that people talk about what is called Gaussian mean shift key or Gaussian minimum shift key? GMSK in the context of digital communication, the word Gaussian there refers to a

Gaussian pattern in the impulse response, whether it is in phase or in amplitude, but there again people really fight hard to realize a Gaussian filter.

So, you see that the Gaussian is difficult to realized physical systems, can only be approximated; so, although it is good news. As we know what the optimal function is, it is bad news that we cannot easily realize, it is, this optimal function by using physical systems; well then that is bad news. Let us bring some good news, if not the Gaussian then can we use a reasonable function? Which we could probably realize with the cascade of two simple systems, the, something of the kind and go close to the Gaussian.

So, in other words, when we started with the hour we are at terrible time bandwidth product infinite. Now, can we do a little better? Suppose, we took a cascade of two systems whose impulse response is essentially that pulse; what I mean by that is instead of taking just one pulse take a cascade of them.

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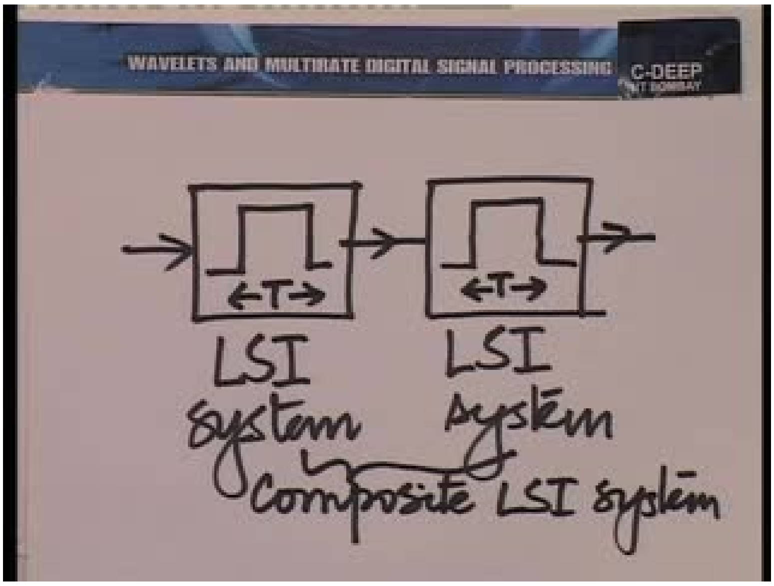

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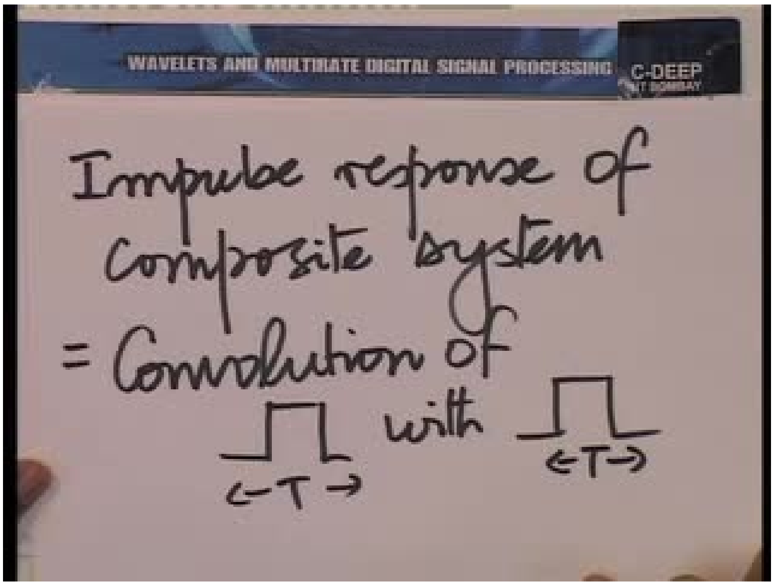

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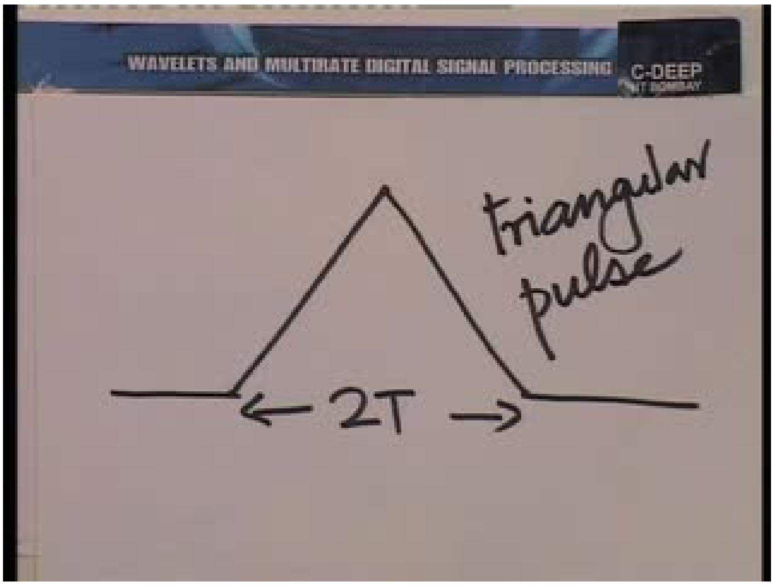

So, we, suppose we have two systems, each of whose impulse response is essentially a pulse, say of the same width. This is the linear shift invariant system, this is another linear shift invariant system, the impulse response here is essentially a pulse, and the impulse response here too is a pulse; both pulse is of the same width that is a t, we cascade them. And we note, of course, that together this also forms a composite LSI system, and the impulse response of that composite LSI system is essentially the convolution. And we know that convolution very well; that convolution looks something like this, I hardly need it to work it out, this is something very familiar to as form any basic course on signals or systems, the convolution looks like this.

Essentially, what is called a triangular pulse? Now, we can gives this a physical interpretation, you know when you have this LSI system with an impulse response equal to a pulse, what you are essentially doing, is a sample and hold process.

So, if an impulse results in a pulse you are essentially sampling a function at a point and holding it, for the duration of that pulse; that is what the physical meaning of that. Impulse response given by a pulse is, so, if a too search sampling in hold, then you are effectively talking about triangular impulse response; so, there is some underline physical meaning.

Now, the natural question to ask is what can we say about the time bandwidth product of this triangular pulse? How bad or good is it compared to the Gaussian? That is the next question that we shall answer.

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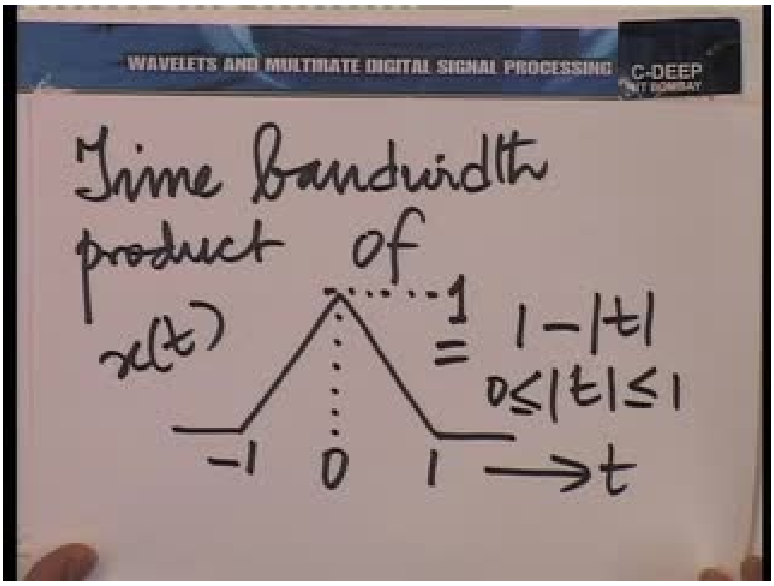

So, the time bandwidth product of this triangular pulse and you do know you remember that we do not need to worry where this triangular pulse lies, so we can as well center it, and put at zero, we do not need to worry how wide this triangular pulse is as long as, we will kept it symmetric. So, we can put this from minus 1 to 1, and we do not need to worry about the height is, and we can as well; therefore, make the height equal to1, good all; this is because of the invariance properties of the time bandwidth product. It is invariant to scaling on the dependent variable, it is invariant to scaling on the independent variable and it is invariant to translation.

So, we shall find out the time bandwidth product of this; in fact, we can describe this function, let us call this function x of t as a function of t, and describe it. It is essentially 1 minus mod t, for mod t between 0 and1.

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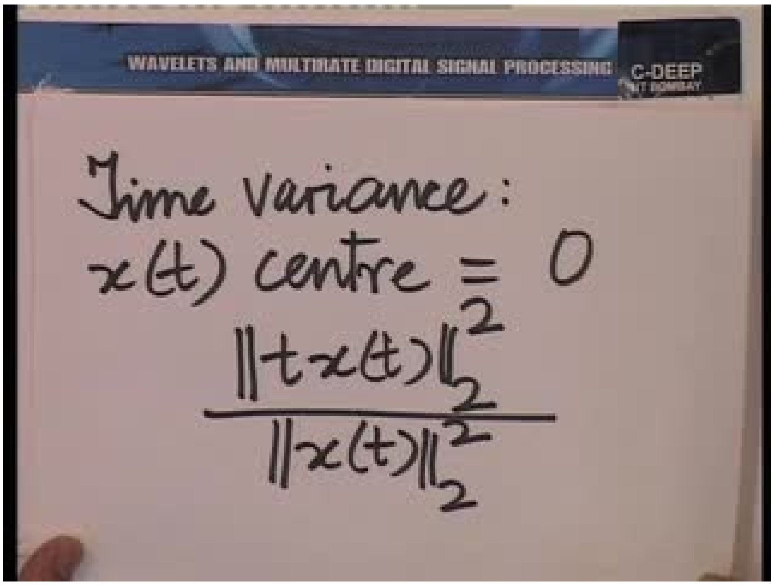

So, how would we find the time bandwidth product? We shall first obtain the time variance, and you will recall that since the function is centered, that means, the centre of x t is at 0. The time variance is going to be described by the norm of t x t the whole squared in l 2 r divided by the norm of x in l 2 r the whole squared.

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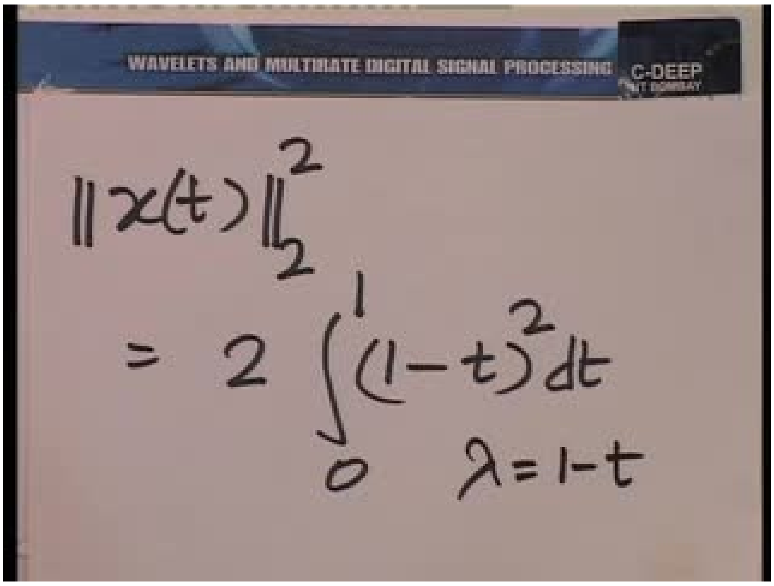

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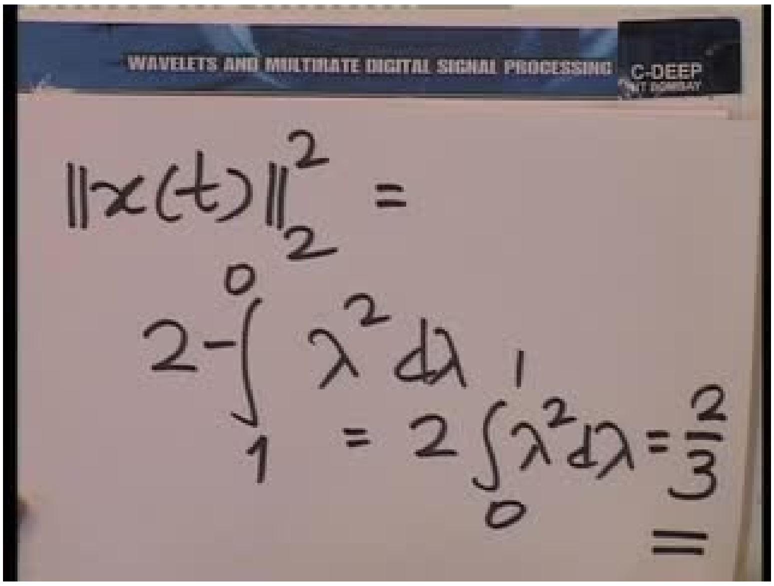

Now, we shall be requiring this norm of x in l 2 r the whole squared again and again, so let us begin by calculating this norm first. The norm of x t in l 2 r is easily seen to be 2 times integral from 0 to 1 1 minus t the whole squared d t, this 2 times comes becomes of the symmetry around t equal to 0. So, essentially the area on the negative and the positive side is the same. Now, this is a easy integral to evaluate, we can easily make the substitution, lambda is 1 minus t, and evaluate this integral, and that is gives us, the norm of x t in l 2 r squared is 2, again this is lambda squared, now d lambda is minus d t.

So, we could write minus d lambda here, but the limits also change from 1 to 0, in that case and therefore, is the same as 2 integral 0 to 1 lambda squared d lambda; this is easy to evaluate, this essentially evaluates to 2 by 3, that is easy to evaluate; lambda 2 by 3 from 0 to 1, anyway, so much, so for the l 2 r norm.

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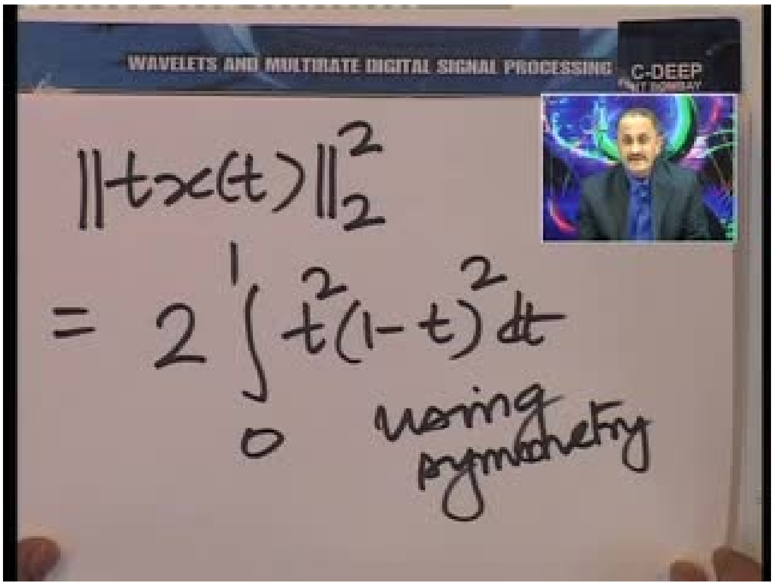

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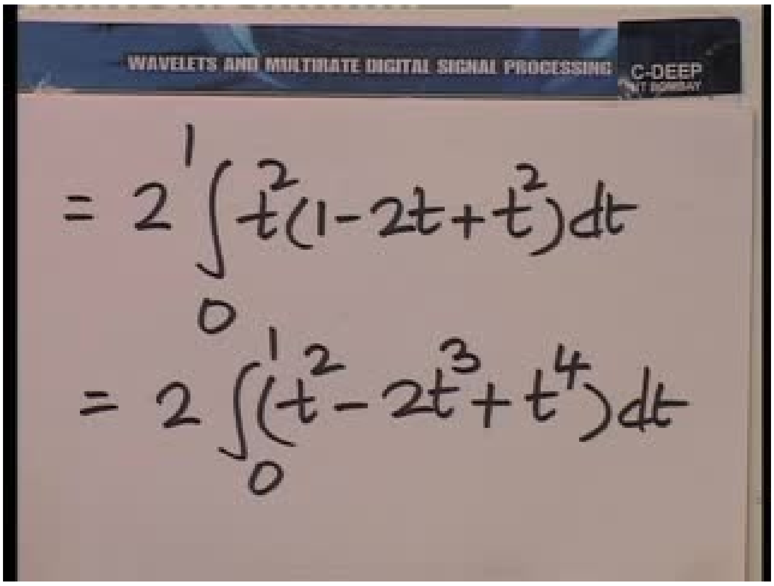

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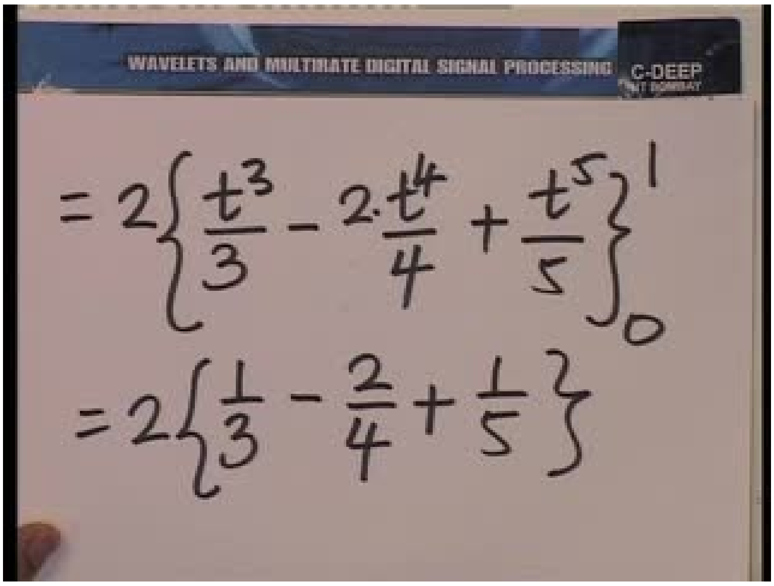

Now, let us take the norm of t x t; so, let us evaluate norm t x t the whole squared, now here, again, we will use symmetry, so it is 2 times the integral from 0 to 1 t times 1minus t d t using symmetry the whole square, of course. Now, here it is not going to help very much to make a substitution of variable, because you know if we substitute lambda is 1 minus t, we will get a 1 minus lambda, here which is not so convenient; let us as keep it as an integral in t. And let us evaluate the integral bravely, so to speak; so that is, 2 integral from 0 to1t squared 1 minus 2 t plus t squared d t which is 2 integral 0 to 1 t squared minus 2 t cubed plus t to the power 4 d t; easy integrals to evaluate and we do that t cube by 3 the t to the power 4 by 4 and t to the power 5 by 5 here, evaluated from 0 to 1; and that is 2 1 by 3 minus 2 by 4 plus 1 by 5, simple enough.

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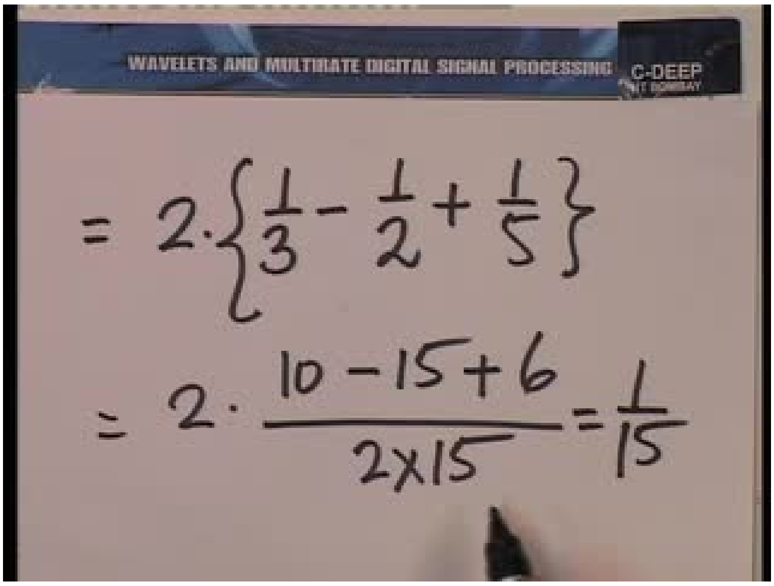

Let us simplify a little bit; so, 1 by 15 is what we have; 16 minus 15 that is 1, 2 and 2 cancel, and there you are. Now, this is the l 2 norm of t x t squared, I mean the l 2 norm of t x t the whole squared in l 2 r, that is, what I mean.

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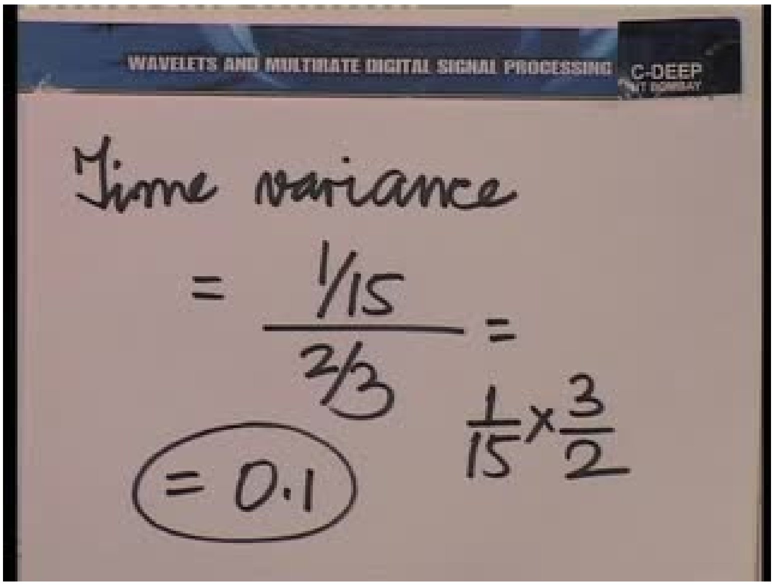

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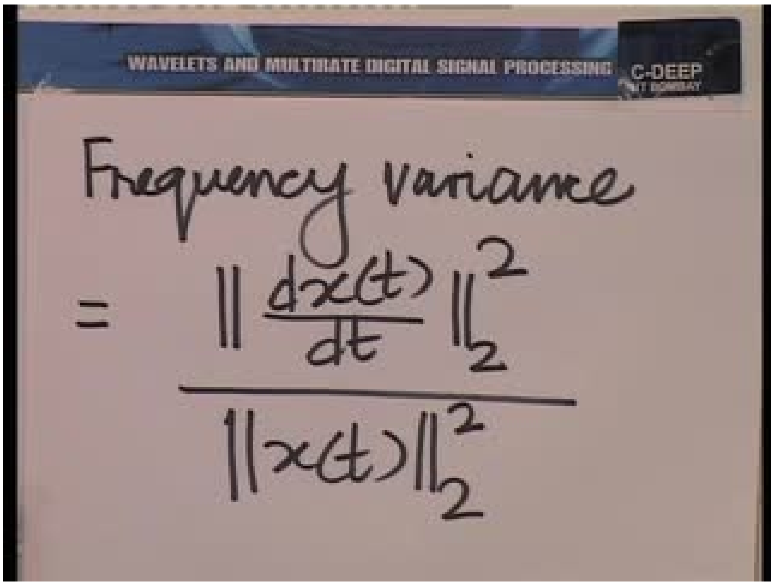

So, we have the time variance ready for us; the time variance is therefore, 1 by 15 divided by 2 by 3, which is 1 by 15 into 3 by 2 or that is 1 by 10. Now, let us look at the frequency domain, in fact the frequency domain will be a little easier, because we are going to make use of the principle of bringing the frequency to the time domain, in calculating variance. As a very easy integral to evaluate, the frequency variance is going to given by the l 2 norm of d x t d t the whole squared divided by the l 2 norm of x t the whole squared. And d x t d t is the very simple function to evaluate; in fact d x t d t has a following appearance.

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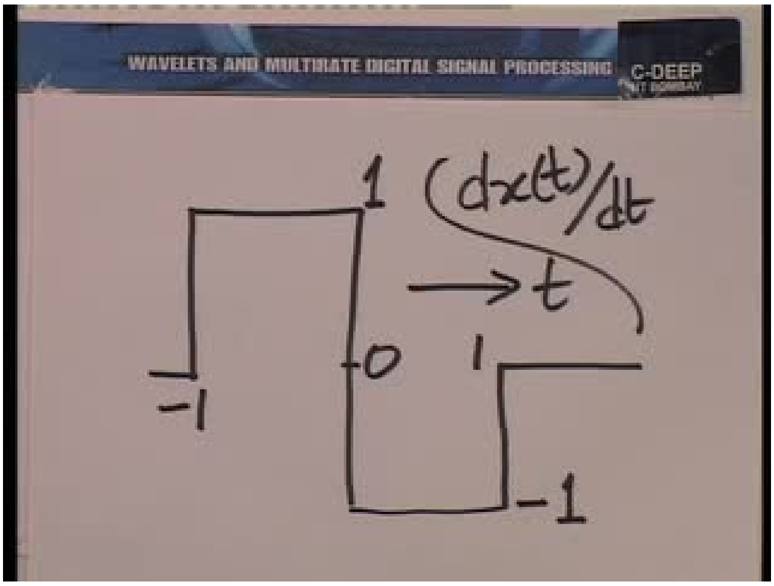

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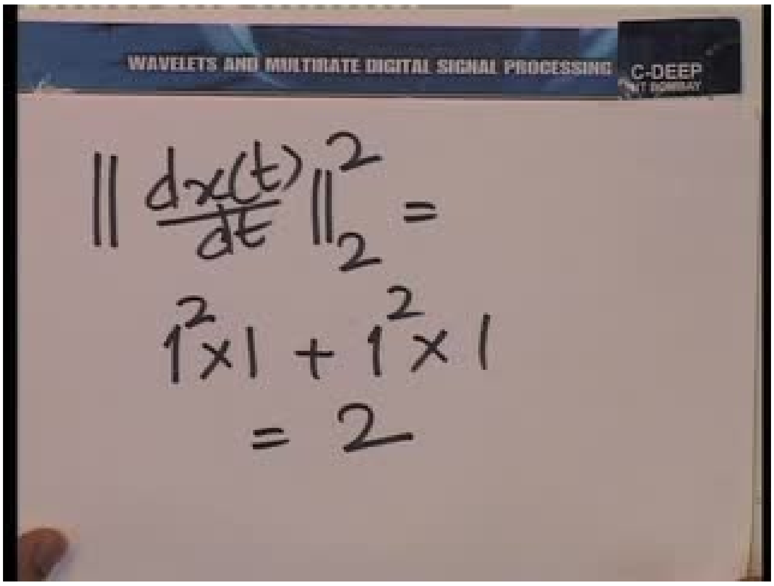

It is interesting, d x t d t here as the appearance of hoar wavelet. Let us very easy to calculate the energy, in this the l 2 norm of d x t d t is simply l 2 norm squared, I mean is simply 1 squared into 1 plus 1 squared into 1, looking at the areas of the rectangles, that is 2. And we already know the l 2 norm of the function in squared 2.

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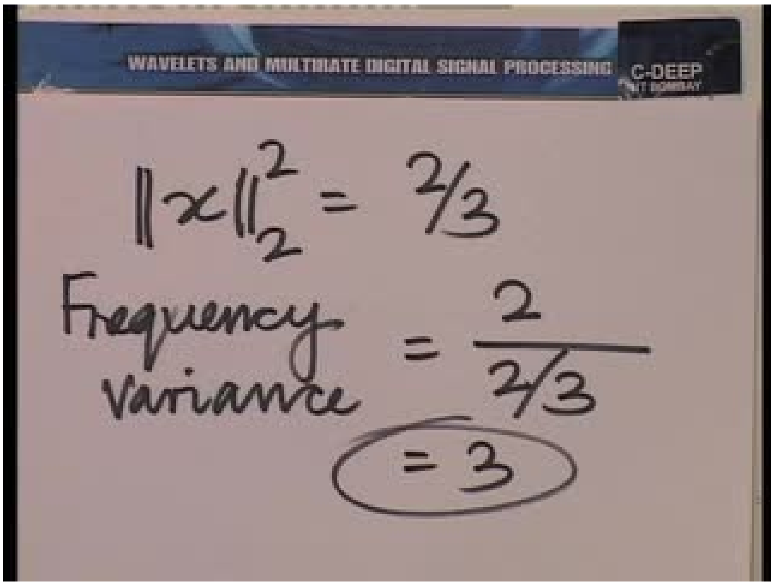

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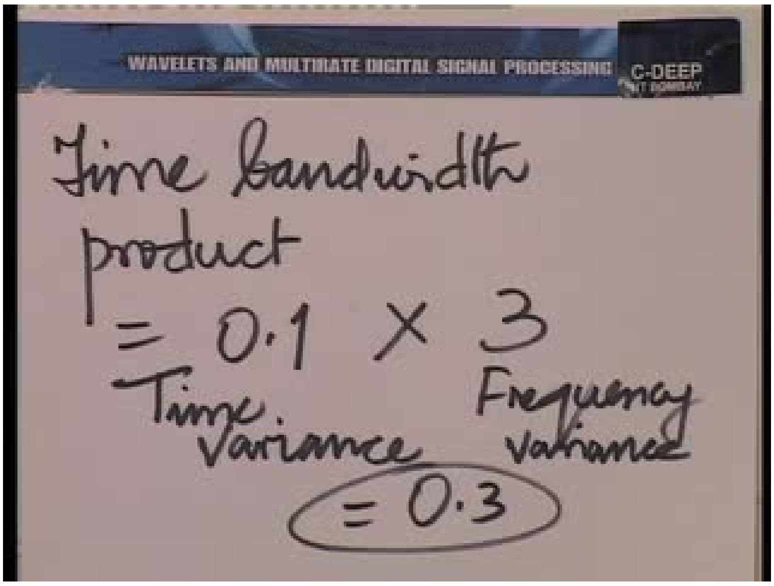

So, we know this, we know the l 2 norm of x it is 2 by 3; and therefore, the frequency variance turns out to be 2 divided by 2 by 3, which is 3. Now, we can calculate the time bandwidth product; so, the time bandwidth product is 0.1, the time variance multiplied by 3, the frequency variance low and behold it is 0.3.

So, that is very good news, actually if you think about it. We know the minimum; we go to is 0.25, if come all the way down to 0.3, pretty good for infinity; if come all the way to 0.3, just by cascading the system with itself once again, not bad at all. So, although there was bad news in the uncertainty principle, that you cannot reduce the simultaneous localization in time and frequency below 0.25, in the sense of the time bandwidth product. There was good news in that you knew what the optimal function was namely the Gaussian; then there was bad news again that the Gaussian was physically unrealizable as a function, but now we have some good news namely, that we can go all the way down to 0.3, by a very meaningful function.

Now, we seem to have an alternation of bad news and good news, and yes that alternation takes one more step. The bad news is that, now when you want to go from 0.3 to 0.5, we are going to have to work really, really, really hard that is what nature does most of the time, I mention this before. Nature brings you tantalizing closed to an ideal and makes you work very hard to go anywhere closer. If you look at many fields, whether it is filter design, whether it is system design, you known to get an acceptable level of system performance, you may have to work just a little bit to get that finish in system performance; one has to really work hard. And the difference between acceptable and fine performance may not be all that much all the time that is true here is. Well to go closed to the uncertainty principle is not to difficult to go any closer is very difficult; in fact, one way of going closer is to repeatedly convolve the pulse with itself.

So, if you took this sample in whole system, and if you made a cascade of one more such system with it you get the triangular pulse. Now, if you take a cascade of three such systems, put one more in cascade, what I mean by that is take this cascade.

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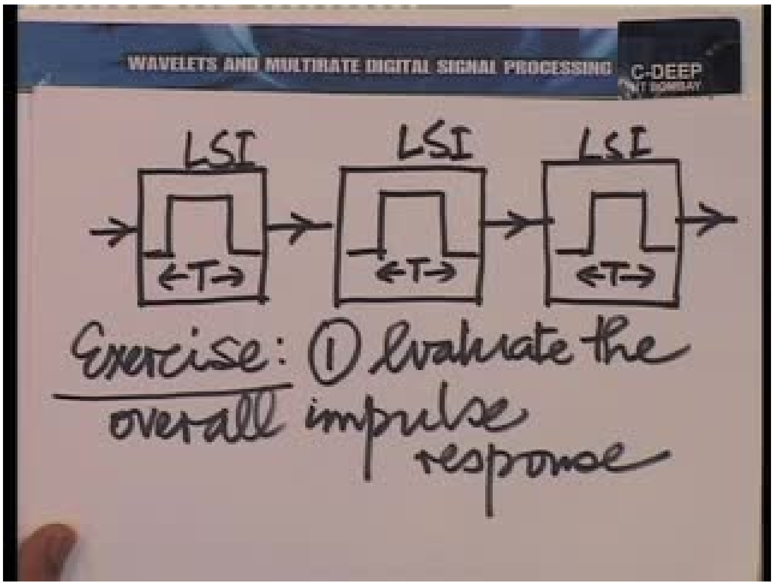

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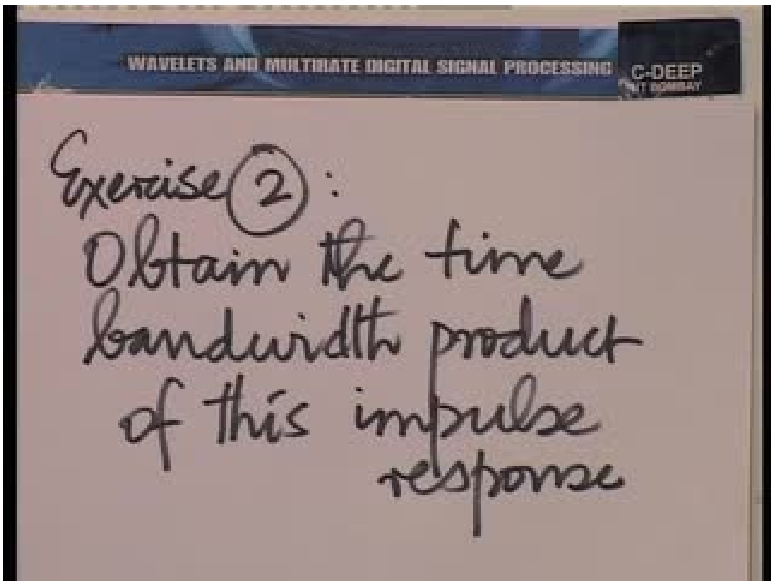

So, I have a cascade of three LSI systems, each of whose impulse response is essentially a pulse. I shall leave a little exercise for you to do here; the exercise is number one, evaluate the overall impulse response; exercise number two, these are all LSI system by the way, linear shifting variance system with the impulse response shown, exercise two, obtain the time bandwidth product of this impulse response. And naturally when you do that you would be inclined to compare it with the number 0.3, you hope there should be less, but I leave it you to see what it actually is. In fact, it will be interesting to do this further, and it will be even more interesting to see if you can come up with an inductive argument that is not easy, by the way, each time you put one in the cascade or you going to do better in the terms of time bandwidth product or you going to go closer to 0.25. I leave it to you to take a couple of steps it should be an interesting thing to do. But anyway, let us be one more remark, you know it is not just of compactly supported function like this, one which has a time bandwidth product of 0.3.

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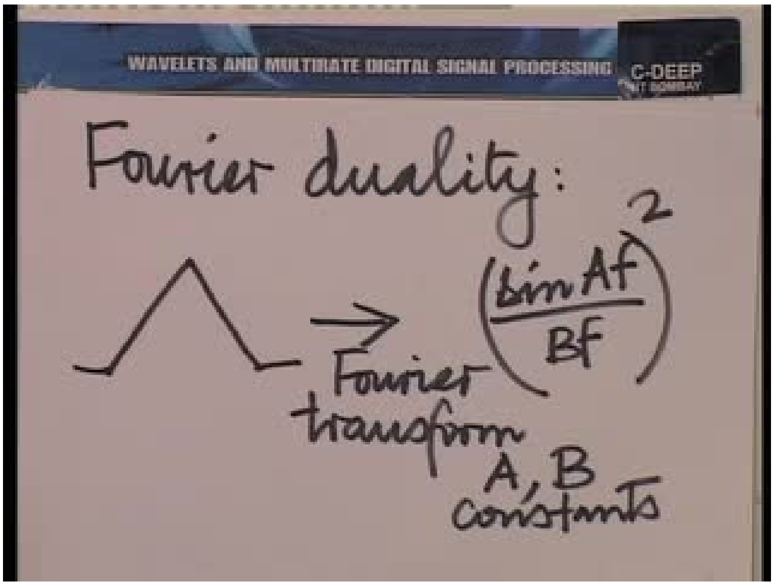

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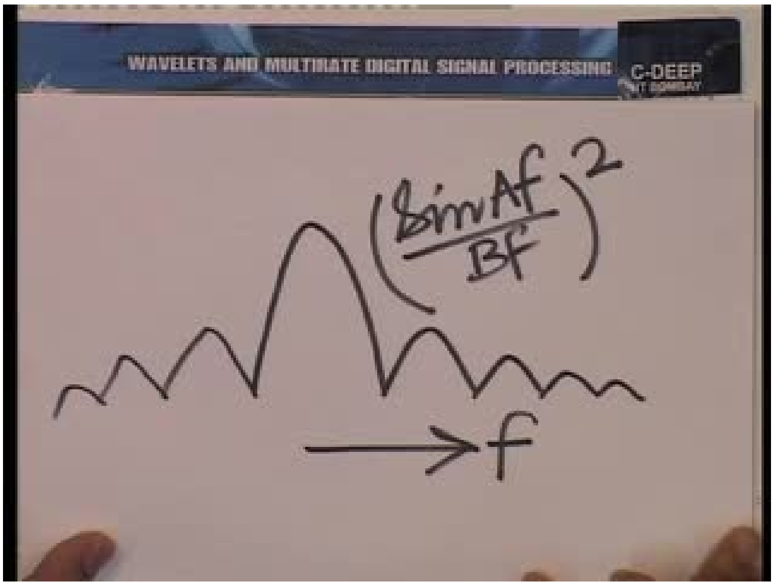

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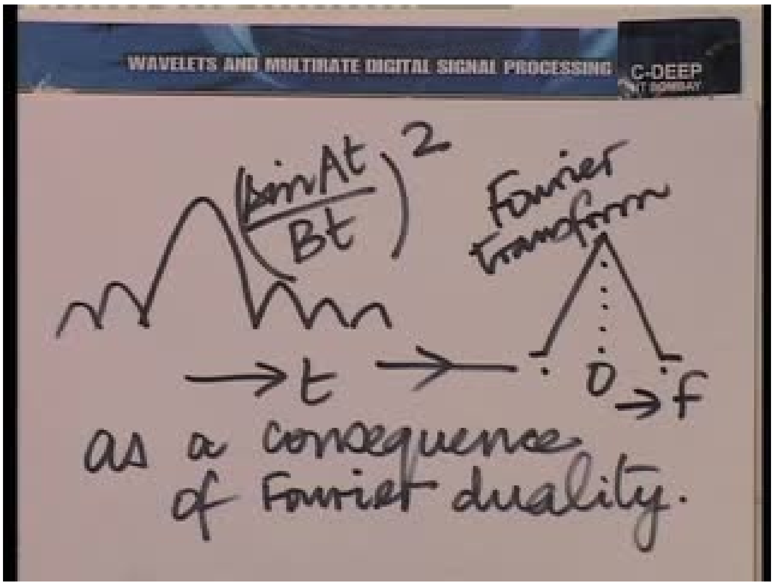

Now, I shall use a very simple argument to show that the same 0.3 can come from a non-compact disported function. We use the principle of Fourier duality, and we know that the fourier transform of this triangular pulse in time is of the form sin A f by B f the whole squared, A and B are constants. You can suitably evaluate the A and B that are not important; what is important? Is the form; so in fact, we can even sketch it. The Fourier transform would look something like this, as the function of f; now, the important question to ask is what the Fourier transform is, if this is treated as the time function? And that is what fourier duality would give us; it would tell us that if we considered a time function like this, something like sin A t by B t the whole squared; its Fourier transform is going to look something like this, I would not need to mark the limits, but this is 0 as a consequence of duality; this is the fourier transform function of f here.

So, what we are calling the time variance for the triangular pulse becomes the frequency variance for this, for this function the sin At by B t, kind of function here. And what we are calling the frequency variance for the triangular pulse becomes the time variance for this; so, in a, in other words, when you take the Fourier transform of a function, and ask what its time bandwidth product? The time bandwidth product is the same.

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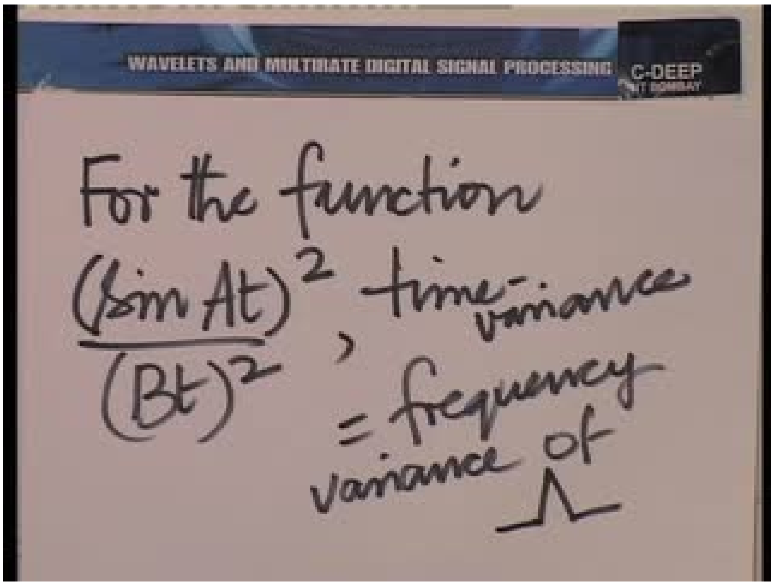

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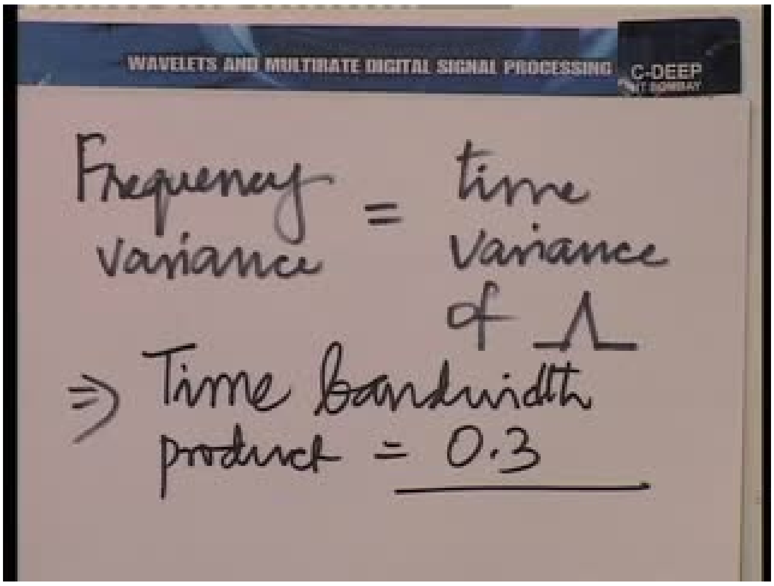

So, for this function, for the function sin A t by B t the whole squared or this time variance is equal to the frequency variance of the triangular pulse; and the frequency variance is equal to the time variance of the same triangular pulse; and therefore, the time bandwidth product is very easy to calculate, in fact, the time bandwidth product would simply be 0.3.

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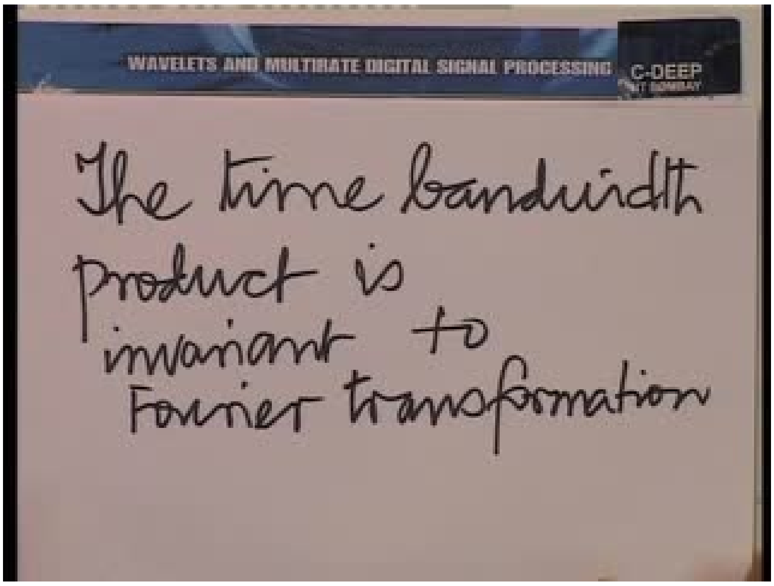

Now, we have a partial answer to the question that we race the last time; can you change the shape and maintain the same time bandwidth product? Yes, you can, in fact, what we have just done, answers many question of brings up many different conclusions; one is we have discovered one more kind of invariance of the time bandwidth product, and let us, write that down the time bandwidth product is invariant to Fourier transformation; and this in the very deep kinds of invariance. One more conclusion that we have drawn from here is that we can have both compactly and non-compactly supported functions; namely, function which are non-zero on a finite interval, and functions are non-zero over an infinite interval with the same time bandwidth product.

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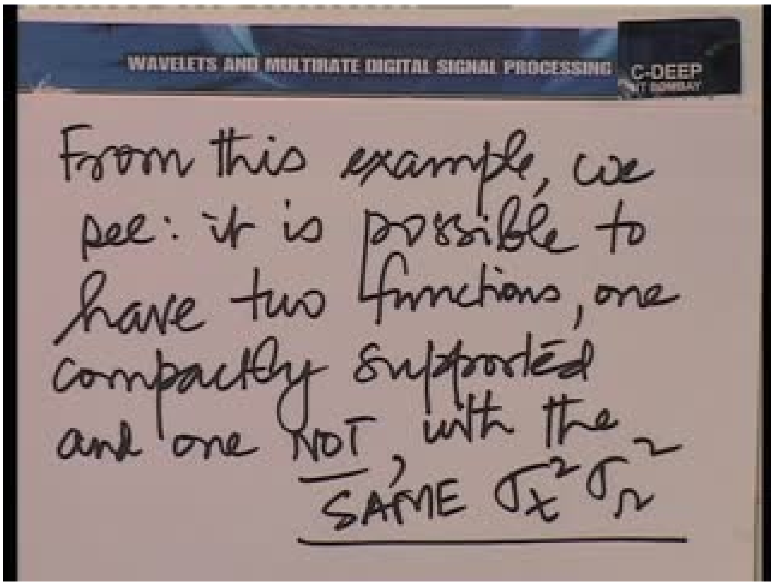

So, it is possible from this example, we see; it is possible to have two functions, one compactly supported, and one not with the same time bandwidth product. Now, with this remark, we would like to take the idea of the time bandwidth product further; now that we have identified two domains, let us put the domains together, and bring out a new domain at two variable domains.

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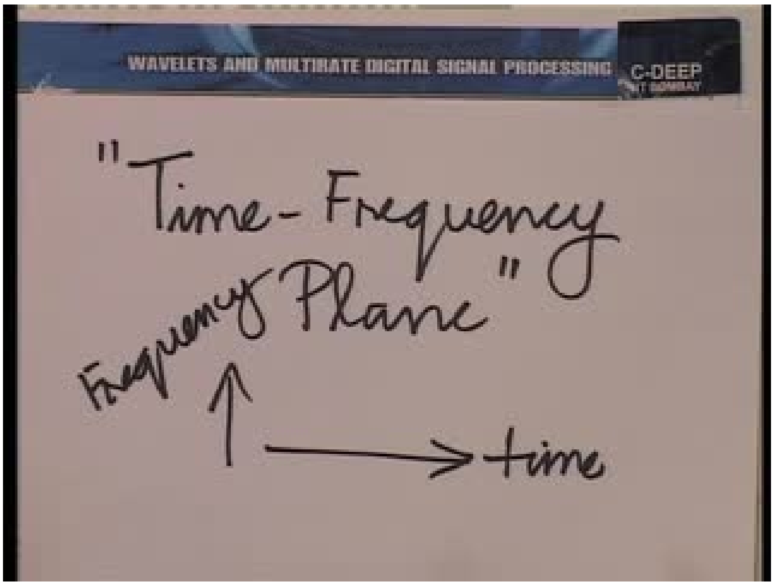

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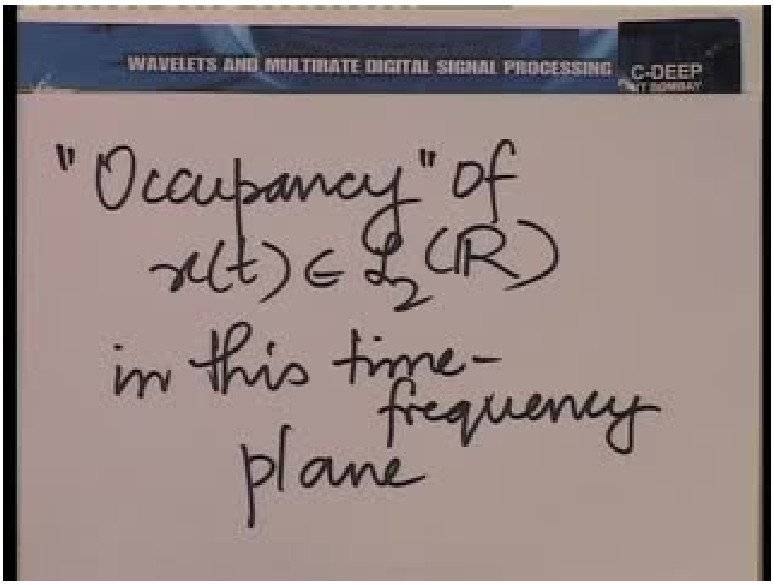

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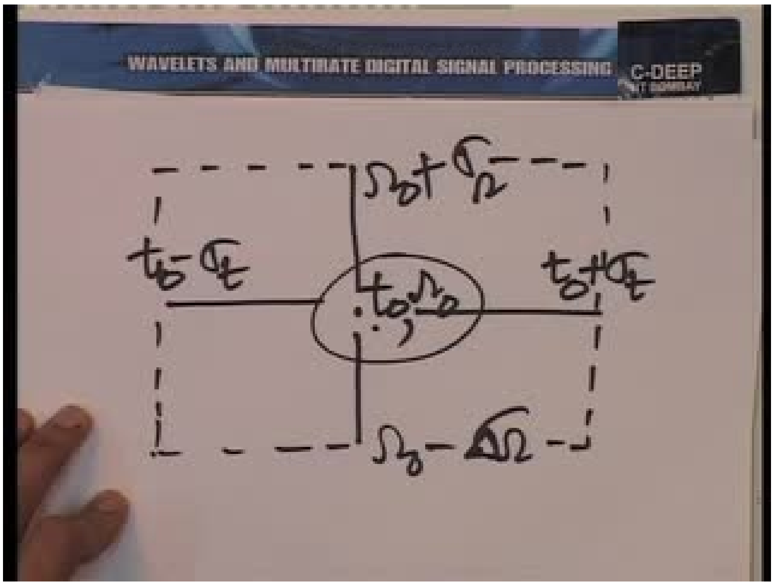

So, we shall hence fourth talk of what is called a time frequency plane; essentially a plane in which one axis say - the horizontal axis represents time and the other axis say - the vertical axis represents frequency. And therefore, what the uncertainty principles says, is that, if you wish to describe the occupancy of a function in this plan, you know, you can think of each function in l 2 r; we can think of the occupancy of x t in this time frequency plane, occupancy is notional; and this occupancy can be thought as being from t 0 the center in time to t 0 plus sigma t on one side ,and t 0 minus sigma t on the other; this is the horizontal axis. And on the vertical axis, we could center it at capital omega 0 namely the frequency center and we could spread it to omega 0 minus delta omega 0 or sigma omega if you please, and above we could take it to omega zero plus sigma omega.

So, this is in some sense notionally the spread, so you could think of that function x t as located in a rectangle, which is centered at t 0 omega 0 here, which has a width or horizontal spread of two times sigma t, and a vertical spread of two times sigma omega.

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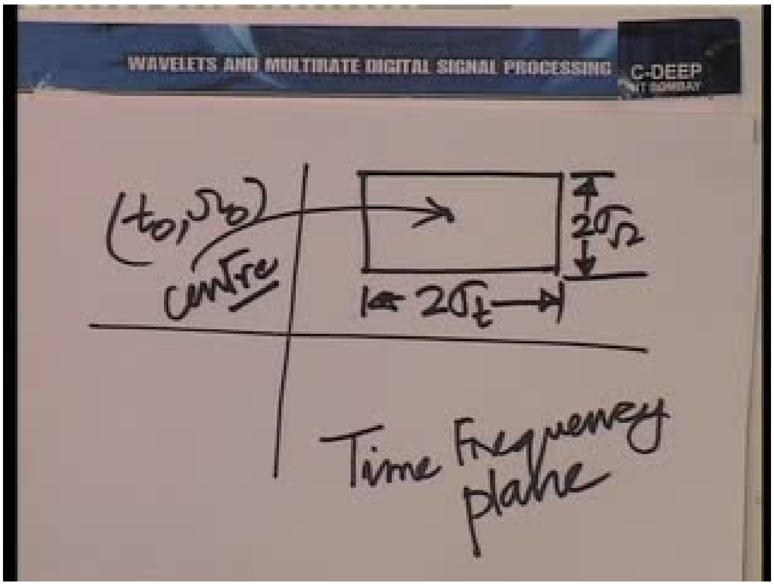

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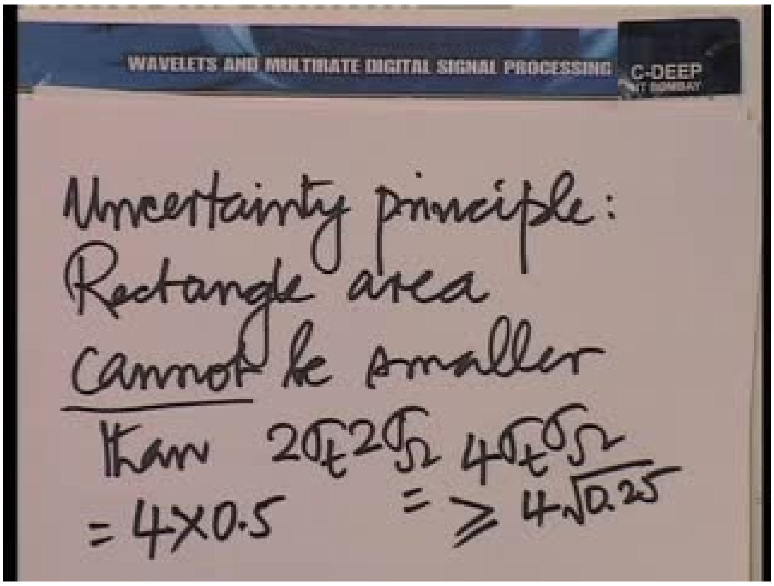

So, let us show the whole time frequency plane in some sense, what we are saying is, here you have the time frequency plane, so let us mark it; this is the time frequency plane notionally. And a function x t occupies a rectangle in this time frequency plane, center t 0 omega 0; spread two sigma t horizontally, two sigma omega vertically. A very interesting concept, a function in a l 2 r lies in a certain region of the time frequency plane, it occupies a certain area in the time frequency plane; and what the uncertainty principle says, is that, this rectangle cannot have an area smaller than a certain number. Uncertainty says - the rectangle area cannot be smaller than, well how much 2 sigma t into 2 sigma omega that is 4 sigma t sigma omega greater than equal to 4 times square root of 0.25, which is 0.5.

So, that is 4 into 0.5 that is the smallest area that it can have two units; the area of the rectangle cannot be smaller than two units; now, within that limitation you can change the width and the height that is also the positive side of the uncertainty principle. And in fact, if you wish to cover the time frequency plane with functions, what you mean by covering a time frequency plane with functions? It means using functions which occupy different such rectangles, in such a way, that it gives you different information in the time and frequency domain about in other function.

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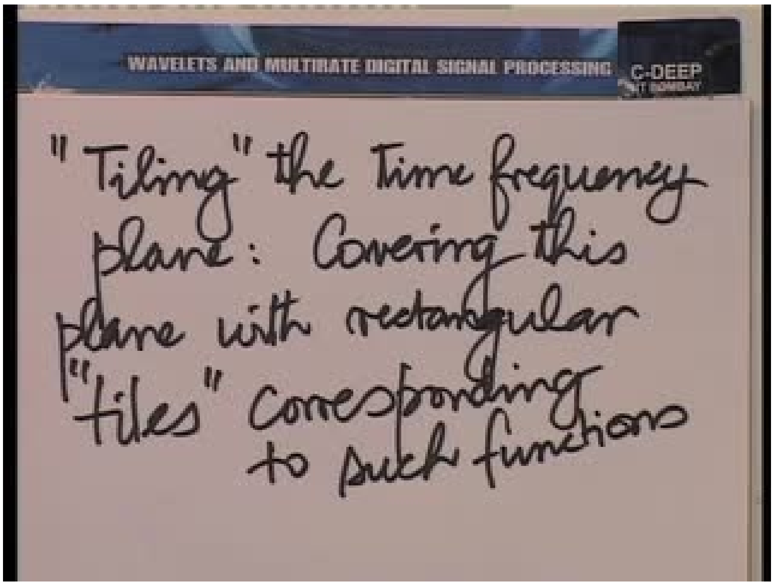

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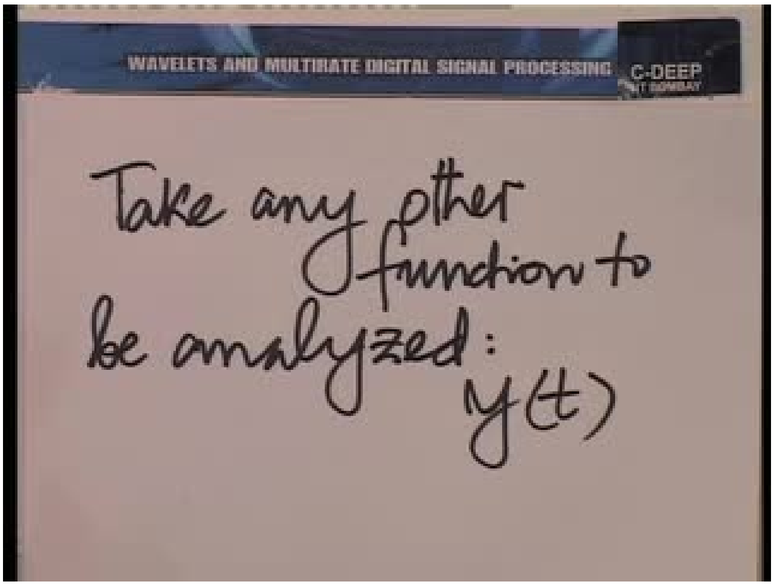

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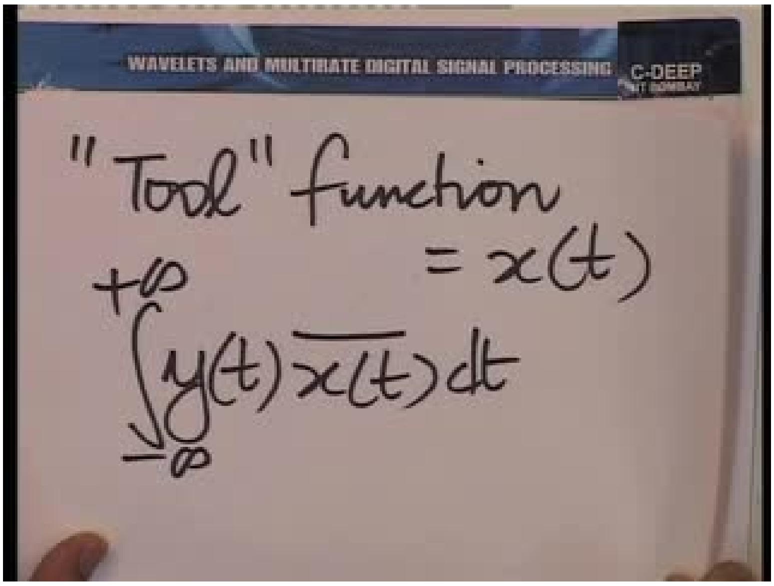

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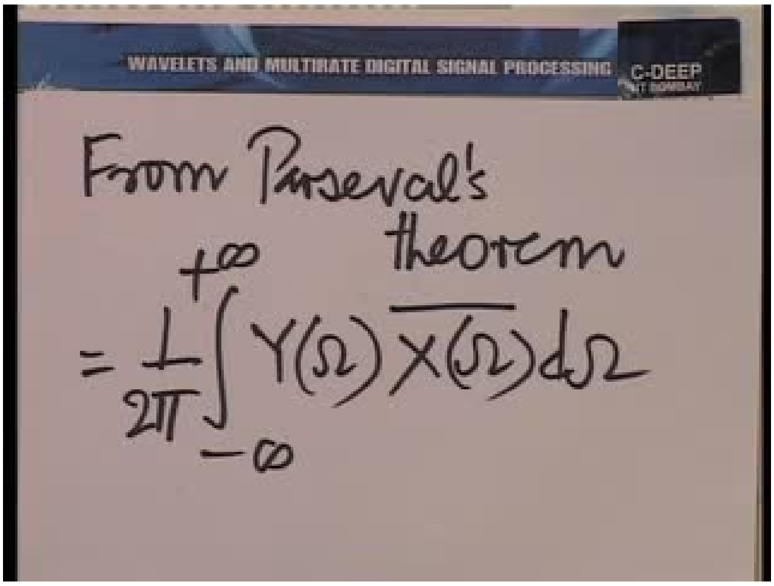

So, now we can talk of what is called tiling the time frequency plane: it essentially means covering this plane, which such rectangles, with rectangles corresponding to functions; in fact, I say rectangular tiles corresponding to such functions. And what we are essentially saying, is that, when we take the dot product of such a function, you known, let us, let us, be specific, so when we take any other function, take any other function to be analyzed; let us say y of t; and let this tool function, so to, is speak be x of t. Then Parseval's theorem says that - if I take the dot product of y t with x t, essentially y t x bar t d t, then the same thing happens in the frequency domain; Parseval's theorem says this is equal to 1 by 2 pi, you do not worry about factor, 1 by 2 pi essentially just normalizing constant, the important thing is inside, Fourier transform of y fourier transform of x complex conjugated integrated over omega.

So, what is means physically is, if I take the dot product, if I take the projection of a function y t, on such a tool function x t in time, I am doing the same projection in frequency; so, by projecting this y t on x t in time, I am essentially extracting information about y t in a time region between t 0 minus sigma t, and t 0 plus sigma t. Parsval's theorem tells be, simultaneously I am also extracting information of the Fourier transform of y in a region capture between capital omega 0 minus sigma omega and capital omega 0 plus sigma omega; simultaneously I am extracting information in the time frequency plane about y t in that rectangle using a tool function.

So, when I take the dot product of y t which such a tool function, I am immediately extracting information about y t in that rectangle of the time frequency plane. And now, we have an interpretation, the rectangular region over which you want to extract information about a function y t cannot be smaller in area than 2 units; well you known the number 2 is not the point, I mean you can always change that number by changing the unit, we use a certain unit here; in fact, we will use angular frequency as for as frequency goes.

If you used hertz frequency would get a different number there that is not the point; the point is that there is a minimum rectangular area over which you can view y t; there is a minimum joint resolution, you know this, in the sense you cannot go finer, then that resolution when you look that two domains together, but the good news, is that, there are many different ways in which you can look at a small domain, when it is within the uncertainty limit. In fact, now tiling has a different interpretation, if I wish to analyze a function, I think of the function in the time domain and in the frequency domain together.

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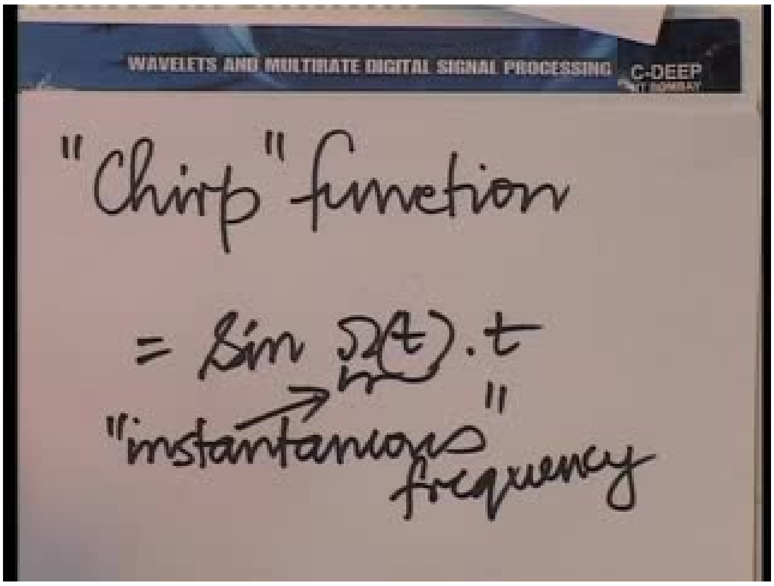

So, essentially I am viewing the function in a joint domain, and I wish to see how the function looks in a joint domain; let be give an example, suppose I have what is called a chirp function, you know a chirp function is named of, to the sound of birds, when birds chirp broadly or crudely, the chirp wave form has a pattern, which is a continuously changing frequency, in time instantaneous frequency.

So, it is something of the form say, sin omega as a function of t t; so, essentially this instantaneous frequency, so to speak of the same way is the function of time. Now, you know an important question in analyzing chirp function that one in count as sometimes in radar or in sonar, is trace this variation of the instantaneous frequency in time, and there the uncertainty principles hits hard.

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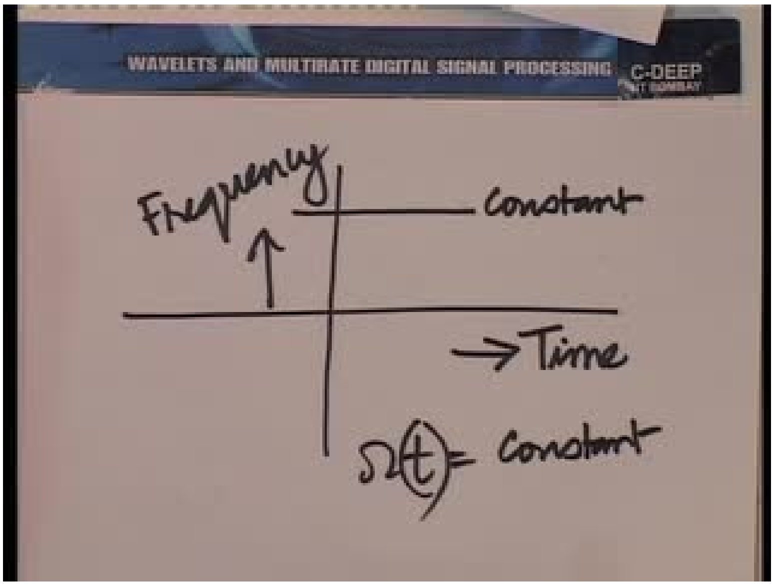

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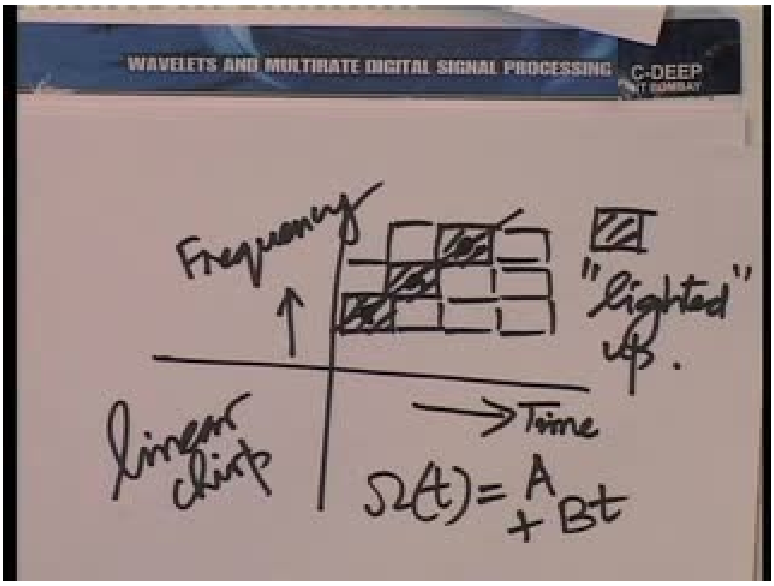

So, what we are saying is, you know in the time frequency plane, suppose omega t is a constant function of time is, is a, is a constant function of time, then we are talking about a frequency independent of time; suppose, that is a linear function of time which is often true, something like say, this omega is a function of t is some A plus B t, this is called a linear chirp. What would you try to do with the tool? You would try to trace this pattern, and that is where the uncertainty principles hits you; it is says that - you can only put rectangles that look something like this, and you can never really trace what is happening with in that rectangle; so, you could totally say if the rectangle, you know suppose you think of putting many rectangles on this time frequency plane, you know, and so, you had rectangle like this put them all over; and you would see that these rectangles are lighted up so to speak; so, you know I am shade these the shaded rectangles are lighted up.

In other words, if I looked at the dot product of this function y t which has this linear chirp nature, with this set up tiles; that means, many of these functions which are different tiles in this time frequency plane. The tiles in which the function essentially is prominent would be lighted up, other piece magnitude would be launch there; the intensity would be launch of the dot product.

Now, all that even indicate, is that, you know it would show you decrease point; so, here for example, if you would back this time frequency plane, each of these rectangles would correspond to single point here; so, if would show this point, this point, and this point, the once at live on the line as lighted up, alright.

So, these, these points would be lighted up here, but you cannot could closer than this point; so, of course, this point would live on what can be seen to be straight line, but you would not know what is happen between the points that what is the uncertainty principle says, you cannot get instantaneous frequency as a function of time exactly, but you can do it as closely as we desire by taking smaller rectangles; and the smallest, this, the, area of rectangles that you take to within, of course, the uncertainty principle the better you can make this estimate. One of the meanings of a time frequency plane, and its timing we shall more in the next lecture.

Thank you